GENERALISED TRAPEZOID TYPE INEQUALITIES FOR VECTOR-VALUED FUNCTIONS AND APPLICATIONS

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Abstract. A generalisation of the trapezoid formula for vector-valued functions and applications for operatorial inequalities and vector-valued integral equations are given.

1. Introduction

Let X be a Banach space and $-\infty < a < b < \infty$. A function $f : [a, b] \to X$ is called *measurable* if there exists a sequence of simple functions $f_n : [a, b] \to X$ which converges punctually almost everywhere on $[a, b]$ at f. We recall that a measurable function $f : [a, b] \to X$ is *Bochner integrable* if and only if its norm function (i.e., the function $t \mapsto \|f(t)\| : [a, b] \to \mathbb{R}_+$) is Lebesgue integrable on $[a, b]$. The Banach space X has the *Radon-Nikodym's property* if every $X-$valued, absolutely continuous function f defined on $[a, b]$ is differentiable almost everywhere on $[a, b]$. For other details about the Radon-Nikodym spaces, see [2, pp. 217-219]. It is known that if $g : [a, b] \to X$ (X being an arbitrary Banach space) is a Bochner integrable function, then its primitive function (i.e., the function given by $f(t) = \int_a^t g(s)\, ds$, $t \in [a, b]$) is differentiable almost everywhere and $f'(t) = g(t)$ almost everywhere on $[a, b]$.

In this paper we point out a generalized trapezoid formula for vector-valued functions and Bochner integral and apply it for operatorial inequalities in Banach spaces and for approximating the solutions of certain integral equations. Some numerical experiments are also provided.

2. Integral Inequalities

The following theorem holds.

Theorem 1. *Let $(X, \|\cdot\|)$ be a Banach space with the Radon-Nikodym property and $f : [a, b] \to X$ be an absolutely continuous function on $[a, b]$ with the property that $f' \in L_\infty([a, b]; X)$, i.e.,*

$$|||f'|||_{[a,b],\infty} := ess \sup_{t\in[a,b]} \|f'(t)\| < \infty.$$

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Then we have the inequalities:

$$\begin{aligned}
&\left\| \frac{(s-a) f(a) + (b-s) f(b)}{b-a} - \frac{1}{b-a} (B) \int_a^b f(t)\, dt \right\| \qquad (2.1)\\
\leq\ & \frac{1}{b-a} \int_a^b |t-s| \left\| f'(t) \right\| dt \\
\leq\ & \frac{1}{2(b-a)} \left[(s-a)^2 \left\| |f'| \right\|_{[a,s],\infty} + (b-s)^2 \left\| |f'| \right\|_{[s,b],\infty} \right] \\
\leq\ & \left[\frac{1}{4} + \left(\frac{s - \frac{a+b}{2}}{b-a} \right)^2 \right] (b-a) \left\| |f'| \right\|_{[a,b],\infty} \\
\leq\ & \frac{1}{2} (b-a) \left\| |f'| \right\|_{[a,b],\infty}
\end{aligned}$$

for any $s \in [a,b]$.

Proof. Using the integration by parts formula, we may write that

$$(B) \int_a^b (t-s) f'(t)\, dt = (b-s) f(b) + (s-a) f(a) - (B) \int_a^b f(t)\, dt \qquad (2.2)$$

for any $s \in [a,b]$.

Taking the norm on (2.2), we get

$$\begin{aligned}
&\left\| (b-s) f(b) + (s-a) f(a) - (B) \int_a^b f(t)\, dt \right\| \\
=\ & \left\| (B) \int_a^b (t-s) f'(t)\, dt \right\| \leq \int_a^b |t-s| \left\| f'(t) \right\| dt =: B(s)
\end{aligned}$$

and the first inequality in (2.1) is proved.

We also have

$$\begin{aligned}
B(s) &= \int_a^s (s-t) \left\| f'(t) \right\| dt + \int_s^b (t-s) \left\| f'(t) \right\| dt \\
&\leq \left\| |f'| \right\|_{[a,s],\infty} \int_a^s (s-t)\, dt + \left\| |f'| \right\|_{[s,b],\infty} \int_s^b (t-s)\, dt \\
&= \frac{1}{2} \left[(s-a)^2 \left\| |f'| \right\|_{[a,s],\infty} + (b-s)^2 \left\| |f'| \right\|_{[s,b],\infty} \right],
\end{aligned}$$

which proves the second inequality in (2.1).

The third and fourth inequalities are obvious and we omit the details. ∎

Corollary 1. *With the assumptions of Theorem 1, we have the trapezoid inequality:*

$$\begin{aligned}(2.3)\quad & \left\| \frac{f(a)+f(b)}{2} - \frac{1}{b-a}(B)\int_a^b f(t)\,dt \right\| \\ & \leq \frac{1}{b-a}\int_a^b \left| t - \frac{a+b}{2} \right| \|f'(t)\|\,dt \\ & \leq \frac{b-a}{2}\left[\||f'|\|_{[a,\frac{a+b}{2}],\infty} + \||f'|\|_{[\frac{a+b}{2},b],\infty} \right] \\ & \leq \frac{1}{4}(b-a)\||f'|\|_{[a,b],\infty}.\end{aligned}$$

Remark 1. *We observe that for the scalar function $B:[a,b]\to\mathbb{R}$ defined above, we have*

$$B'(s) = \int_a^s \|f'(t)\|\,dt - \int_s^b \|f'(t)\|\,dt, \quad s\in(a,b) \tag{2.4}$$

and

$$B''(s) = 2\|f'(s)\| \geq 0, \quad s\in(a,b), \tag{2.5}$$

showing that $B(\cdot)$ is convex on $[a,b]$.
If $s_m\in(a,b)$ is such that

$$\int_a^{s_m} \|f'(t)\|\,dt = \int_{s_m}^b \|f'(t)\|\,dt, \tag{2.6}$$

then

$$\begin{aligned}\inf_{s\in[a,b]} B(s) &= B(s_m) = \frac{1}{b-a}\int_a^b |t-s_m|\,\|f'(t)\|\,dt \\ &= \frac{1}{b-a}\left[\int_{s_m}^b t\|f'(t)\|\,dt - \int_a^{s_m} t\|f'(t)\|\,dt \right] \\ &= \frac{1}{b-a}\int_a^b sgn(t-s_m)\|f'(t)\|\,dt.\end{aligned}$$

Consequently, for a $s_m\in(a,b)$ satisfying (2.6), we have

$$\begin{aligned}(2.7)\quad & \left\| \frac{(s_m-a)f(a)+(b-s_m)f(b)}{b-a} - \frac{1}{b-a}(B)\int_a^b f(t)\,dt \right\| \\ & \leq \frac{1}{b-a}\int_a^b sgn(t-s_m)\|f'(t)\|\,dt.\end{aligned}$$

The version in terms of the $p-$norms, $p\in[1,\infty)$ of the derivative f' is embodied in the following theorem.

Theorem 2. *Let $(X,\|\cdot\|)$ be a Banach space with the Radon-Nikodym property and $f:[a,b]\to X$ be an absolutely continuous function on $[a,b]$ with the property that $f'\in L_p([a,b];X)$, $p\in[1,\infty)$, i.e.,*

$$\||f'|\|_{[a,b],p} := \left(\int_a^b \|f'(t)\|^p\,dt \right)^{\frac{1}{p}} < \infty. \tag{2.8}$$

Then we have the inequalities:

$$\left\| \frac{(s-a) f(a) + (b-s) f(b)}{b-a} - \frac{1}{b-a} (B) \int_a^b f(t)\, dt \right\| \tag{2.9}$$

$$\leq \frac{1}{b-a} \int_a^b |t-s| \, \| f'(t) \| \, dt$$

$$\leq \begin{cases} \frac{1}{b-a} \left[(s-a) \, |||f'|||_{[a,s],1} + (b-s) \, |||f'|||_{[s,b],1} \right] \\ \qquad \text{if } f' \in L_1([a,b]; X); \\ \\ \frac{1}{(b-a)(q+1)^{\frac{1}{q}}} \left[(s-a)^{\frac{1}{q}+1} |||f'|||_{[a,s],p} + (b-s)^{\frac{1}{q}+1} |||f'|||_{[s,b],p} \right] \\ \qquad \text{if } f' \in L_p([a,b]; X), \ p>1, \ \frac{1}{p}+\frac{1}{q}=1; \end{cases}$$

$$\leq \begin{cases} \left[\frac{1}{2} + \left| \frac{s - \frac{a+b}{2}}{b-a} \right| \right] |||f'|||_{[a,b],1} \\ \qquad \text{if } f' \in L_1([a,b]; X); \\ \\ \frac{1}{(q+1)^{\frac{1}{q}}} \left[\left(\frac{s-a}{b-a} \right)^{\frac{1}{q}+1} + \left(\frac{b-s}{b-a} \right)^{\frac{1}{q}+1} \right] (b-a)^{\frac{1}{q}} |||f'|||_{[a,b],p} \\ \qquad \text{if } f' \in L_p([a,b]; X), \ p>1, \ \frac{1}{p}+\frac{1}{q}=1; \end{cases}$$

for any $s \in (a,b)$.

Proof. We have

$$\begin{aligned} B(s) &= \int_a^s (s-t) \| f'(t) \| \, dt + \int_s^b (t-s) \| f'(t) \| \, dt \\ &\leq (s-a) \int_a^s \| f'(t) \| \, dt + (b-s) \int_s^b \| f'(t) \| \, dt \\ &= (s-a) \, |||f'|||_{[a,s],1} + (b-s) \, |||f'|||_{[s,b],1} . \end{aligned}$$

Using Hölder's integral inequality, we also have

$$\begin{aligned} B(s) &\leq \left(\int_a^s (s-t)^q \, dt \right)^{\frac{1}{q}} \left(\int_a^s \| f'(t) \|^p \, dt \right)^{\frac{1}{p}} \\ &\quad + \left(\int_s^b (t-s)^q \, dt \right)^{\frac{1}{q}} \left(\int_s^b \| f'(t) \|^p \, dt \right)^{\frac{1}{p}} \\ &= \frac{(s-a)^{\frac{1}{q}+1}}{(q+1)^{\frac{1}{q}}} |||f'|||_{[a,s],p} + \frac{(b-s)^{\frac{1}{q}+1}}{(q+1)^{\frac{1}{q}}} |||f'|||_{[s,b],p} \end{aligned}$$

and the first inequality in (2.5) is proved.

Now, we observe that

$$\begin{aligned} & (s-a) \, |||f'|||_{[a,s],1} + (b-s) \, |||f'|||_{[s,b],1} \\ \leq \ & \max(s-a, b-s) \left[|||f'|||_{[a,s],1} + |||f'|||_{[s,b],1} \right] \\ = \ & \left[\frac{1}{2}(b-a) + \left| s - \frac{a+b}{2} \right| \right] |||f'|||_{[a,b],1} \end{aligned}$$

and, by the discrete Hölder's inequality

$$\begin{aligned} & (s-a)^{\frac{1}{q}+1} \left\|\left|f'\right|\right\|_{[a,s],p} + (b-s)^{\frac{1}{q}+1} \left\|\left|f'\right|\right\|_{[s,b],p} \\ \leq & \left[\left((s-a)^{\frac{1}{q}+1}\right)^q + \left((b-s)^{\frac{1}{q}+1}\right)^q\right]^{\frac{1}{q}} \times \left[\left\|\left|f'\right|\right\|_{[a,s],p}^p + \left\|\left|f'\right|\right\|_{[s,b],p}^p\right]^{\frac{1}{p}} \\ = & \left[(s-a)^{q+1} + (b-s)^{q+1}\right]^{\frac{1}{q}} \left\|\left|f'\right|\right\|_{[a,b],p} \end{aligned}$$

and the last part of (2.5) is also proved. ∎

The following trapezoid type inequality holds.

Corollary 2. *With the assumptions of Theorem 2, we have the inequalities:*

$$\begin{aligned} & \left\| \frac{f(a)+f(b)}{2} - \frac{1}{b-a}(B)\int_a^b f(t)\,dt \right\| \qquad (2.10) \\ \leq & \frac{1}{b-a}\int_a^b \left| t - \frac{a+b}{2} \right| \left\| f'(t) \right\| dt \\ \leq & \begin{cases} \frac{1}{2} \left\|\left|f'\right|\right\|_{[a,b],1} & \text{if } f' \in L_1([a,b];X); \\ \frac{(b-a)^{\frac{1}{q}}}{2^{1+\frac{1}{q}}(q+1)^{\frac{1}{q}}} \left[\left\|\left|f'\right|\right\|_{[a,\frac{a+b}{2}],p} + \left\|\left|f'\right|\right\|_{[\frac{a+b}{2},b],p}\right] \\ \qquad \text{if } f' \in L_p([a,b];X),\ p>1,\ \frac{1}{p}+\frac{1}{q}=1; \end{cases} \\ \leq & \begin{cases} \frac{1}{2} \left\|\left|f'\right|\right\|_{[a,b],1} & \text{if } f' \in L_1([a,b];X); \\ \frac{1}{2(q+1)^{\frac{1}{q}}}(b-a)^{\frac{1}{q}} \left[\left\|\left|f'\right|\right\|_{[a,b],p}\right] \\ \qquad \text{if } f' \in L_p([a,b];X),\ p>1,\ \frac{1}{p}+\frac{1}{q}=1; \end{cases} \end{aligned}$$

Remark 2. *The above results both generalise and extend for vector-valued functions the results in* [1].

3. Applications for the Operator Inequality

Let X be an arbitrary Banach space and $\mathcal{L}(X)$ the Banach space of all bounded linear operators on X. We recall that if $T \in \mathcal{L}(X)$, then its operatorial norm is defined by

$$\|T\| = \sup\{\|Tx\| : x \in X, \|x\| \leq 1\}.$$

We denote by $r(T)$, $\rho(T)$, $\sigma(T)$ the spectral radius, the resolvent set and the spectrum of T, respectively. It is well-known that $\rho(T)$ is the set of all complex numbers λ such that $\lambda I - T$ is an invertible operator in $\mathcal{L}(X)$. Here $T^0 := I$ is the identity operator in $\mathcal{L}(X)$. The spectrum of T is $\sigma(T) := \mathbb{C}\backslash\rho(T)$ and the spectral radius of T is given by the following formulae

$$r(T) = \sup\{|\lambda| : \lambda \in \sigma(T)\} = \lim_{n\to\infty} \|T^n\|^{\frac{1}{n}} = \inf_{\substack{n\in\mathbb{N} \\ n\geq 1}} \|T^n\|^{\frac{1}{n}}.$$

It is clear that $r(T) \leq \|T\|$.

If $r(T) < 1$, then the series $\left(\sum_{n\geq 0} T^n\right)$ converges absolutely and its sum is $(I-T)^{-1}$. Indeed, if m is a strictly positive integer number such that $\|T^m\| < 1$ and $p > 1$, then:

$$\begin{aligned}\sum_{n=0}^{\infty}\|T^n\| &\leq \left(\left\|T^0\right\|+\cdots+\left\|T^{m-1}\right\|\right)\sum_{k=0}^{\infty}\|T^m\|^k \\ &= \left(\left\|T^0\right\|+\cdots+\left\|T^{m-1}\right\|\right)\cdot\frac{1}{1-\|T^m\|},\end{aligned}$$

and

$$(I-T)\left(I+T+T^2+\cdots+T^{mp-1}\right) = I - T^{mp} \to I \quad \text{when} \quad p\to\infty$$

because

$$\|T^{mp}\| \leq \|T^m\|^p \to 0 \quad \text{when} \quad p \to \infty.$$

Now, let $T \in \mathcal{L}(X)$ such that $0 < r(T) < 1$ and let $0 < a < b < \frac{1}{r(T)}$. It is clear that $r(tT) = tr(T)$ for all $t > 0$. In the following we will consider some operator-valued functions defined on $[a,b]$ and we write for them the inequalities from Theorem 1.

The series $\left(\sum_{n\geq 0}(tT)^n\right)$ converges absolutely and uniformly on $[a,b]$ and its sum is given by

$$s(t) := \sum_{n=0}^{\infty}(tT)^n = [I-(tT)]^{-1} = t^{-1}R\left(t^{-1},T\right),$$

where

$$R(\lambda,t) := (\lambda I - T)^{-1}, \quad (\lambda \in \rho(T)),$$

is the resolvent operator of T.

1. Let $0 < a < b < \|T\|^{-1} \leq (r(T))^{-1}$. Consider the function f defined by

$$\tau \mapsto f(\tau) := s^2(\tau) : [a,b] \to \mathcal{L}(X).$$

In order to apply Theorem 1 for f, we remark that:

(a)

$$\begin{aligned}\frac{d}{d\tau}\left[R\left(\frac{1}{\tau},T\right)\right] &= \lim_{t\to\tau}\frac{R\left(\frac{1}{t},T\right)-R\left(\frac{1}{\tau},T\right)}{t-\tau} \\ &= \lim_{t\to\tau}\frac{1}{t\tau}R\left(\frac{1}{t},T\right)R\left(\frac{1}{\tau},T\right) \\ &= \frac{1}{\tau^2}R^2\left(\frac{1}{\tau},T\right) = f(\tau),\ \tau\in[a,b].\end{aligned}$$

(b)

$$\frac{d}{d\tau}[f(\tau)] = -\frac{2}{\tau^3}R^2\left(\frac{1}{\tau},T\right)+\frac{2}{\tau^4}R^3\left(\frac{1}{\tau},T\right) = \frac{2}{\tau}s^2(\tau)[I-s(\tau)].$$

Moreover,

$$\|s(\tau)\| \leq \sum_{n=0}^{\infty}\|\tau T\|^n = (1-\tau\|T\|)^{-1}$$

and

$$\|I - s(\tau)\| \leq \tau \|T\| \cdot \sum_{n=0}^{\infty} \|\tau T\|^n = \tau \|T\| (1 - \tau \|T\|)^{-1}$$

and thus

$$\|f'(\tau)\| \leq 2 \|T\| (1 - \tau \|T\|)^{-3}, \text{ for all } \tau \in [a, b].$$

Then from the second estimate of (2.1) we obtain

$$\left\| \frac{s-a}{a^2} R^2\left(\frac{1}{a}, T\right) + \frac{b-s}{b^2} R^2\left(\frac{1}{b}, T\right) - \frac{b-a}{ab} R\left(\frac{1}{a}, T\right) R\left(\frac{1}{b}, T\right) \right\| \tag{3.1}$$
$$\leq \left[(s-a)^2 \cdot \frac{\|T\|}{(1 - s\|T\|)^3} + (b-s)^2 \cdot \frac{\|T\|}{(1 - b\|T\|)^3} \right].$$

If T is a real number, $0 < T < 1$ and $0 < a \leq s \leq b < \frac{1}{T}$ then from (3.1) we get the inequality

$$\left| \frac{s-a}{(1-aT)^2} + \frac{b-s}{(1-bT)^2} - \frac{b-a}{(1-aT)(1-bT)} \right| \leq \frac{T(s-a)^2}{(1-sT)^3} + \frac{T(b-s)^2}{(1-bT)^3}.$$

2. Let a and b be two real numbers with $a < b$ and $U \in \mathcal{L}(X)$ be a non-null operator. We recall that the series $\left(\sum_{n \geq 0} \frac{(tU)^n}{n!}\right)$ converges absolutely and locally uniformly for $t \in \mathbb{R}$ with respect to the operatorial norm in $\mathcal{L}(X)$. From the third estimate of (2.9), it follows that

$$\left\| \frac{(s-a)e^{aU} + (b-s)e^{bU}}{b-a} - \frac{1}{b-a} \int_a^b e^{tU} dt \right\| \tag{3.2}$$
$$\leq \left[\frac{1}{2} + \left| \frac{s - \frac{a+b}{2}}{b-a} \right| \right] (b-a) \cdot p(a, b, U),$$

where

$$p(a,b,U) = \begin{cases} e^{b\|U\|} - e^{a\|U\|}, & \text{if } a \geq 0; \\ e^{-a\|U\|} - e^{-b\|U\|}, & \text{if } b \leq 0; \\ e^{b\|U\|} + e^{-a\|U\|} - 2, & \text{if } a \leq 0 \leq b. \end{cases}$$

If $s = \frac{a+b}{2}$ and U is an invertible operator in $\mathcal{L}(X)$, then from (3.2) we get the following inequality

$$\left\| \frac{e^{aU} + e^{bU}}{2} - U^{-1} \frac{e^{bU} - e^{aU}}{b-a} \right\| \leq \frac{1}{2}(b-a)\, p(a,b,U).$$

3. Let $a, b \in \mathbb{R}$ with $a < b$ and A, B two linear and bounded operators acting on X such that $\|A\| \neq \|B\|$. Then the following inequality holds:

$$\left\| \frac{e^{(b-a)A}(B-A) + (B-A)e^{(b-a)B}}{2} - \left[e^{(b-a)B} - e^{(b-a)A} \right] \right\| \tag{3.3}$$
$$\leq \frac{b-a}{2} \|B - A\| \cdot (\|A\| + \|B\|) \cdot \frac{e^{(b-a)\|B\|} - e^{(b-a)\|A\|}}{\|B\| - \|A\|}.$$

In order to prove the inequality (3.3), we consider the function

$$f:[a,b]\to\mathcal{L}(X),\ f(t)=e^{(b-t)A}(B-A)e^{(t-a)B}$$

and we apply the first estimate of (2.1) for $s=\frac{a+b}{2}$.
We have that

$$\begin{aligned}\int_a^b f(t)\,dt &= \int_a^b e^{(b-t)A}\frac{d}{dt}\left[e^{(t-a)B}\right]dt+\int_a^b\frac{d}{dt}\left[e^{(b-t)A}\right]e^{(t-a)B}dt \\ &= 2\left[e^{(b-a)B}-e^{(b-a)A}\right]-\int_a^b f(t)\,dt\end{aligned} \tag{3.4}$$

and

$$\begin{aligned}\|f'(t)\| &= \|-Af(t)+f(t)B\| \\ &\le \frac{(\|A\|+\|B\|)\|B-A\|}{\|B\|-\|A\|}\cdot e^{(b-t)\|A\|}(\|B\|-\|A\|)(B-A)e^{(t-a)\|B\|}.\end{aligned}$$

Using (3.4), it follows that

$$\begin{aligned}&\int_a^b |t-s|\,\|f'(t)\|\,dt \\ \le\ &\max\{b-s,s-a\}\cdot\frac{(\|A\|+\|B\|)\|B-A\|}{\|B\|-\|A\|}\cdot\left[e^{(b-a)\|B\|}-e^{(b-a)\|A\|}\right].\end{aligned}$$

Now the inequality (3.3) can be easily obtained from the first estimate of (2.1) if we put $s=\frac{a+b}{2}$.

4. A Quadrature Formula of Generalised Trapezoid Type

Now, let $I_n: a=x_0<x_1<\cdots<x_{n-1}<x_n=b$ be a partitioning of the interval $[a,b]$ and defined $h_i=x_{i+1}-x_i$, $\nu(h):=\max\{h_i|i=0,\dots,n-1\}$. Consider for the mapping $f:[a,b]\to X$, where X is a Banach space with the Radon-Nicodym property, the following *generalised trapezoid rule:*

$$T_n(f,I_n,\boldsymbol{\xi}):=\sum_{i=0}^{n-1}\left[(\xi_i-x_i)f(x_i)+(x_{i+1}-\xi_i)f(x_{i+1})\right], \tag{4.1}$$

where $\boldsymbol{\xi}:=(\xi_0,\dots,\xi_{n-1})$ and $\xi_i\in[x_i,x_{i+1}]$ $(i=0,\dots,n-1)$ are intermediate (arbitrarily chosen) points.

The following theorem holds.

Theorem 3. *Let f be as in Theorem 1. Then we have*

$$(B)\int_a^b f(t)\,dt=T_n(f,I_n,\boldsymbol{\xi})+R_n(f,I_n,\boldsymbol{\xi}), \tag{4.2}$$

where $T_n(f, I_n, \boldsymbol{\xi})$ is the generalised trapezoid rule defined in (4.1) and the remainder $R_n(f, I_n, \boldsymbol{\xi})$ in (4.2) satisfies the bound

$$\begin{aligned}
&\|R_n(f, I_n, \boldsymbol{\xi})\| \\
\leq\ & \sum_{i=0}^{n-1} \int_{x_{i,}}^{x_{i+1}} |t-\xi_i| \, \|f'(t)\| \, dt \\
\leq\ & \frac{1}{2} \sum_{i=0}^{n-1} \left[(\xi_i - x_i)^2 \, |||f'|||_{[x_i,\xi_i],\infty} + (x_{i+1}-\xi_i)^2 \, |||f'|||_{[\xi_i, x_{i+1}],\infty} \right] \\
\leq\ & \sum_{i=0}^{n-1} \left[\frac{1}{4} h_i^2 + \left(\xi_i - \frac{x_i + x_{i+1}}{2} \right)^2 \right] |||f'|||_{[x_i, x_{i+1}],\infty} \\
\leq\ & \frac{1}{2} \sum_{i=0}^{n-1} h_i^2 \, |||f'|||_{[x_i, x_{i+1}],\infty} \leq \frac{1}{2} \, |||f'|||_{[a,b],\infty} \sum_{i=0}^{n-1} h_i^2 \\
\leq\ & \frac{1}{2} (b-a) \, \nu(h) \, |||f'|||_{[a,b],\infty} .
\end{aligned} \tag{4.3}$$

Proof. Apply the inequality (2.1) on the interval $[x_i, x_{i+1}]$ to obtain

$$\begin{aligned}
&\left\| (\xi_i - x_i) f(x_i) + (x_{i+1} - \xi_i) f(x_{i+1}) - (B) \int_{x_{i,}}^{x_{i+1}} f(t) \, dt \right\| \\
\leq\ & \int_{x_{i,}}^{x_{i+1}} (t - \xi_i) \, \|f'(t)\| \, dt \\
\leq\ & \frac{1}{2} \left[(\xi_i - x_i)^2 \, |||f'|||_{[x_i,\xi_i],\infty} + (x_{i+1}-\xi_i)^2 \, |||f'|||_{[\xi_i, x_{i+1}],\infty} \right] \\
\leq\ & \left[\frac{1}{4} h_i^2 + \left(\xi_i - \frac{x_i + x_{i+1}}{2} \right)^2 \right] |||f'|||_{[x_i, x_{i+1}],\infty} \\
\leq\ & \frac{1}{2} h_i^2 \, |||f'|||_{[x_i, x_{i+1}],\infty}
\end{aligned} \tag{4.4}$$

for any $i = 0, \ldots, n-1$.

Summing over i from 0 to $n-1$ and using the generalised triangle inequality for sums, we obtain (4.3). ∎

If we consider the trapezoid formula given by

$$T_n(f, I_n) := \sum_{i=0}^{n-1} h_i \left[\frac{f(x_i) + f(x_{i+1})}{2} \right], \tag{4.5}$$

then we may state the following corollary.

Corollary 3. *With the assumptions in Theorem 1, we have*

$$(B) \int_a^b f(t) \, dt = T_n(f, I_n) + W_n(f, I_n), \tag{4.6}$$

where $T_n(f, I_n)$ is the vector-valued trapezoid quadrature rule given in (4.5) and the remainder $W_n(f, I_n)$ satisfies the estimate

$$\begin{aligned}
(4.7)\quad \|W_n(f, I_n)\| &\leq \sum_{i=0}^{n-1} \int_{x_{i,}}^{x_{i+1}} \left| t - \frac{x_i + x_{i+1}}{2} \right| \|f'(t)\| \, dt \\
&\leq \frac{1}{8} \sum_{i=0}^{n-1} h_i^2 \left[|||f'|||_{\left[x_i, \frac{x_i + x_{i+1}}{2}\right], \infty} + |||f'|||_{\left[\frac{x_i + x_{i+1}}{2}, x_{i+1}\right], \infty} \right] \\
&\leq \frac{1}{4} \sum_{i=0}^{n-1} h_i^2 \, |||f'|||_{[x_i, x_{i+1}], \infty} \leq \frac{1}{4} |||f'|||_{[a,b], \infty} \sum_{i=0}^{n-1} h_i^2 \\
&\leq \frac{1}{4} |||f'|||_{[a,b], \infty} \, \nu(h).
\end{aligned}$$

Remark 3. *It is obvious that $\|W_n(f, I_n)\| \to 0$ as $\nu(h) \to 0$, showing that $T_n(f, I_n)$ is an approximation for the Bochner integral $(B) \int_a^b f(t) \, dt$ with order one accuracy.*

Remark 4. *Similar bounds for the remainders $R_n(f, I_n, \boldsymbol{\xi})$ and $W_n(f, I_n)$ may be obtained in terms of the $p-$norm $(p \in [1, \infty))$, but we omit the details.*

5. Applications for Vector-Valued Integral Equations

We consider the Voltera integral equation:

$$(A, f) \qquad u(t) = f(t) + \int_0^t K(t - \tau) A u(\tau) \, d\tau, \quad t \geq 0,$$

where A is a closed linear operator on a Banach space X, f is a $X-$valued, continuous function defined on $\mathbb{R}_+ := [0, \infty)$ and $K(\cdot)$ is a locally integrable and non-null scalar kernel on $\mathbb{R}_+$. A strongly continuous family $\{U(t) : t \geq 0\} \subset \mathcal{L}(X)$ (that is, for any $x \in X$ the maps $t \mapsto U(t)x : \mathbb{R}_+ \to X$ are continuous) is said to be a *solution family* for (A, f) if

$$\begin{aligned}
(5.1) \qquad AU(t)x &= U(t)Ax \quad \text{for all} \quad x \in D(A), \quad t \geq 0, \quad \text{and} \\
(5.2) \qquad U(t)x &= x + A \int_0^t K(t - \tau) U(\tau) x d\tau, \quad x \in X, \ t \geq 0.
\end{aligned}$$

For example, if A is the infinitesimal generator of the strongly continuous semigroup $\mathbf{T} = \{T(t) : t \geq 0\} \subset \mathcal{L}(X)$, then the family $\mathbf{T}$ is a solution family for (A, f), i.e., (5.1) and (5.2) hold, see [4], [5].

Also, if B is the generator of the *strongly continuous cosine function* $\mathcal{C} := \{C(t) : t \in \mathbb{R}\} \subset \mathcal{L}(X)$ then the family $\{C(t) : t \geq 0\}$ is a solution family for (B, f), see for example [7], [3].

Let $h > 0$. An $X-$valued, continuous function $v(\cdot)$ defined on $[0, h]$ is called a *mild solution* of (A, f) if,

$$(5.3) \qquad v(t) = f(t) + A \int_0^t K(t - \tau) v(\tau) \, d\tau, \quad \text{for all } t \in [0, h].$$

We denote by $W^{1,1}([0, h], X)$ the space of all functions $f \in L^1([0, h], X)$ for which there exists $g \in L^1([0, h], X)$ such that

$$(5.4) \qquad f(t) = f(0) + \int_0^t g(s) \, ds, \quad \text{for all } t \in [0, h].$$

Lemma 1. *Let $f \in W^{1,1}([0,h], X)$, $K(\cdot)$ a function of bounded variation on $[0,h]$ and A a closed, densely defined linear operator acting on X. In these conditions the integral equation (A, f) has a unique solution $v(\cdot)$. Moreover, there exists a solution family $\{V(t) : t \geq 0\} \subset \mathcal{L}(X)$ such that*

$$v(t) = V(t) f(0) + \int_0^t V(t-\tau) f'(\tau) d\tau, \quad t \in [0,h]. \tag{5.5}$$

Here, we only prove the fact that the map given in (5.5) is a solution for the equation (A, f), i.e., it verifies the relation (5.3). For more details, we refer the reader to [6, Proposition 1.2]. Using (5.5) and (5.2) we have that:

$$\begin{aligned}
& A \int_0^t K(t-\tau) v(\tau) d\tau \\
= \; & A \int_0^t K(t-\tau) V(\tau) f(0) d\tau + \int_0^t \left[K(t-\tau) A \int_0^\tau V(\tau - r) f'(\tau) dr \right] d\tau \\
= \; & V(t) f(0) - f(0) + \int_0^t \left(\int_0^t 1_{[0,\tau]}(r) K(t-\tau) AV(\tau - r) f'(\tau) dr \right) d\tau \\
= \; & V(t) f(0) - f(0) + \int_0^t \left(\int_r^t K(t-\tau) AV(\tau - r) f'(\tau) d\tau \right) dr \\
= \; & V(t) f(0) - f(0) + \int_0^t \left(\int_0^{t-r} K(t-r-\sigma) AV(\sigma) f'(r) d\sigma \right) dr \\
= \; & V(t) f(0) - f(0) + \int_0^t (V(t-r) f'(r) - f'(r)) dr \\
= \; & V(t) f(0) - f(0) + \int_0^t V(t-r) f'(r) dr - f(t) + f(0) \\
= \; & v(t) - f(t),
\end{aligned}$$

i.e., (5.3) holds. Here $1_{[0,\tau]}$ is the characteristic function of the interval $[0,\tau]$.

Let $0 = \lambda_0 < \lambda_1 < \cdots < \lambda_{n-1} < \lambda_n = 1$, $\mu_i \in [\lambda_i, \lambda_{i+1}]$, $i \in \{0, 1, \ldots, n-1\}$ and $T > 0$. We preserve all hypothesis about f, $K(\cdot)$ and A from Lemma 1. In addition, we consider that the functions $V(\cdot)$ and $g(\cdot)$ (for g see (5.4)) are continuously differentiable on $[0,T]$. Then the solution $v(\cdot)$ of (A, f) given by (5.5), can be represented as

$$v(t) = V(t) f(0) + T_n(\boldsymbol{\lambda}, \boldsymbol{\mu}, t) + R_n(\boldsymbol{\lambda}, \boldsymbol{\mu}, t), \quad t \in [0,T],$$

where

$$\begin{aligned}
& T_n(\boldsymbol{\lambda}, \boldsymbol{\mu}, t) \\
= \; & t \sum_{i=0}^{n-1} \{ (\mu_i - \lambda_i) V[t(1-\lambda_i)] g(\lambda_i t) + (\lambda_{i+1} - \mu_i) V[t(1-\lambda_{i+1})] g(\lambda_{i+1} t) \}
\end{aligned} \tag{5.6}$$

and the remainder $R_n(\boldsymbol{\lambda}, \boldsymbol{\mu}, t)$ satisfies the estimate

$$\|R_n(\boldsymbol{\lambda}, \boldsymbol{\mu}, t)\| \leq \frac{1}{2} t^2 \nu(\boldsymbol{\lambda}) \cdot \rho(t). \tag{5.7}$$

Here

$$\rho(t) := \|V'\|_{[0,t],\infty} \cdot \|g\|_{[0,t],\infty} + \|V\|_{[0,t],\infty} \cdot \|g'\|_{[0,t],\infty}.$$

Indeed, for a fixed $t>0$, consider the function

$$s\mapsto G(s):=V(t-s)\,g(s),\quad s\in[0,t].$$

Then G is differentiable on $[0,t]$ and

$$\frac{dG(s)}{ds}=-V'(t-s)\,g(s)+V(t-s)\,g'(s)$$

for each $s\in[0,t]$. Moreover,

$$\begin{aligned}\left\|\frac{dG(s)}{ds}\right\| &\le \|V'(t-s)\|\cdot\|g(s)\|+\|V(t-s)\|\cdot\|g'(s)\|\\ &\le \rho(t),\quad \text{for all}\ \ s\in[0,t].\end{aligned}$$

Now it is easy to see that (5.7) follows by the later estimate of (4.3) if we put $x_i=t\cdot\lambda_i$.

Using Corollary 3, the solution $v(\cdot)$ of (A,f) can be represented as

$$\begin{aligned}&v(t) \quad (5.8)\\ &=\frac{t}{2n}\sum_{i=0}^{n-1}\left\{V\left[\frac{t(n-i)}{n}\right]f'\left(\frac{it}{n}\right)+V\left[\frac{t(n-i-1)}{n}\right]f'\left[\frac{(i+1)t}{n}\right]\right\}+W_n,\end{aligned}$$

where $\|W_n\|\le\frac{t}{4n}\cdot\rho(t)$.

For the proof of (5.8), it is sufficient to apply Corollary 3, with f replaced by G and x_i replaced by $\frac{i\cdot t}{n}$.

6. Numerical Examples

1. Let $X=\mathbb{R}^2$, $x=(\xi,\eta)\in X$, $\|x\|_2=\sqrt{\xi^2+\eta^2}$. We consider the linear, 2-dimensional, inhomogeneous differential system

$$\left\{\begin{array}{llll}\dot{u}_1= & -u_1 & & +e^{-t}\\ \dot{u}_2= & & -2u_2 & +\sin t \qquad (t\ge 0).\\ u_1(0)= & u_2(0) & =0 & \end{array}\right. \tag{6.1}$$

If we let $A=\begin{pmatrix}-1&0\\0&-2\end{pmatrix}$; $u(t)=(u_1(t),u_2(t))$; $g(t)=(e^{-t},\sin t)$, $V(t)=e^{tA}=\begin{pmatrix}e^{-t}&0\\0&e^{-2t}\end{pmatrix}$, $K(t)\equiv 1$ and $f(t)=\int_0^t g(\tau)\,d\tau=(1-e^{-t},1-\cos t)$, then the above system can be expressed by the integral equation

$$u(t)=f(t)+A\int_0^t K(t-\tau)\,u(\tau)\,d\tau,\quad t\ge 0. \tag{6.2}$$

The exact solution of (6.1) or (6.2) is

$$\begin{aligned}u(t) &= e^{tA}f(0)+\int_0^t e^{(t-\tau)A}g(\tau)\,d\tau\\ &=\left(te^{-t};\frac{1}{5}\left(e^{-2t}-\cos t+2\sin t\right)\right).\end{aligned} \tag{6.3}$$

From (5.8) we obtain the following approximating formula for $u(\cdot)$:

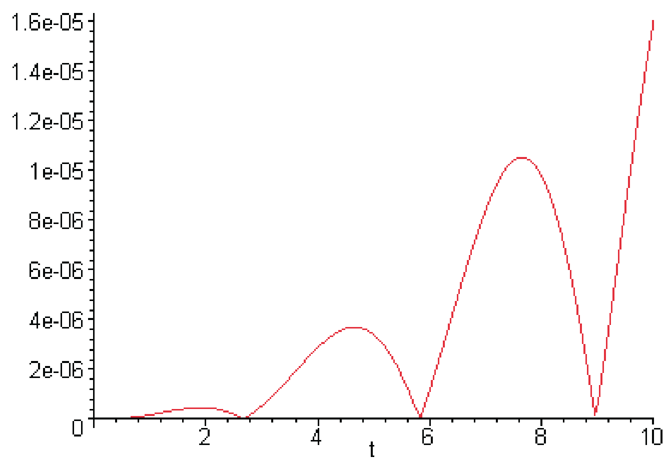
$$u_1(t) = \frac{t}{2n}\sum_{i=0}^{n-1}\left[e^{\frac{-t(n-i)}{n}}\cdot e^{\frac{-ti}{n}} + e^{\frac{-t(n-i-1)}{n}}\cdot e^{\frac{-t(i+1)}{n}}\right] + W_n^{(1)},$$

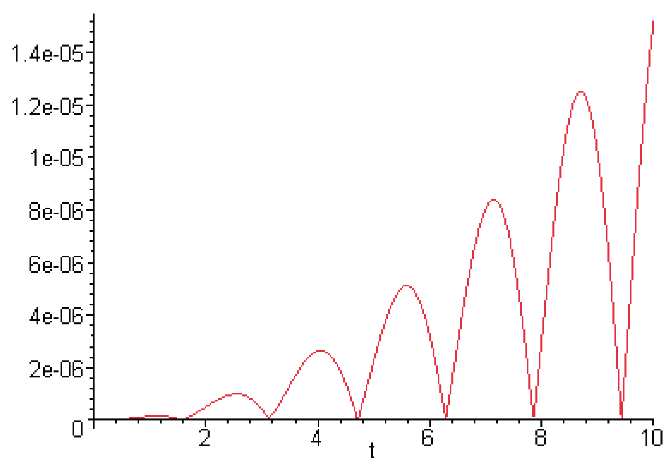
$$u_2(t) = \frac{t}{2n}\sum_{i=0}^{n-1}\left[e^{\frac{-2t(n-i)}{n}}\cdot \sin\left(\frac{ti}{n}\right) + e^{\frac{-2t(n-i-1)}{n}}\cdot \sin\left(\frac{t(i+1)}{n}\right)\right] + W_n^{(2)},$$

where the remainder $W_n = \left(W_n^{(1)}, W_n^{(2)}\right)$ satisfies the estimate

$$\|W_n\|_2 := \sqrt{\left(W_n^{(1)}\right)^2 + \left(W_n^{(2)}\right)^2} \leq \frac{t}{4n}\cdot\rho(t).$$

The Figure 1 contains the behaviour of the error $\varepsilon_n(t) := \|W_n\|_2$.





2. Let X, A and u be as in **1.**, $B=-A^2$,

$$\begin{aligned} V(t) &= C(t) := \sum_{n=0}^{\infty} (-1)^n \frac{(tA)^{2n}}{(2n)!} = \begin{pmatrix} \cos t & 0 \\ 0 & \cos 2t \end{pmatrix}, \quad K(t)=t; \\ u_0 &= (1,0),\ u_1=(0,1) \quad \text{and} \quad f(t)=u_0+tu_1. \end{aligned}$$

Consider the system:

$$\begin{cases} \ddot{u}_1 = -u_1 \\ \ddot{u}_2 = -4u_2 \\ u_1(0)=1; \quad u_2(0)=0 \\ \dot{u}_1(0)=0; \quad \dot{u}_2(0)=1. \end{cases}$$

The above differential system can be written as the following integral equation

$$u(t) = f(t) + B\int_0^t (t-\tau)\, u(\tau)\, d\tau, \quad t \geq 0.$$

The exact solution of the above integral equation is

$$\begin{aligned} u(t) &= C(t)\,u_0 + \int_0^t C(t-\tau)\,u_1 d\tau \\ &= (\cos t, 0) + \left(0, \frac{1}{2}\sin 2t\right) \\ &= \left(\cos t, \frac{1}{2}\sin 2t\right). \end{aligned} \tag{6.4}$$

From (5.8) and (6.4) we also obtain the following approximating formula for $u(\cdot)$:

$$\begin{aligned} u_1(t) &= \cos t + R_n^{(1)} \\ u_2(t) &= \frac{t}{2n}\sum_{i=0}^{n-1}\left\{\cos\left[\frac{2t(n-i)}{n}\right] + \cos\left[\frac{2t(n-i-1)}{n}\right]\right\} + R_n^{(2)}, \end{aligned}$$

where the remainder $R_n = \left(R_n^{(1)}, R_n^{(2)}\right)$ satisfies the estimate

$$\|R_n\|_2 = \sqrt{\left(R_n^{(1)}\right)^2 + \left(R_n^{(2)}\right)^2} \leq \frac{t}{4n}\cdot\rho(t).$$

The Figure 2 contains the behaviour of the error

$$\varepsilon_n(t) := \|R_n\|_2.$$

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