OSTROWSKI'S INEQUALITY FOR VECTOR-VALUED FUNCTIONS AND APPLICATIONS

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Abstract. Some Ostrowski type inequalities for vector-valued functions are obtained. Applications for operatorial inequalities and numerical approximation for the solutions of certain differential equations in Banach spaces are also given.

1. Introduction

The concepts of Riemann and Lebesgue integrability are well known for a scalar-valued function $F:[a,b]\to\mathbb{K}$, where $\mathbb{K}$ is the field of real or complex numbers and $-\infty<a<b<\infty$. It is known, for example, that if F is an absolutely continuous function, then it is differentiable almost everywhere and its derivative function $f:=F'$ is a Lebesgue integrable function. Moreover, in this case, the following fundamental formula of calculus, holds:

$$F(t)=F(a)+(L)\int_a^t f(s)\,ds,\qquad\text{for all } t\in[a,b],\tag{1.1}$$

where $(L)\int_a^t f(s)\,ds$ is Lebesgue's integral. If we replace $\mathbb{K}$ with a real or complex linear space X, that is, if F is a vector-valued function, then the above result will not hold. More precisely, if X is a Banach space, then the concept of Lebesgue integrability can be replaced with the concept of Bochner integrability (see for example [3], [11], [2]). However, there exist $X-$valued functions defined on $[a,b]$ which are absolutely continuous, and the set of points $t\in[a,b]$ for which f is not differentiable with respect to t, is of non-null Lebesgue measure.

A Banach space X with the property that every absolutely continuous $X-$valued function is almost everywhere differentiable is said to be a *Radon-Nikodym* space [5, pp. 217–219] or [11, 2]. For example, every reflexive Banach space (in particular, every Hilbert space) is a Radon-Nikodym space, but the space $L_\infty[0,1]$ of all $\mathbb{K}-$valued, essentially bounded functions defined on the interval $[0,1]$, endowed with the norm

$$\|g\|_\infty:=ess\sup_{t\in[0,1]}|g(t)|$$

is a Banach space which is not a Radon-Nikodym space.

However, if $f:[a,b]\to X$ (where X is an arbitrary Banach space) is a Bochner integrable function on $[a,b]$, then the function

$$t\mapsto F(t):=(B)\int_a^t f(s)\,ds:[a,b]\to X$$

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is differentiable almost everywhere on $[a,b]$, i.e., $F' = f$ a.e. and (1.1) holds. It should be noted that the integral is being considered in the Bochner sense.

A function $f : [a,b] \to X$ is *measurable* if there exists a sequence of simple functions (f_n) (with $f_n : [a,b] \to X$) which converges punctually a.e. at f on $[a,b]$.

It is well-known that a measurable function $f : [a,b] \to X$ is Bochner integrable if and only if its norm, i.e., the function $t \longmapsto \|f\|(t) := \|f(t)\| : [a,b] \to \mathbb{R}_+$ is Lebesgue integrable on $[a,b]$, (see for example [10]).

It is known that if f is a scalar-valued and Riemann integrable function on $[a,b]$, then its primitive function, that is, the function $t \mapsto F(t) := (R)\int_a^t f(s)\,ds : [a,b] \to \mathbb{K}$ is differentiable almost everywhere and (1.1) holds a.e. on $[a,b]$. Such a result, however, is not valid for vector-valued functions. For example, the function $f : [0,1] \to L_\infty[0,1]$ given by $f(t) = 1_{[0,t]}(\cdot)$, $t \in [0,1]$ (where $1_{[0,t]}$ is the characteristic function of the interval $[0,t]$) is a Riemann integrable vector valued function and its Riemann integral is given by

$$F(t) := (R)\int_0^t f(s)\,ds = (t - \cdot)\,1_{[0,t]}(\cdot)\,, \quad t \in [0,1]\,. \tag{1.2}$$

The function $F : [0,1] \to L_\infty[0,1]$, defined in (1.2) is absolutely continuous (in fact, it is even Lipschitz continuous on $[0,1]$) but nowhere differentiable because

$$\frac{F(t+h) - F(t)}{h}(\cdot) = 1_{[0,t]}(\cdot) + \frac{1}{h}(t+h-\cdot)\,1_{[t,t+h]}(\cdot)$$

does not converge in $L_\infty[0,1]$ as $h \to 0$ for any $0 \le t \le 1$.

Another example can be found in [11, p. 172].

In Section 2, we will use the integration by parts formula. This holds under the following general conditions:

Let $-\infty < a < b < \infty$ and f, g be two mappings defined on $[a,b]$ such that f is $\mathbb{C}$-valued and g is X-valued, where X is a real or complex Banach space. If f, g are differentiable on $[a,b]$ and their derivatives are Bochner integrable on $[a,b]$, then

$$(B)\int_a^b f'g = f(b)\,g(b) - f(a)\,g(b) - (B)\int_a^b fg'.$$

Using this in Section 2, we obtain some Ostrowski type inequalities for vector-valued functions and show that the mid-point inequality is the best possible inequality in the class. In Section 3, a quadrature formula of the Riemann type for the Bochner integral and the error bounds are considered. Section 4 is devoted to operator inequalities that can be obtained via Ostrowski type inequalities for vector-valued functions for which, in the last section, a numerical approximation for the mild solution of inhomogeneous vector-valued differential equations is given. In the last section, two numerical examples are considered.

For some results on the Ostrowski inequality for real-valued functions, see [1], [4], [8] and [9], and the references therein.

2. Ostrowski's Inequality for the Bochner Integral

The following theorem concerning a version of Ostrowski's inequality for vector-valued functions holds.

Theorem 1. *Let $(X;\|\cdot\|)$ be a Banach space with the Radon-Nikodym property and $f : [a,b] \to X$ an absolutely continuous function on $[a,b]$ with the property that*

$f' \in L_\infty([a,b];X)$, *i.e.*,

$$|||f'|||_{[a,b],\infty} := ess \sup_{t\in[a,b]} \|f'(t)\| < \infty.$$

Then we have the inequalities:

$$\left\| f(s) - \frac{1}{b-a}(B)\int_a^b f(t)\,dt \right\| \tag{2.1}$$
$$\begin{aligned}
&\le \frac{1}{b-a}\left[\int_a^s (t-a)\|f'(t)\|\,dt + \int_s^b (b-t)\|f'(t)\|\,dt\right] \\
&\le \frac{1}{2(b-a)}\left[(s-a)^2|||f'|||_{[a,s],\infty} + (b-s)^2|||f'|||_{[s,b],\infty}\right] \\
&\le \left[\frac{1}{4} + \left(\frac{s-\frac{a+b}{2}}{b-a}\right)^2\right](b-a)|||f'|||_{[a,b],\infty} \\
&\le \frac{1}{2}(b-a)|||f'|||_{[a,b],\infty}
\end{aligned}$$

for any $s \in [a,b]$, *where* $(B)\int_a^b f(t)\,dt$ *is the Bochner integral of* f.

Proof. Using the integration by parts formula, we may write that

$$(B)\int_a^s (t-a)f'(t)\,dt = (s-a)f(s) - (B)\int_a^s f(t)\,dt$$

and

$$(B)\int_s^b (b-t)f'(t)\,dt = (b-s)f(s) - (B)\int_s^b f(t)\,dt,$$

for any $s \in [a,b]$; from which we get the identity:

$$(b-a)f(s) - (B)\int_a^b f(t)\,dt \tag{2.2}$$
$$= (B)\int_a^s (t-a)f'(t)\,dt + (B)\int_s^b (b-t)f'(t)\,dt.$$

Taking the norm on X, we obtain

$$\begin{aligned}
\left\|(b-a)f(s) - (B)\int_a^b f(t)\,dt\right\| &= \left\|(B)\int_a^s (t-a)f'(t)\,dt + (B)\int_s^b (b-t)f'(t)\,dt\right\| \\
&\le \left\|(B)\int_a^s (t-a)f'(t)\,dt\right\| + \left\|(B)\int_s^b (b-t)f'(t)\,dt\right\| \\
&\le \int_a^s (t-a)\|f'(t)\|\,dt + \int_s^b (b-t)\|f'(t)\|\,dt \\
&= :B(s),
\end{aligned}$$

which proves the first inequality in (2.1).

We also have

$$\int_a^s (t-a)\|f'(t)\|\,dt \le |||f'|||_{[a,s],\infty}\int_a^s (t-a)\,dt = |||f'|||_{[a,s],\infty}\cdot\frac{(s-a)^2}{2}$$

and

$$\int_s^b (b-t)\|f'(t)\|\,dt \le |||f'|||_{[s,b],\infty}\int_s^b (b-t)\,dt = |||f'|||_{[s,b],\infty}\cdot\frac{(b-s)^2}{2}$$

from whence, by addition, we get the second part of (2.1).

Since

$$\max\left\{|||f'|||_{[a,s],\infty}, |||f'|||_{[s,b],\infty}\right\} \le |||f'|||_{[a,b],\infty}$$

and, by the parallelogram identity for real numbers, we have,

$$\frac{1}{2}\left[(s-a)^2+(b-s)^2\right] = \frac{1}{4}(b-a)^2+\left(s-\frac{a+b}{2}\right)^2$$

then the last part of (2.1) is also proved. ∎

Remark 1. *We observe that for the scalar function $B:[a,b]\to\mathbb{R}$, we have*

$$B'(s) = (s-a)\|f'(s)\| - (b-s)\|f'(s)\| = 2\left(s-\frac{a+b}{2}\right)\|f'(s)\|$$

for any $s\in[a,b]$, showing that B is monotonic nonincreasing on $\left[a,\frac{a+b}{2}\right]$ and monotonic nondecreasing on $\left[\frac{a+b}{2},b\right]$ and

$$\begin{aligned}\inf_{s\in[a,b]} B(s) &= B\left(\frac{a+b}{2}\right)\\ &= \frac{1}{b-a}\left[\int_a^{\frac{a+b}{2}}(t-a)\|f'(t)\|\,dt+\int_{\frac{a+b}{2}}^b (b-t)\|f'(t)\|\,dt\right].\end{aligned} \tag{2.3}$$

Consequently, the best inequalities we can obtain from (2.1) are embodied in the following corollary.

Corollary 1. *With the assumptions of Theorem 1, we have the inequality:*

$$\begin{aligned}&\left\|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}(B)\int_a^b f(t)\,dt\right\|\\ &\le \frac{1}{b-a}\left[\int_a^{\frac{a+b}{2}}(t-a)\|f'(t)\|\,dt+\int_{\frac{a+b}{2}}^b (b-t)\|f'(t)\|\,dt\right]\\ &\le \frac{b-a}{2}\left[|||f'|||_{\left[a,\frac{a+b}{2}\right],\infty}+|||f'|||_{\left[\frac{a+b}{2},b\right],\infty}\right]\\ &\le \frac{1}{4}(b-a)\,|||f'|||_{[a,b],\infty}.\end{aligned} \tag{2.4}$$

Bounds involving the $p-$norms, $p\in[1,\infty)$, of the derivative f', are embodied in the following theorem.

Theorem 2. *Let $(X,\|\cdot\|)$ be a Banach space with the Radon-Nikodym property and $f:[a,b]\to X$ be an absolutely continuous function on $[a,b]$ with the property that $f'\in L_p([a,b];X)$, $p\in[1,\infty)$, i.e.,*

$$|||f'|||_{[a,b],p} := \left(\int_a^b \|f'(t)\|^p\,dt\right)^{\frac{1}{p}} < \infty. \tag{2.5}$$

Then we have the inequalities

$$\begin{aligned}
&\left\| f(s) - \frac{1}{b-a}(B)\int_a^b f(t)\,dt \right\| \qquad (2.6)\\
&\leq \frac{1}{b-a}\left[\int_a^s (t-a)\|f'(t)\|\,dt + \int_s^b (b-t)\|f'(t)\|\,dt\right]\\
&\leq \begin{cases} \dfrac{1}{b-a}\left[(s-a)\,|||f'|||_{[a,s],1} + (b-s)\,|||f'|||_{[s,b],1}\right] \\ \qquad\qquad \text{if } f' \in L_1([a,b];X); \\[2ex] \dfrac{1}{(b-a)(q+1)^{\frac{1}{q}}}\left[(s-a)^{\frac{1}{q}+1}|||f'|||_{[a,s],p} + (b-s)^{\frac{1}{q}+1}|||f'|||_{[s,b],p}\right] \\ \qquad\qquad \text{if } p>1,\ \frac{1}{p}+\frac{1}{q}=1 \text{ and } f' \in L_p([a,b];X) \end{cases}\\
&\leq \begin{cases} \left[\dfrac{1}{2} + \left|\dfrac{s-\frac{a+b}{2}}{b-a}\right|\right]|||f'|||_{[a,b],1} \qquad \text{if } f' \in L_1([a,b];X); \\[2ex] \dfrac{1}{(q+1)^{\frac{1}{q}}}\left[\left(\dfrac{s-a}{b-a}\right)^{q+1} + \left(\dfrac{b-s}{b-a}\right)^{q+1}\right]^{\frac{1}{q}}(b-a)^{\frac{1}{q}}|||f'|||_{[a,b],p} \\ \qquad\qquad \text{if } f' \in L_p([a,b];X). \end{cases}
\end{aligned}$$

Proof. We have

$$\int_a^s (t-a)\|f'(t)\|\,dt \leq (s-a)\int_a^s \|f'(t)\|\,dt = (s-a)\,|||f'|||_{[a,s],1}$$

and

$$\int_s^b (b-t)\|f'(t)\|\,dt \leq (b-s)\int_s^b \|f'(t)\|\,dt = (b-s)\,|||f'|||_{[s,b],1}$$

and the first part of the second inequality in (2.6) is proved.

Using Hölder's integral inequality for scalar functions we have (for $p>1$, $\frac{1}{p}+\frac{1}{q}=1$) that

$$\begin{aligned}
\int_a^s (t-a)\|f'(t)\|\,dt &\leq \left(\int_a^s |t-a|^q\,dt\right)^{\frac{1}{q}}\left(\int_a^s \|f'(t)\|^p\,dt\right)^{\frac{1}{p}}\\
&= \frac{(s-a)^{\frac{1}{q}+1}}{(q+1)^{\frac{1}{q}}}|||f'|||_{[a,s],p}
\end{aligned}$$

and

$$\begin{aligned}
\int_s^b (b-t)\|f'(t)\|\,dt &\leq \left(\int_s^b |b-t|^q\,dt\right)^{\frac{1}{q}}\left(\int_s^b \|f'(t)\|^p\,dt\right)^{\frac{1}{p}}\\
&= \frac{(b-s)^{\frac{1}{q}+1}}{(q+1)^{\frac{1}{q}}}|||f'|||_{[s,b],p},
\end{aligned}$$

giving the second part of the second inequality.

Since

$$
\begin{aligned}
& (s-a)\,|||f'|||_{[a,s],1}+(b-s)\,|||f'|||_{[s,b],1} \\
\leq\ & \max\{s-a,b-s\}\left[|||f'|||_{[a,s],1}+|||f'|||_{[s,b],1}\right] \\
=\ & \left[\frac{1}{2}(b-a)+\left|s-\frac{a+b}{2}\right|\right]|||f'|||_{[a,b],1}\,,
\end{aligned}
$$

the first part of the third inequality in (2.6) is proved.

For the last part, we note that for any α, β, γ, $\delta>0$ and $p>1$, $\frac{1}{p}+\frac{1}{q}=1$ we have:

$$
(\alpha^q+\beta^q)^{\frac{1}{q}}(\gamma^p+\delta^p)^{\frac{1}{p}}\geq\alpha\gamma+\beta\delta,
$$

and then:

$$
\begin{aligned}
& (s-a)^{1+\frac{1}{q}}\,|||f'|||_{[a,s],p}+(b-s)^{1+\frac{1}{q}}\,|||f'|||_{[s,b],p} \\
\leq\ & \left[(s-a)^{q\left(1+\frac{1}{q}\right)}+(b-s)^{q\left(1+\frac{1}{q}\right)}\right]^{\frac{1}{q}}\left[|||f'|||^p_{[a,s],p}+|||f'|||^p_{[s,b],p}\right]^{\frac{1}{p}} \\
=\ & \left[(s-a)^{1+q}+(b-s)^{1+q}\right]^{\frac{1}{q}}\left[\int_a^s\|f'(s)\|^p\,ds+\int_s^b|||f'(s)|||^p\,ds\right]^{\frac{1}{p}} \\
=\ & \left[(s-a)^{1+q}+(b-s)^{1+q}\right]^{\frac{1}{q}}|||f'|||_{[a,b],p}\,.
\end{aligned}
$$

The theorem is completely proved. ∎

Remark 2. *The above theorem both generalises and extends for vector-valued functions the results in* [6] *and* [7]*.*

The best inequalities we can obtain from (2.6) in the sense of providing the tightest bound are embodied in the following corollary concerning the mid-point rule.

Corollary 2. *With the assumptions in Theorem 3, we have*

$$
\begin{aligned}
& \left\|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}(B)\int_a^b f(t)\,dt\right\| \qquad (2.7)\\
\leq\ & \frac{1}{b-a}\left[\int_a^{\frac{a+b}{2}}(t-a)\|f'(t)\|\,dt+\int_{\frac{a+b}{2}}^b(b-t)\|f'(t)\|\,dt\right]
\end{aligned}
$$

$$\leq \begin{cases} \dfrac{1}{2}\,|||f'|||_{[a,b],1} & \textit{if } \; f' \in L_1\left([a,b];X\right); \\[2ex] \dfrac{(b-a)^{\frac{1}{q}}}{2^{1+\frac{1}{q}}\,(q+1)^{\frac{1}{q}}}\left[|||f'|||_{\left[a,\frac{a+b}{2}\right],p} + |||f'|||_{\left[\frac{a+b}{2},b\right],p}\right] & \\[2ex] \qquad \textit{if } \; p>1,\ \frac{1}{p}+\frac{1}{q}=1 \textit{ and } f' \in L_p\left([a,b];X\right) & \end{cases}$$

$$\leq \begin{cases} \dfrac{1}{2}\,|||f'|||_{[a,b],1} & \textit{if } \; f' \in L_1\left([a,b];X\right); \\[2ex] \dfrac{1}{2\,(q+1)^{\frac{1}{q}}}\,(b-a)^{\frac{1}{q}}\,|||f'|||_{[a,b],p} & \\[2ex] \qquad \textit{if } \; p>1,\ \frac{1}{p}+\frac{1}{q}=1 \textit{ and } f' \in L_p\left([a,b];X\right). & \end{cases}$$

3. A Quadrature Formula of the Riemann Type

Now, let $I_n : a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$ be a partitioning of the interval $[a,b]$ and define $h_i = x_{i+1} - x_i$, $\nu(h) := \max\{h_i | i = 0, \dots, n-1\}$. Consider the mapping $f : [a,b] \to X$, where X is a Banach space with the Radon-Nikodym property. Define the Riemann sum by:

$$A_n\left(f, I_n, \boldsymbol{\xi}\right) := \sum_{i=0}^{n-1} h_i f\left(\xi_i\right), \tag{3.1}$$

where $\boldsymbol{\xi} = \left(\xi_0, \dots, \xi_{n-1}\right)$ and $\xi_i \in [x_i, x_{i+1}]$ $(i = 0, \dots, n-1)$ are intermediate (arbitrarily chosen) points.

The following theorem holds.

Theorem 3. *Let f be as in Theorem 1. Then we have:*

$$(B)\int_a^b f(t)\,dt = A_n\left(f, I_n, \boldsymbol{\xi}\right) + R_n\left(f, I_n, \boldsymbol{\xi}\right), \tag{3.2}$$

where $A_n\left(f, I_n, \boldsymbol{\xi}\right)$ is the Riemann quadrature given by (3.1) and the remainder $R_n\left(f, I_n, \boldsymbol{\xi}\right)$ in (3.2) satisfies the bound

$$\begin{aligned} & \left\|R_n\left(f, I_n, \boldsymbol{\xi}\right)\right\| \\ \leq\ & \sum_{i=0}^{n-1}\left[\int_{x_i}^{\xi_i}\left(t-x_i\right)\left\|f'(t)\right\|dt + \int_{\xi_i}^{x_{i+1}}\left(x_{i+1}-t\right)\left\|f'(t)\right\|dt\right] \\ \leq\ & \frac{1}{2}\sum_{i=0}^{n-1}\left[\left(\xi_i - x_i\right)^2 |||f'|||_{[x_i,\xi_i],\infty} + \left(x_{i+1}-\xi_i\right)^2 |||f'|||_{[\xi_i,x_{i+1}],\infty}\right] \\ \leq\ & \sum_{i=0}^{n-1}\left[\frac{1}{4}h_i^2 + \left(\xi_i - \frac{x_i + x_{i+1}}{2}\right)^2\right]|||f'|||_{[x_i,x_{i+1}],\infty} \\ \leq\ & \frac{1}{2}\sum_{i=0}^{n-1} h_i^2\,|||f'|||_{[x_i,x_{i+1}],\infty} \\ \leq\ & \frac{1}{2}\,|||f'|||_{[a,b],\infty}\sum_{i=0}^{n-1} h_i^2 \leq \frac{1}{2}\,(b-a)\,\nu(h)\,|||f'|||_{[a,b],\infty}. \end{aligned} \tag{3.3}$$

Proof. Apply the inequality (2.1) on the interval $[x_i, x_{i+1}]$ to obtain

$$\begin{aligned}
&\left\| h_i f(\xi_i) - \int_{x_i}^{x_{i+1}} f(t)\, dt \right\| \\
\leq & \int_{x_i}^{\xi_i} (t - x_i) \|f'(t)\|\, dt + \int_{\xi_i}^{x_{i+1}} (x_{i+1} - t) \|f'(t)\|\, dt \\
\leq & \frac{1}{2} \left[(\xi_i - x_i)^2 \| \|f'\| \|_{[x_i,\xi_i],\infty} + (x_{i+1} - \xi_i)^2 \| \|f'\| \|_{[\xi_i,x_{i+1}],\infty} \right] \\
\leq & \left[\frac{1}{4} + \left(\frac{\xi_i - \frac{x_i + x_{i+1}}{2}}{h_i} \right)^2 \right] h_i^2 \| \|f'\| \|_{[x_i,x_{i+1}],\infty} \\
\leq & \frac{1}{2} h_i^2 \| \|f'\| \|_{[x_i,x_{i+1}],\infty}
\end{aligned} \tag{3.4}$$

for any $i = 0, \ldots, n-1$.

Summing over i from 0 to $n-1$ and using the generalised triangle inequality for norms, we obtain (3.3). ∎

If we consider the *midpoint quadrature rule* given by

$$M_n(f, I_n) := \sum_{i=0}^{n-1} h_i f\left(\frac{x_i + x_{i+1}}{2} \right) \tag{3.5}$$

then we may state the following corollary.

Corollary 3. *With the assumptions in Theorem 1, we have*

$$(B) \int_a^b f(t)\, dt = M_n(f, I_n) + W_n(f, I_n) \tag{3.6}$$

where $M_n(f, I_n)$ is the vector-valued midpoint quadrature rule given in (3.5) and the remainder $W_n(f, I_n)$ satisfies the estimate:

$$\begin{aligned}
&\|W_n(f, I_n)\| \\
\leq & \sum_{i=0}^{n-1} \left[\int_{x_i}^{\frac{x_i + x_{i+1}}{2}} (t - x_i) \|f'(t)\|\, dt + \int_{\frac{x_i + x_{i+1}}{2}}^{x_{i+1}} (x_{i+1} - t) \|f'(t)\|\, dt \right] \\
\leq & \frac{1}{8} \sum_{i=0}^{n-1} h_i^2 \left[\| \|f'\| \|_{\left[x_i, \frac{x_i + x_{i+1}}{2}\right],\infty} + \| \|f'\| \|_{\left[\frac{x_i + x_{i+1}}{2}, x_{i+1}\right],\infty} \right] \\
\leq & \frac{1}{4} \sum_{i=0}^{n-1} h_i^2 \| \|f'\| \|_{[x_i,x_{i+1}],\infty} \leq \frac{1}{4} \| \|f'\| \|_{[a,b],\infty} \sum_{i=0}^{n-1} h_i^2 \\
\leq & \frac{1}{4} (b - a) \| \|f'\| \|_{[a,b],\infty} \nu(h).
\end{aligned} \tag{3.7}$$

Remark 3. *It is obvious that $\|W_n(f, I_n)\| \to 0$ as $\nu(h) \to 0$, showing that $M_n(f, I_n)$ is an approximation for the Bochner integral $(B) \int_a^b f(t)\, dt$ with order one accuracy.*

Remark 4. *Similar bounds for the remainder $R_n(f, I_n, \boldsymbol{\xi})$ and $W_n(f, I_n)$ may be obtained in terms of the $p-$norms $(p \in [1, \infty))$, but we omit the details.*

4. Applications for the Operator Inequality

Let X be an arbitrary Banach space and $\mathcal{L}(X)$ the Banach space of all bounded linear operators on X. We recall that if $A \in \mathcal{L}(X)$ then its operatorial norm is defined by

$$\|A\| = \sup\{\|Ax\| : x \in X,\ \|x\| \leq 1\}.$$

We recall also that the series $\left(\sum_{n\geq 0} \frac{(tA)^n}{n!}\right)$ converges absolutely and locally uniformly for $t \in \mathbb{R}$. If we denote by e^{tA} its sum, then

$$\left\|e^{tA}\right\| \leq e^{t\|A\|}, \quad \text{for all } t \geq 0. \tag{4.1}$$

Another definition of e^{tA} is given in the next section.

Proposition 1. *Let X be a Banach space, $A \in \mathcal{L}(X)$ and $0 \leq a < b < \infty$. Then for each $s \in [a, b]$, we have:*

$$\begin{aligned} &\left\|e^{sA} - \frac{1}{b-a}\int_a^b e^{tA}dt\right\| \\ &\leq \frac{1}{b-a}\left[(2s-a-b)\,e^{s\|A\|} + \frac{1}{\|A\|}\left(e^{a\|A\|} + e^{b\|A\|} - 2e^{s\|A\|}\right)\right]. \end{aligned} \tag{4.2}$$

Proof. We apply Theorem 1 with X replaced by $\mathcal{L}(X)$ and $f(t) = e^{tA}$. Note that in this case the function f is continuously differentiable, so that it is not necessary that X be a Radon-Nikodym space. We have, by (4.1), that

$$\begin{aligned} \int_a^s (t-a)\,\|f'(t)\|\,dt &\leq \|A\|\int_a^s (t-a)\,e^{t\|A\|}dt \\ &= (s-a)\,e^{s\|A\|} - \frac{1}{\|A\|}\left(e^{a\|A\|} - e^{s\|A\|}\right), \end{aligned}$$

and

$$\begin{aligned} \int_s^b (b-t)\,\|f'(t)\|\,dt &\leq \|A\|\int_s^b (b-t)\,e^{t\|A\|}dt \\ &= -(b-s)\,e^{s\|A\|} + \frac{1}{\|A\|}\left(e^{b\|A\|} - e^{s\|A\|}\right). \end{aligned}$$

On adding the two above inequalities, we obtain the desired inequality (4.2). ∎

Corollary 4. *With the assumptions in Proposition 1, we have the following inequality*

$$\left\|e^{\frac{a+b}{2}A} - \frac{1}{b-a}\int_a^b e^{tA}dt\right\| \leq \frac{1}{(b-a)\,\|A\|}\left(e^{\frac{a}{2}\|A\|} - e^{\frac{b}{2}\|A\|}\right)^2. \tag{4.3}$$

Let $GL(X)$ be the subset of $\mathcal{L}(X)$ consisting of all invertible operators. It is known that $GL(X)$ is an open set in $\mathcal{L}(X)$.

Using (4.3), we may state the following result as well.

Corollary 5. *Let $A \in GL(X)$. Then the following inequality holds:*

$$\begin{aligned} \left\|Ae^{\frac{a+b}{2}A} - \frac{1}{b-a}\left(e^{bA} - e^{aA}\right)\right\| &\leq \|A\|\left\|e^{\frac{a+b}{2}A} - \frac{1}{b-a}A^{-1}\left(e^{bA} - e^{aA}\right)\right\| \\ &\leq \frac{1}{b-a}\left(e^{\frac{a}{2}\|A\|} - e^{\frac{b}{2}\|A\|}\right)^2. \end{aligned}$$

Proof. The first inequality is obvious. For the second inequality we remark that

$$\int_a^b e^{tA}dt = A^{-1}\left(e^{bA} - e^{aA}\right)$$

and apply Corollary 4. ∎

Remark 5. *As a consequence of Corollary 5, we can obtain the well-known inequality for real numbers* $e^y \geq 1 + y$ *for each* $y \in \mathbb{R}$. *Indeed, if* $A = x \in (0, \infty)$, *then*

$$\left| xe^{\frac{a+b}{2}x} - \frac{1}{b-a}\left(e^{bx} - e^{ax}\right)\right| \leq \frac{1}{b-a}\left(e^{\frac{a}{2}x} - e^{\frac{b}{2}x}\right)^2.$$

which is equivalent to

$$e^{\frac{a-b}{2}x} \geq 1 + \frac{a-b}{2}x \quad \text{and} \quad e^{\frac{b-a}{2}x} \geq 1 + \frac{b-a}{2}x.$$

Another example of an operatorial inequality is embodied in the following proposition.

Proposition 2. *Let* X *be a Banach space,* $A \in \mathcal{L}(X)$ *and* $0 \leq a < b < \infty$. *Then for each* $s \in [a, b]$, *we have:*

$$\left\| \sin(sA) - \frac{1}{b-a}\int_a^b \sin(tA)\,dt \right\| \leq \left[\frac{1}{4} + \left(\frac{s - \frac{a+b}{2}}{b-a}\right)^2\right](b-a)\|A\|. \tag{4.4}$$

Proof. We apply the first inequality from Theorem 1 for

$$f(t) = \sin(tA) := \sum_{n=0}^{\infty} (-1)^n \frac{(tA)^{2n+1}}{(2n+1)!}.$$

We have

$$\left\|(\sin(tA))'\right\| = \|A\cos(tA)\| \leq \|A\|.$$

Then

$$\int_a^s (t-a)\|f'(t)\|\,dt \leq \|A\| \cdot \frac{(s-a)^2}{2}$$

and

$$\int_s^b (b-t)\|f'(t)\|\,dt \leq \|A\| \cdot \frac{(s-b)^2}{2}.$$

On adding the above inequalities, we obtain the desired result (4.4). Here, $\cos(tA) = \sum_{n=0}^{\infty} (-1)^n \frac{(tA)^{2n}}{(2n)!}$. ∎

Corollary 6. *With the assumptions as in Proposition 2, we have the following inequality:*

$$\left\| \sin\left(\frac{a+b}{2} \cdot A\right) - \frac{1}{b-a}\int_a^b \sin(tA)\,dt \right\| \leq \frac{(b-a)^2}{4} \cdot \|A\|.$$

If in addition $A \in GL(X)$, then

$$\begin{aligned}
&\left\| A \sin\left(\frac{a+b}{2} \cdot A\right) + \frac{1}{b-a}\left[\cos(bA) - \cos(aA)\right]\right\| \\
\leq\ & \|A\| \cdot \left\| \sin\left(\frac{a+b}{2} \cdot A\right) + \frac{1}{b-a} A^{-1}\left[\cos(bA) - \cos(aA)\right]\right\| \\
\leq\ & \frac{(b-a)^2}{4} \cdot \|A\|^2 .
\end{aligned}$$

Remark 6. *In particular, for $A = x \in \mathbb{R}\backslash\{0\}$, it follows that*

$$\left| \sin\left(\frac{a+b}{2} \cdot x\right)\left[1 - \frac{\sin\frac{(b-a)x}{2}}{\frac{(b-a)x}{2}}\right]\right| \leq \frac{(b-a)^2}{4}|x| . \tag{4.5}$$

The similar result for $\cos(tA)$ will be summarised next.

Proposition 3. *With the above notations, we have:*

(i) $\left\|\cos(sA) - \frac{1}{b-a}\int_a^b \cos(tA)\,dt\right\| \leq \left[\frac{1}{4} + \left(\frac{s - \frac{a+b}{2}}{b-a}\right)^2\right](b-a)\|A\| .$

(ii) $\left\|\cos\left(\frac{a+b}{2} \cdot A\right) - \frac{1}{b-a}\int_a^b \cos(tA)\,dt\right\| \leq \frac{(b-a)^2}{4}\|A\| .$

If, in addition $A \in GL(X)$, then

(iii)
$$\begin{aligned}
&\left\| A \cos\left(\frac{a+b}{2} \cdot A\right) - \frac{1}{b-a}\left[\sin(bA) - \sin(aA)\right]\right\| \\
&\leq \|A\| \left\| \cos\left(\frac{a+b}{2} \cdot A\right) - \frac{1}{b-a} \cdot A^{-1}\left[\sin(bA) - \sin(aA)\right]\right\| \\
&\leq \frac{(b-a)^2}{4}\|A\|^2 .
\end{aligned}$$

Remark 7. *In particular, for $A = x \in \mathbb{R}\backslash\{0\}$, it follows that*

$$\left| \cos\left(\frac{a+b}{2} \cdot x\right) \cdot \left[1 - \frac{\sin\frac{(b-a)}{2} \cdot x}{\frac{(b-a)}{2} \cdot x}\right]\right| \leq \frac{(b-a)^2}{4}|x| . \tag{4.6}$$

Remark 8. *Taking the square of both sides of the inequalities (4.5) and (4.6) and then adding them, we obtain*

$$\left|1 - \frac{\sin\frac{(b-a)}{2} \cdot x}{\frac{(b-a)}{2} \cdot x}\right| \leq \frac{\sqrt{2}}{4}(b-a)^2|x| , \quad \textit{for all } x \in \mathbb{R}^* .$$

In particular, if $b - a = 2$, then

$$|\sin x - x| \leq \sqrt{2}x^2, \quad \textit{for all } x \in \mathbb{R},$$

which is an interesting scalar inequality.

Another type of example is considered in the following.

A densely defined linear operator A on a Banach space X is said to be *sectorial* [13] if $(0, \infty) \subset \rho(A)$ and there exists $M = M_A > 0$ such that

$$\|R(t, A)\| \leq \frac{M}{1+t}, \text{ for all } t > 0, \tag{4.7}$$

where $R(t, A) := (tI - A)^{-1}$ is the resolvent operator of A.

Proposition 4. *Let A be a sectorial operator on a Banach space X. Then for $0 \le a \le s \le b < \infty$, we have:*

(i) $\left\|R^2(s,A) - R(a,A)R(b,A)\right\| \le \frac{M^3}{(b-a)(s+1)^2} \cdot \left[\frac{(s-a)^2}{a+1} + \frac{(b-s)^2}{b+1}\right]$;
and
(ii) $\left\|R^2\left(\frac{a+b}{2},A\right) - R(a,A)R(b,A)\right\| \le \frac{M^3(b-a)}{(a+1)(b+1)(a+b+2)}$.

Proof. By the resolvent identity

$$R(t,A) - R(s,A) = (s-t)R(t,A)R(s,A),$$

it follows that

$$\frac{d}{dt}[R(t,A)] = -R^2(t,A).$$

We apply Theorem 1 in Section 2 for $f(t) = R^2(t,A)$ giving, from (4.7)

$$\left\|\frac{d}{dt}\left[R^2(t,A)\right]\right\| = \left\|-2R^3(t,A)\right\| \le \frac{2M^3}{(t+1)^3}.$$

Further,

$$\begin{aligned}
&\frac{1}{b-a}\left[\int_a^s (t-a)\left\|f'(t)\right\|dt + \int_s^b (b-t)\left\|f'(t)\right\|dt\right] \\
\le\ & \frac{2M^3}{b-a}\left[\int_a^s \frac{(t-a)}{(1+t)^3}dt + \int_s^b \frac{(b-t)}{(1+t)^3}dt\right] \\
\le\ & \frac{2M^3}{b-a}\left[\frac{(s-a)^2}{2(a+1)(s+1)^2} + \frac{(b-s)^2}{2(b+1)(s+1)^2}\right] \\
\le\ & \frac{M^3}{(b-a)(s+1)^2}\left[\frac{(s-a)^2}{a+1} + \frac{(b-s)^2}{b+1}\right].
\end{aligned}$$

Statement (i) is thus proved. Taking $s = \frac{a+b}{2}$ gives (ii). ∎

Remark 9. *If $A = x \in (-\infty, 0)$, then we can choose $M_x = \sup_{t>0}\left[\frac{t+1}{t-x}\right] = -\frac{1}{x}$ and from (i) we obtain the interesting inequality:*

$$(a-x)(b-x)(a+b-2x)^2 \ge (-x)^3(b-a)(a+1)(b+1)(a+b+2),$$

for all $x \le 0$ and all $0 \le a < b < \infty$.

5. Applications for Vector-Valued Differential Equations

Many problems of mathematical physics can be modelled using the following abstract Cauchy problem

$$(ACP_x) \qquad \begin{cases} \dot{u}(t) = Au(t), & t \ge 0 \\ u(0) = x, \end{cases}$$

where A is a linear, usually unbounded, operator with domain $D(A)$ on a Banach space X. For every particular mathematical physics problem, X is a suitable Banach space of functions and A is a partial differential operator. By the *classical solution* for (ACP_x), we mean a continuous differentiable function $u_x : [0,\infty) \to D(A)$ which satisfies (ACP_x). A continuous function $u : [0,\infty) \to X$ is said to be a *mild* *solution* for (ACP_x) if there exists a sequence $(x_n)_{n\in\mathbb{N}}$ with $x_n \in D(A)$ such that for each n the problem (ACP_x) has a classical solution $u_{x_n}(\cdot)$ with $\lim_{n\to\infty} u_{x_n}(t) = u(t)$ locally uniform on $[0,\infty)$. We say that the abstract Cauchy problem associated with a linear operator A is *well-posed* if for each initial value $x \in D(A)$ the problem (ACP_{x_n}) has a unique classical solution. An example of an operator A for which the associated abstract Cauchy problem is well-posed is presented in the following.

Let X be a Banach space and $\mathcal{L}(X)$ the space of all bounded linear operators. We denote by $\|\cdot\|$ the norms of vectors and operators. A family $\mathbf{T} = \{T(t)\}_{t\geq 0} \subset \mathcal{L}(X)$ is called a semigroup of operators if the following conditions hold:

(S_1) $T(0) = I$, I is the identity operator on X;
(S_2) $T(t+s) = T(t) \circ T(s)$ for all $t, s \geq 0$.

A semigroup $\mathbf{T}$ is said to be *uniformly continuous* if the mapping $t \longmapsto T(t) : [0,\infty) \to \mathcal{L}(X)$ is continuous at $t_0 = 0$ (or equivalently, is continuous on $\mathbb{R}_+$) in the operatorial norm in $\mathcal{L}(X)$.

A semigroup $\mathbf{T}$ is said to be *strongly continuous* (or C_0-semigroup) if the mapping $t \longmapsto T(t)x : [0,\infty) \to X$ is continuous at $t_0 = 0$ (or equivalently on $\mathbb{R}_+$) for all $x \in X$. It is well known [12] that if $\mathbf{T}$ is a uniformly continuous semigroup, then there exists an operator $A \in \mathcal{L}(X)$ such that

$$T(t) = e^{tA} := \sum_{n=0}^{\infty} \frac{(tA)^n}{n!}; \quad t \geq 0.$$

In this case, the problem (ACP_x) associated with A has a unique classical (or mild) solution and it is given by

$$u_x(t) = u(t) = e^{tA}x, \;\; t \geq 0.$$

If $\mathbf{T}$ is a C_0-semigroup, then its generator A with its domain $D(A)$ are given by

$$D(A) = \left\{ x \in X : \lim_{t\downarrow 0} \frac{T(t)x - x}{t} \text{ exists in } X \right\}$$

and

$$Ax = \lim_{t\downarrow 0} \frac{T(t)x - x}{t}, \quad x \in D(A).$$

It is easy to see that the function $t \mapsto T(t)x$ is differentiable on $\mathbb{R}_+$ for all $x \in D(A)$. It is well-known ([13], [12]) that the generator A is a closed and densely defined operator (i.e., $D(A)$ is dense in X). In this case, the abstract Cauchy problem associated with A is well-posed. The classical solution is given by $u_x(t) = T(t)x$ for $x \in D(A)$ and the mild solution is given by $u(t) = T(t)x$ for $x \in X$. The converse result is also true.

For example, if A is a linear operator with domain $D(A)$, the abstract Cauchy problem associated with A is well-posed and the resolvent set of A $(\rho(A))$ is nonempty, then A is the generator for a strongly continuous semigroup $\mathbf{T}$ ([13], [12]). Every C_0-semigroup $\mathbf{T}$ has a growth bound. That is, there exist $M > 0$ and $\omega \in \mathbb{R}$ such that

$$\|T(t)\| \leq Me^{\omega t}, \;\; \text{for all } t \geq 0. \tag{5.1}$$

Let $f : \mathbb{R}_+ \to X$ be a locally Bochner integrable function. We consider the inhomogeneous abstract Cauchy problem

$$(A, f, 0, x) \qquad \begin{cases} \dot{u}(t) = Au(t) + f(t) \ , & t \geq 0 \\ \\ u(0) = x \ , \end{cases}$$

where A is the generator of a strongly continuous semigroup $\mathbf{T}$ and $x \in X$. The function $T(t - \cdot) f(\cdot)$ is measurable, because if $\{f_n\}$ is a sequence of simple functions, then $g_n(\cdot) := T(t - \cdot) f_n(\cdot)$ are measurable for each $n \in \mathbb{N}$ (we used the strong continuity of $\mathbf{T}$), and $g_n(s) \to T(t-s) f(s)$ as $n \to \infty$, a.e. on $[0, t]$. Moreover, the function $T(t - \cdot) f(\cdot)$ is Bochner integrable on $[0, t]$, because $\|T(t - \cdot) f(\cdot)\| \leq Me^{\omega t} \|f(\cdot)\|$ and the function f is Bochner integrable on $[0, t]$.

The mild solution of the problem $(A, f, 0, x)$ can be represented by

$$u(t) = x + (B) \int_0^t T(t-s) f(s) \, ds, \ \ t \geq 0, \ x \in X,$$

We may state the following theorem in approximating the mild solutions of the inhomogeneous system $(A, f, 0, x)$.

Theorem 4. *Let* $0 = \lambda_0 < \lambda_1 < \cdots < \lambda_{n-1} < \lambda_n = 1$ *and* $\mu_i \in [\lambda_i, \lambda_{i+1}]$ $\left(i = \overline{0, n-1}\right)$. *If either*

- (i) $\mathbf{T}$ *is a uniformly continuous semigroup and* f *is a differentiable continuous* $X-$*valued function (*X *is an arbitrary Banach space)*
  or
- (ii) $\mathbf{T}$ *is a strongly continuous semigroup,* f *is differentiable continuous and* $f(t) \in D(A)$ *for all* $t \geq 0$, *and* $Af(\cdot)$ *is a locally bounded function on* $[0, \infty)$

hold, then the mild solution $u(\cdot)$ *of* $(A, f, 0, x)$ *can be represented as*

$$u(t) = x + S_n(\boldsymbol{\lambda}, \boldsymbol{\mu}, t) + Q_n(\boldsymbol{\lambda}, \boldsymbol{\mu}, t), \ \ t \geq 0, \tag{5.2}$$

where

$$S_n(\boldsymbol{\lambda}, \boldsymbol{\mu}, t) := t \sum_{i=0}^{n-1} (\lambda_{i+1} - \lambda_i) T[(1 - \mu_i) t] f(\mu_i t) \tag{5.3}$$

and the remainder $Q_n(\boldsymbol{\lambda}, \boldsymbol{\mu}, t)$ *satisfies, in the first case, the estimates*

$$\begin{aligned} &\|Q_n(\boldsymbol{\lambda}, \boldsymbol{\mu}, t)\| \\ \leq\ & t^2 e^{\|A\| t} \left[\|A\| \, |||f|||_{[0,t],\infty} + |||f'|||_{[0,t],\infty}\right] \\ & \times \sum_{i=0}^{n-1} \left[\frac{1}{4} (\lambda_{i+1} - \lambda_i)^2 + \left(\mu_i - \frac{\lambda_i + \lambda_{i+1}}{2}\right)^2\right] \\ \leq\ & \frac{1}{2} t^2 e^{\|A\| t} \left[\|A\| \, |||f|||_{[0,t],\infty} + |||f'|||_{[0,t],\infty}\right] \sum_{i=0}^{n-1} (\lambda_{i+1} - \lambda_i)^2 \\ \leq\ & \frac{1}{2} \nu(\boldsymbol{\lambda}) t^3 e^{\|A\| t} \left[\|A\| \, |||f|||_{[0,t],\infty} + |||f'|||_{[0,t],\infty}\right], \end{aligned} \tag{5.4}$$

where $\nu(\boldsymbol{\lambda}) := \max_{i=\overline{0,n-1}} (\lambda_{i+1} - \lambda_i)$, *and, in the second case, the estimates*

$$\begin{aligned}
&\|Q_n(\boldsymbol{\lambda},\boldsymbol{\mu},t)\| \\
\leq\ & Mt^2 e^{\omega t}\left[|||Af(\cdot)|||_{[0,t],\infty} + |||f'|||_{[0,t],\infty}\right] \\
&\times \sum_{i=0}^{n-1}\left[\frac{1}{4}(\lambda_{i+1}-\lambda_i)^2 + \left(\mu_i - \frac{\lambda_i+\lambda_{i+1}}{2}\right)^2\right] \\
\leq\ & \frac{1}{2}t^2 M e^{\omega t}\left[|||Af(\cdot)|||_{[0,t],\infty} + |||f'|||_{[0,t],\infty}\right]\sum_{i=0}^{n-1}(\lambda_{i+1}-\lambda_i)^2 \\
\leq\ & \frac{1}{2}\nu(\boldsymbol{\lambda})\, t^3 e^{\omega t}\left[|||Af(\cdot)|||_{[0,t],\infty} + |||f'|||_{[0,t],\infty}\right],
\end{aligned} \tag{5.5}$$

for each $t \in [0,\infty)$, *where* ω *is a positive number such that the estimate (5.1) holds.*

Proof. For a fixed $t > 0$, consider the function $g(s) := T(t-s) f(s)$, $s \in [0,t]$. Then g is differentiable on $(0,t)$ and

$$\frac{dg(s)}{ds} = \frac{d}{ds}[T(t-s)f(s)] = -AT(t-s)f(s) + T(t-s)f'(s),$$

for each $s \in (0,t)$.

We have, in the first case, that

$$\begin{aligned}
\left|\left|\left|\frac{dg}{ds}\right|\right|\right|_{[0,t],\infty} &\leq |||AT(t-\cdot)f(\cdot)|||_{[0,t],\infty} + |||T(t-\cdot)f'(\cdot)|||_{[0,t],\infty} \\
&\leq \|A\| e^{\|A\|t} |||f|||_{[0,t],\infty} + e^{\|A\|t} |||f'|||_{[0,t],\infty} \\
&= e^{\|A\|t}\left[\|A\|\, |||f|||_{[0,t],\infty} + |||f'|||_{[0,t],\infty}\right],
\end{aligned}$$

for any $t \in [0,\infty)$.

In the second case, we have in a similar manner, that

$$\left|\left|\left|\frac{dg}{ds}\right|\right|\right|_{[0,t],\infty} \leq Me^{\omega t}\left[|||Af(\cdot)|||_{[0,t],\infty} + |||f'(\cdot)|||_{[0,t],\infty}\right],$$

for each $t \in [0,\infty)$.

Now, consider the partitioning of the interval $[0,t]$ given by $x_i := \lambda_i t$ $\left(i=\overline{0,n-1}\right)$ where $0 = \lambda_0 < \lambda_1 < \cdots < \lambda_{n-1} < \lambda_n = 1$ and the intermediate points $\xi_i = \mu_i t$ $\left(i = \overline{0,n-1}\right)$ where $\mu_i \in [\lambda_i, \lambda_{i+1}]$ $\left(i=\overline{0,n-1}\right)$. If we apply Theorem 3 for $a = 0$, $b = t$, x_i, ξ_i $\left(i=\overline{0,n-1}\right)$ and g as defined above, then we deduce the representation (5.2) and the remainder $Q_n(\boldsymbol{\lambda},\boldsymbol{\mu},t)$ satisfies either the estimate (5.4) or the estimate (5.5). ∎

If we define the quadrature formula

$$M_n(\boldsymbol{\lambda},t) := t\sum_{i=0}^{n-1}(\lambda_{i+1}-\lambda_i)\, T\left[\left(1-\frac{\lambda_i+\lambda_{i+1}}{2}\right)t\right] f\left(\frac{\lambda_i+\lambda_{i+1}}{2}\cdot t\right), \tag{5.6}$$

then we may state the following corollary.

Corollary 7. *Let* $0 = \lambda_0 < \lambda_1 < \cdots < \lambda_{n-1} < \lambda_n = 1$. *If either (i) or (ii) in Theorem 4 hold, then the mild solution* $u(\cdot)$ *of* $(A,f,0,x)$ *can be represented as*

$$u(t) = x + M_n(\boldsymbol{\lambda},t) + L_n(\boldsymbol{\lambda},t), \tag{5.7}$$

where $M_n(\boldsymbol{\lambda}, t)$ is as given in (5.6) and the remainder $L_n(\boldsymbol{\lambda}, t)$ satisfies, in the first case, the estimates

$$\begin{aligned}
\|L_n(\boldsymbol{\lambda}, t)\| &\leq \frac{1}{4} t^2 e^{\|A\| t} \left[\|A\| \, |||f|||_{[0,t],\infty} + |||f'|||_{[0,t],\infty}\right] \sum_{i=0}^{n-1} h_i^2 \\
&\leq \frac{1}{4} t^3 \nu(h) e^{\|A\| t} \left[\|A\| \, |||f|||_{[0,t],\infty} + |||f'|||_{[0,t],\infty}\right],
\end{aligned} \tag{5.8}$$

where $h_i := \lambda_{i+1} - \lambda_i > 0$ $\left(i = \overline{0, n-1}\right)$, and, in the second case, the estimates:

$$\begin{aligned}
\|L_n(\boldsymbol{\lambda}, t)\| &\leq \frac{1}{4} M t^2 e^{\omega t} \left[|||Af(\cdot)|||_{[0,t],\infty} + |||f'|||_{[0,t],\infty}\right] \sum_{i=0}^{n-1} h_i^2 \\
&\leq \frac{1}{4} M \nu(h) t^3 e^{\omega t} \left[|||Af(\cdot)|||_{[0,t],\infty} + |||f'|||_{[0,t],\infty}\right]
\end{aligned} \tag{5.9}$$

for each $t \in (0, \infty)$.

Remark 10. *In practical applications, it is easier to consider a uniform partitioning of $[0, t]$ given by*

$$E_n : x_i = \left(\frac{i}{n}\right) \cdot t, \quad i = \overline{0, n},$$

and then (5.6) becomes

$$M_n(t) := \frac{t}{n} \sum_{i=0}^{n-1} T\left[\left(\frac{2n-2i-1}{2n}\right) t\right] f\left[\left(\frac{2i+1}{2n}\right) t\right]. \tag{5.10}$$

In this case, we have the representation of $u(\cdot)$ given by

$$u(t) = x + M_n(t) + V_n(t), \tag{5.11}$$

where the approximation $M_n(\cdot)$ is as defined above in (5.10) and the remainder $V_n(\cdot)$ satisfies the error bounds

$$\|V_n(t)\| \leq \frac{1}{4n} t^3 e^{\|A\| t} \left[\|A\| \, |||f|||_{[0,t],\infty} + |||f'|||_{[0,t],\infty}\right] \tag{5.12}$$

in the first case, and

$$\|V_n(t)\| \leq \frac{1}{4n} M t^3 e^{\omega t} \left[|||Af(\cdot)|||_{[0,t],\infty} + |||f'|||_{[0,t],\infty}\right] \tag{5.13}$$

in the second case, for each $t \in [0, \infty)$.

6. Numerical Examples

Let $X = \mathbb{R}^2$, $x = (\xi, \eta) \in \mathbb{R}^2$, $\|x\|_2 = \sqrt{\xi^2 + \eta^2}$. We consider the linear, 2-dimensional, inhomogeneous differential systems

$$\begin{cases}
\dot{u}_1(t) = u_1(t) \qquad\qquad + \sin t & \\
 & t \geq 0 \\
\dot{u}_2(t) = \qquad\qquad -u_2(t) + \cos t & \\
u_1(0) = u_2(0) = 0 &
\end{cases}.$$

If we let $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $f(t) = (\sin t, \cos t)$, $x_0 = (0,0)$ and identify (ξ, η) by $\begin{pmatrix} \xi \\ \eta \end{pmatrix}$, then the above system is the Cauchy problem $(A, f, 0, x_0)$. We have: $e^{tA} = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$,

$$
\begin{aligned}
(6.1) \qquad u(t) &= \int_0^t e^{(t-s)A} f(s)\, ds \\
&= \left(\int_0^t e^{(t-s)} \sin s ds, \int_0^t e^{-(t-s)} \cos s ds \right) \\
&= \left(\frac{1}{2}\left(e^t - \sin t - \cos t\right), \frac{1}{2}\left(\sin t + \cos t - e^{-t}\right) \right).
\end{aligned}
$$

Now, if we consider

$$
\tilde{M}_n(t) := \frac{t}{n} \sum_{i=0}^{n-1} \left[e^{\left[\left(\frac{2n-2i-1}{2n}\right)t\right]} \sin\left[\left(\frac{2i+1}{2n}\right)t\right], e^{-\left[\left(\frac{2n-2i-1}{2n}\right)t\right]} \cos\left[\left(\frac{2i+1}{2n}\right)t\right] \right]
$$

then, by (5.11), the exact solution $u(\cdot)$ given in (6.1) may be represented by

$$
u(t) = \tilde{M}_n(t) + \tilde{V}_n(t) \ \text{ for any } \ t \geq 0. \tag{6.2}
$$

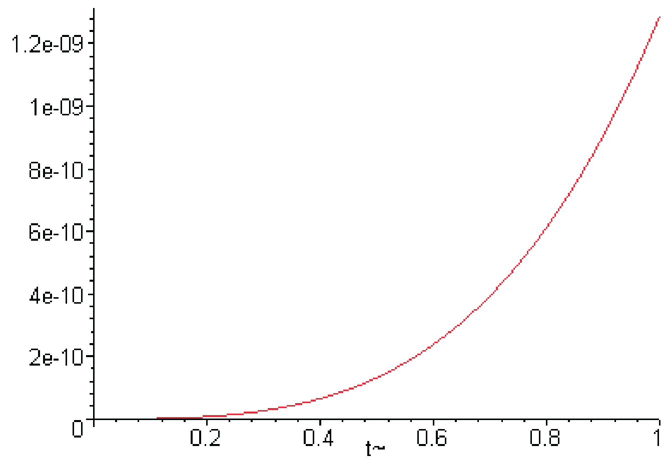
and, by (5.12), we know that

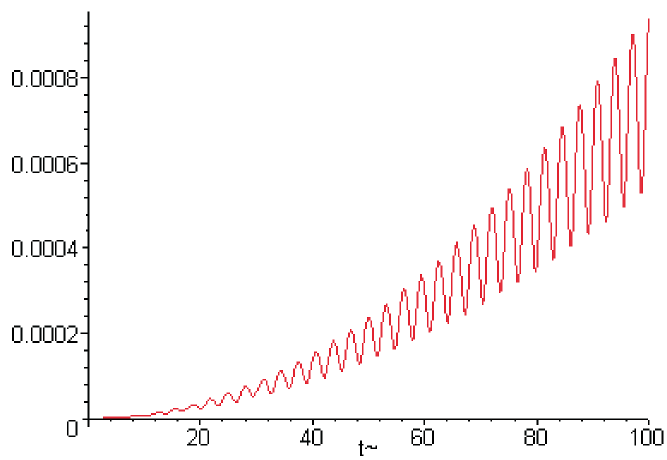
$$
\lim_{n\to\infty} \left\| \tilde{V}_n(t) \right\|_2 = 0 \ \text{ for each } \ t \geq 0. \tag{6.3}
$$

We have

$$
\begin{aligned}
B_n(t) \quad : \quad &= \left\| \tilde{V}_n(t) \right\|_2 \\
= \quad & \left\{ \left[\frac{1}{2}\left(e^t - \sin t - \cos t\right) - \frac{t}{n} \sum_{i=0}^{n-1} e^{\left[\left(\frac{2n-2i-1}{2n}\right)t\right]} \sin\left[\left(\frac{2i+1}{2n}\right)t\right] \right]^2 \right. \\
& \left. + \left[\frac{1}{2}\left(\sin t + \cos t - e^{-t}\right) - \frac{t}{n} \sum_{i=0}^{n-1} e^{-\left[\left(\frac{2n-2i-1}{2n}\right)t\right]} \cos\left[\left(\frac{2i+1}{2n}\right)t\right] \right]^2 \right\}^{\frac{1}{2}}.
\end{aligned}
$$

If we implement $B_n(\cdot)$ for $n = 10^6$ and $t \in [0,1]$, then the plot of the error in approximating the exact value of $u(\cdot)$ by its approximation $\tilde{M}_n(\cdot)$ on the interval $[0,1]$ is embodied in Figure 1.





Let us now consider another system

$$\left\{\begin{array}{llll} \dot{u}_1(t)= & -u_1(t) & & +\sin t \\ \dot{u}_2(t)= & & -2u_2(t) & +\cos t \\ u_1(0)= & u_2(0)=0. & & \end{array}\right. . \tag{6.4}$$

The solution of this system is given by

$$u(t)=\left(\frac{1}{2}\left(e^{-t}+\sin t-\cos t\right),\frac{1}{5}\left(-2e^{-2t}+\sin t+2\cos t\right)\right). \tag{6.5}$$

Now, if we consider

$$\tilde{M}_n(t):=\frac{t}{n}\sum_{i=0}^{n-1}\left[e^{-\left[\left(\frac{2n-2i-1}{2n}\right)t\right]}\sin\left[\left(\frac{2i+1}{2n}\right)t\right],e^{-2\left[\left(\frac{2n-2i-1}{2n}\right)t\right]}\cos\left[\left(\frac{2i+1}{2n}\right)t\right]\right]$$

then by (5.11) the exact solution of the system (6.4), given in (6.5) may be represented as in (6.2), and by (5.13), we know that

$$\lim_{n\to\infty}\left\|\tilde{V}_n(t)\right\|_2=0$$

for any t on $[0,\infty)$. We have

$$\begin{aligned} B_n(t) \quad : &= \left\|\tilde{V}_n(t)\right\|_2 \\ &= \left\{\left[\frac{1}{2}\left(e^t-\sin t-\cos t\right)-\frac{t}{n}\sum_{i=0}^{n-1}e^{-\left[\left(\frac{2n-2i-1}{2n}\right)t\right]}\sin\left[\left(\frac{2i+1}{2n}\right)t\right]\right]^2\right. \\ &\quad \left.+\left[\frac{1}{5}\left(-2e^{-2t}+\sin t+2\cos t\right)-\frac{t}{n}\sum_{i=0}^{n-1}e^{-2\left[\left(\frac{2n-2i-1}{2n}\right)t\right]}\cos\left[\left(\frac{2i+1}{2n}\right)t\right]\right]^2\right\}^{\frac{1}{2}}. \end{aligned}$$

If we implement $B_n(\cdot)$ for $n=10^3$, then the plot of the error in approximating the exact value $u(\cdot)$ by its approximation $\tilde{M}_n(\cdot)$ on the interval $[0,100]$ is embodied in Figure 2.

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