OSTROWSKI'S INEQUALITY FOR VECTOR-VALUED FUNCTIONS OF BOUNDED SEMIVARIATION AND APPLICATIONS

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Abstract. An Ostrowski type inequality for vector-valued functions of bounded semivariation and its applications for linear operator inequalities and differential equations in Banach spaces are given.

1. Introduction

Let X be a real or complex Banach space and X^* its topological dual space, i.e., the space consisting of all bounded linear functionals $x^* : X \to \mathbb{K}$. Let $-\infty < a < b < \infty$ be two real numbers. A function $f : [a, b] \to X$ is said to be:

(i) of *bounded variation* if there exists an $M \geq 0$ such that for all partitions $\Pi : a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b$ we have

$$\sum_{i=1}^{n} \| f(t_i) - f(t_{i-1}) \| \leq M.$$

(ii) of *bounded semivariation* if there exists an $M \geq 0$ such that for each natural non-null number N and all mutual disjoint intervals $(s_1, t_1), (s_2, t_2), \ldots, (s_N, t_N)$ with $(s_i, t_i) \subset [a, b]$ for every $i \in \{1, \ldots, N\}$ we have

$$\left\| \sum_{i=1}^{N} (f(t_i) - f(s_i)) \right\| \leq M.$$

(iii) of *weakly bounded variation* if the function $x^* \circ f$ is of bounded variation for each $x^* \in X^*$.

It is clear that if f is of bounded variation, then it is of bounded semivariation. Moreover, if f is of bounded variation, then it is of weakly bounded variation, because for every $x^* \in X^*$, $\|x^*\| \leq 1$, we have

$$|x^* (f(t_i) - f(t_{i-1}))| \leq \| f(t_i) - f(t_{i-1}) \|, \quad \text{for all} \quad i = \overline{1, n}.$$

In fact, a function $f : [a, b] \to X$ is of bounded semivariation if and only if f is of weakly bounded variation [2].

Let $\Pi : a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b$ be a partition of an interval $[a, b]$. We denote by $\nu(\Pi) := \max \{t_i - t_{i-1},\ i \in 1, 2, \ldots, n\}$ the norm of Π. Let $f : [a, b] \to X$ and $g : [a, b] \to \mathbb{C}$ be two functions. The function g is Riemann-Stieltjes integrable

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with respect to f on $[a,b]$ if for all $\boldsymbol{\xi}=(\xi_1,\xi_2,\dots,\xi_n)$ with $t_{i-1}\le\xi_i\le t_i$ for all $i=\overline{1,n}$, the limit

$$\lim_{\nu(\Pi)\to 0}\sum_{i=1}^{n} g(\xi_i)\left[f(t_i)-f(t_{i-1})\right]$$

exists in X. Such a limit is denoted by $\int_a^b gdf$ and is called the Riemann-Stieltjes integral of g with respect to f on $[a,b]$.

It is easy to see that if g is Riemann-Stieltjes integrable with respect to f, then f is Riemann-Stieltjes with respect to g. In addition, the following formula

$$\int_a^b fdg = g(b)f(b)-g(a)f(a)-\int_a^b gdf$$

holds.

If one of the functions f,g is continuous and the other is of bounded semivariation, then each of them is Riemann-Stieltjes integrable with respect to the other [2]. In particular, if $f:[a,b]\to X$ is of bounded semivariation, then f is Riemann integrable on $[a,b]$.

If $f:[a,b]\to X$ is of bounded semivariation then its *totally weak variation* (which is denoted as follows by $w-\bigvee_a^b(f)$) is finite, i.e., there exists $M>0$ such that

$$\begin{aligned} w-\bigvee_a^b(f) \quad &: = \sup\left\{\sum_{i=1}^{n}\left|x^*\left(f(t_i)-f(t_{i-1})\right)\right|, p\in\Pi([a,b]), x^*\in X^*, \|x^*\|\le 1\right\} \\ &= M<\infty, \end{aligned}$$

where $p: a=t_0<t_1<\cdots<t_{n-1}<t_n=b$ and $\Pi([a,b])$ is the set of all partitions of the interval $[a,b]$.

Indeed, the set of all bounded linear operators $T_{p,f}:X^*\to\mathbb{C}$, given by

$$T_{p,f}(x^*) := \sum_{i=1}^{n} x^*\left(f(t_i)-f(t_{i-1})\right), \quad p\in\Pi([a,b]),$$

is uniformly punctually bounded, i.e., for each $x^*\in X^*$ there exists $K(x^*)>0$ such that

$$\left|T_{p,f}(x^*)\right|\le K(x^*)<\infty, \quad \text{for all } p\in\Pi([a,b]).$$

Then from the *uniform boundedness principle* it follows that there exists $K>0$ such that

$$\left|T_{p,f}(x^*)\right|\le K\|x^*\|, \quad \text{for all } p\in\Pi([a,b]),$$

i.e., the desired statement holds.

Having considered all the above, we can now formulate the following result.

Lemma 1. *If $g:[a,b]\to\mathbb{C}$ is a continuous function and $f:[a,b]\to X$ is of bounded semivariation, then*

$$\left\|\int_a^b gdf\right\|\le\sup_{t\in[a,b]}|g(t)|\left(w-\bigvee_a^b(f)\right). \tag{1.1}$$

Proof. Let $\Pi: a=t_0<t_1<\cdots<t_{n-1}<t_n=b$ be an arbitrary partition of the interval $[a,b]$ and $x^*\in X^*$ with $\|x^*\|\le 1$. Then for every intermediate point

$\xi_i \in [t_{i-1}, t_i]$, we have:

$$\begin{aligned}\left|x^*\left(\sum_{i=1}^n g(\xi_i)(f(t_i)-f(t_{i-1}))\right)\right| &\le \sum_{i=1}^n |g(\xi_i)|\,|x^*(f(t_i)-f(t_{i-1}))| \\ &\le \sup_{t\in[a,b]}|g(t)|\sum_{i=1}^n |x^*(f(t_i)-f(t_{i-1}))| \\ &\le \sup_{t\in[a,b]}|g(t)|\left(w-\bigvee_a^b(f)\right).\end{aligned}$$

Then, using a well-known fact (see for example [4, p. 135]), namely that for $x \in X$ one has

$$\|x\| = \sup\{|x^*(x)| : x^* \in X^*, \|x^*\| \le 1\},$$

it follows that

$$\left\|\sum_{i=1}^n g(\xi_i)(f(t_i)-f(t_{i-1}))\right\| \le \sup_{t\in[a,b]}|g(t)|\left(w-\bigvee_a^b(f)\right).$$

Taking the limit as $\nu(\Pi) \to 0$ in the previous inequality and using the fact that g is Riemann-Stieltjes integrable with respect to f, (1.1) follows. ∎

The following result easily follows using some elementary considerations and the fact that (1.1) holds for scalar valued functions.

Lemma 2. *Let $-\infty < a \le c \le b < \infty$ and $f : [a,b] \to X$ be a function which is of bounded semivariation on $[a,b]$ and of bounded semivariation on $[c,b]$. Then f is of bounded semivariation on $[a,b]$ and*

$$w-\bigvee_a^b(f) = \left(w-\bigvee_a^c(f)\right) + \left(w-\bigvee_c^b(f)\right).$$

In this paper we point out an inequality of Ostrowski type for vector-valued functions of bounded semivariation and apply it for operator inequalities and for approximating the solutions of certain differential equations in Banach spaces.

For the Ostrowski type inequalities for scalar-valued functions, see [1], [6] and [7].

2. An Ostrowski Type Inequality

The following theorem holds.

Theorem 1. *Let X be a Banach space and $f : [a,b] \to X$ a mapping of bounded semivariation on $[a,b]$. Then for all $s \in [a,b]$, we have the inequalities*

$$\begin{aligned}&\left\|\int_a^b f(t)\,dt - (b-a)f(s)\right\| \qquad (2.1)\\ \le\ & (s-a)\left(w-\bigvee_a^s(f)\right) + (b-s)\left(w-\bigvee_s^b(f)\right) \\ \le\ & \left[\frac{1}{2}(b-a) + \left|s-\frac{a+b}{2}\right|\right]\left(w-\bigvee_a^b(f)\right).\end{aligned}$$

The constant $\frac{1}{2}$ in the second inequality is the best possible one.

Proof. Using the integration by parts formula for Riemann-Stieltjes integrals, we have

$$\int_a^s (t-a)\,df(t) = (s-a)\,f(s) - \int_a^s f(t)\,dt$$

and

$$\int_s^b (t-b)\,df(t) = (b-s)\,f(s) - \int_s^b f(t)\,dt.$$

If we add the two equalities, we obtain

$$(b-a)\,f(s) - \int_a^b f(t)\,dt = \int_a^s (t-a)\,df(t) + \int_s^b (t-b)\,df(t) \tag{2.2}$$

for any $s \in [a,b]$.

Taking the norm on (2.2), we get

$$\begin{aligned} & \left\| (b-a)\,f(s) - \int_a^b f(t)\,dt \right\| \\ \leq\; & \left\| \int_a^s (t-a)\,df(t) \right\| + \left\| \int_s^b (t-b)\,df(t) \right\| \\ \leq\; & \sup_{t\in[a,s]} (t-a) \left(w - \bigvee_a^s (f) \right) + \sup_{t\in[s,b]} (b-t) \left(w - \bigvee_s^b (f) \right) \\ =\; & (s-a) \left(w - \bigvee_a^s (f) \right) + (b-s) \left(w - \bigvee_s^b (f) \right) \end{aligned}$$

where, for the last inequality, we have applied Lemma 1. Thus, the first inequality in (2.1) is proved.

Using Lemma 2, we may write that

$$\begin{aligned} & (s-a) \left(w - \bigvee_a^s (f) \right) + (b-s) \left(w - \bigvee_s^b (f) \right) \\ \leq\; & \max\{s-a, b-s\} \left[\left(w - \bigvee_a^s (f) \right) + \left(w - \bigvee_s^b (f) \right) \right] \\ \leq\; & \left[\frac{1}{2}(b-a) + \left| s - \frac{a+b}{2} \right| \right] \left(w - \bigvee_a^b (f) \right) \end{aligned}$$

and the last part of (2.1) is proved.

The fact that $\frac{1}{2}$ is the best constant follows in the same manner as in [5] and we omit the details. ∎

Corollary 1. *With the assumptions in Theorem 1, we have*

$$\left\| \int_a^b f(t)\,dt - (b-a)\,f\left(\frac{a+b}{2}\right) \right\| \leq \frac{1}{2}(b-a) \left(w - \bigvee_a^b (f) \right). \tag{2.3}$$

The constant $\frac{1}{2}$ is best possible.

Remark 1. *If $f:[a,b]\to X$ is of bounded variation on $[a,b]$, then*

$$\left\|\int_a^b f(t)\,dt-(b-a)f(s)\right\| \leq (s-a)\bigvee_a^s(f)+(b-s)\bigvee_s^b(f) \leq \left[\frac{1}{2}(b-a)+\left|s-\frac{a+b}{2}\right|\right]\bigvee_a^b(f). \tag{2.4}$$

In particular, if f is differentiable and the derivative $f':[a,b]\to X$ is continuous, then

$$\begin{aligned}&\left\|\int_a^b f(t)\,dt-(b-a)f(s)\right\| \\ \leq\ & (s-a)\int_a^s\|f'(t)\|\,dt+(b-s)\int_s^b\|f'(t)\|\,dt \\ \leq\ & \left[\frac{1}{2}(b-a)+\left|s-\frac{a+b}{2}\right|\right]\int_a^b\|f'(t)\|\,dt.\end{aligned} \tag{2.5}$$

Remark 2. *When X is $\mathbb{K}$, the field of scalars, then the inequality (2.4) becomes a known result obtained in* [5].

In the following we will present three examples in which we apply Theorem 1 and its consequence from (2.5).

Let $X=L^2([0,1],\mathbb{R})$. We consider the function $f:[0,1]\to X$ given by $f(t)=t\cdot 1_{[0,t]}$, $t\in[0,1]$. Here $1_{[0,t]}$ is the characteristic function on the interval $[0,t]$.

Let $\Pi: 0=t_0<t_1<\cdots<t_{n-1}<t_n=1$ be an arbitrary partition of the interval $[0,1]$. Then for all $x^*\in L^2([0,1],\mathbb{R})=X^*$, we have:

$$\begin{aligned}&\sum_{i=1}^n|x^*(f(t_i)-f(t_{i-1}))| \\ =\ & \sum_{i=1}^n\left|\int_0^1 x^*(s)\left[(f(t_i))(s)-(f(t_{i-1}))(s)\right]ds\right| \\ =\ & \sum_{i=1}^n\left|\int_0^{t_{i-1}}x^*(s)\cdot t_i s ds-\int_0^{t_{i-1}}x^*(s)\cdot t_{i-1}s ds+\int_{t_{i-1}}^{t_i}x^*(s)\cdot t_i s ds\right| \\ \leq\ & \sum_{i=1}^n\left[(t_i-t_{i-1})\int_0^{t_{i-1}}|x^*(s)|\,ds+\int_{t_{i-1}}^{t_i}|x^*(s)|\,ds\right] \\ \leq\ & 2\int_0^1|x^*(s)|\,ds\leq 2\left(\int_0^1|x^*(s)|^2\,ds\right)^{\frac{1}{2}}=2\|x^*\|_2.\end{aligned}$$

Taking the supremum for all $x^*\in X^*$ with $\|x^*\|_2\leq 1$, we obtain that $w-\bigvee_0^1(f)\leq 2$, which shows that f is of bounded semivariation. On the other hand,

$$\begin{aligned}\|f(t_i)-f(t_{i-1})\|_2^2 &= \int_0^1|(f(t_i)-f(t_{i-1}))(s)|^2\,ds \\ &= \int_0^{t_{i-1}}(t_i-t_{i-1})^2 s^2 ds+\int_{t_{i-1}}^{t_i}(t_i s)^2\,ds \\ &= (t_i-t_{i-1})^2\frac{t_{i-1}^3}{3}+\frac{t_i^3}{3}\left(t_i^3-t_{i-1}^3\right).\end{aligned}$$

If we choose $t_i = \frac{i}{n^p}$, $i = 0, 1, 2 \ldots, n$, then

$$\begin{aligned}
&\sum_{i=1}^{n} \|f(t_i) - f(t_{i-1})\|_2 \\
\geq\ & \sum_{i=1}^{n} (t_i - t_{i-1})\, t_{i-1} \sqrt{\frac{t_{i-1}}{3}} = \frac{1}{n^p} \sum_{i=1}^{n} \frac{i-1}{n^p} \sqrt{\frac{i-1}{3n^p}} \\
\geq\ & \frac{1}{n^{2p}\sqrt{3n^p}} \sum_{i=1}^{n-1} i = \frac{n(n-1)}{2n^{2p}\sqrt{3n^p}} \to \infty
\end{aligned}$$

as $n \to \infty$, if p is a suitable positive number.

Proposition 1. *With the above notations the following inequality holds:*

$$w - \bigvee_0^1 (f) \geq \frac{\sqrt{35s^5 - 30s^3 + 8}}{\sqrt{15}\,(1 + |2s-1|)}, \quad \textit{for all} \quad s \in [0,1]. \tag{2.6}$$

Proof. We apply Theorem 1 for our function f, $a = 0$ and $b = 1$. Then, for all $0 \leq s \leq 1$, we have:

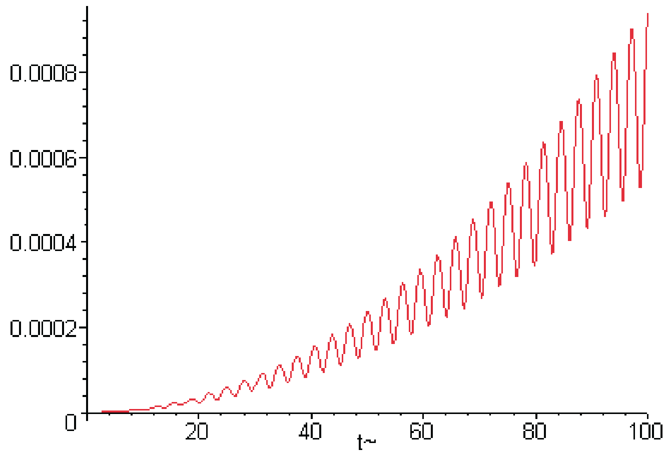
$$\begin{aligned}
\left\| \int_0^1 t \cdot 1_{[0,t]} dt - f(s) \right\|_2^2 &= \int_0^1 \left\{ \left(\int_0^1 t \cdot 1_{[0,t]} dt \right)(\xi) - [f(s)](\xi) \right\}^2 d\xi \\
&= \int_0^1 \left\{ \int_0^1 t \cdot 1_{[0,t]}(\xi)\, dt - [f(s)](\xi) \right\}^2 d\xi \\
&= \int_0^s \left(\int_1^\xi t dt - s\xi \right)^2 d\xi + \int_s^1 \left(\int_\xi^1 t dt \right)^2 d\xi \\
&= \int_0^s \left(\frac{1-\xi^2}{2} - s\xi \right)^2 d\xi + \int_s^1 \left(\frac{1-\xi^2}{2} \right)^2 d\xi \\
&= \frac{1}{60} \left(35s^5 - 30s^3 + 8 \right).
\end{aligned}$$

and the proposition is proved. ∎

Remark 3. *Using the plot of the function $g(s)$ in the right hand side of the inequality (2.6), we will obtain the estimate*

$$w - \bigvee_0^1 (f) \geq \sup_{s \in [0,1]} \frac{\sqrt{35s^5 - 30s^3 + 8}}{\sqrt{15}\,(1 + |2s-1|)} = .5968668193$$

(see Figure 1).



Proposition 2. *Let X be a Banach space, A a linear and bounded operator on X and $-\infty < a < b < \infty$. Then for each $s \in [a,b]$, we have:*

$$\left\| \int_a^b e^{tA} dt - (b-a) e^{sA} \right\| \tag{2.7}$$
$$\leq \begin{cases} \left[\frac{1}{2}(b-a) + \left| s - \frac{a+b}{2} \right| \right] \left[e^{b\|A\|} - e^{a\|A\|} \right], & \text{if} \quad a \geq 0; \\ \left[\frac{1}{2}(b-a) + \left| s - \frac{a+b}{2} \right| \right] \left[e^{-a\|A\|} - e^{-b\|A\|} \right], & \text{if} \quad b \leq 0; \\ \left[\frac{1}{2}(b-a) + \left| s - \frac{a+b}{2} \right| \right] \left[e^{b\|A\|} + e^{-a\|A\|} - 2 \right], & \text{if} \quad a \leq 0 \leq b. \end{cases}$$

Proof. Let $\mathcal{L}(X)$ be the Banach space of all bounded linear operators on X endowed with the operatorial norm. We recall that if $A \in \mathcal{L}(X)$, then its operatorial norm is defined by

$$\|A\| = \sup \{ \|Ax\| : x \in X, \ \|x\| \leq 1 \}.$$

We recall also that the series $\left(\sum_{n \geq 1} \frac{(tA)^n}{n!} \right)$ converges absolutely and locally uniformly for $t \in \mathbb{R}$. Let e^{tA} be its sum. It is easy to see that $\left\| e^{tA} \right\| \leq e^{|t|\|A\|}$ for every $t \in \mathbb{R}$ and $\left(e^{tA} \right)' = Ae^{tA}$ for all $t \in \mathbb{R}$. Then applying the inequality from (2.5) with X replaced by $\mathcal{L}(X)$ and $f(t) = e^{tA}$, we get

$$\begin{aligned} \left\| \int_a^b e^{tA} dt - (b-a) e^{sA} \right\| &\leq \left[\frac{1}{2}(b-a) + \left| s - \frac{a+b}{2} \right| \right] \cdot \int_a^b \left\| Ae^{tA} \right\| dt \\ &\leq \left[\frac{1}{2}(b-a) + \left| s - \frac{a+b}{2} \right| \right] \|A\| \int_a^b e^{|t|\|A\|} dt. \end{aligned}$$

Now the estimate (2.7) can be obtained using elementary calculus. We omit the details. ∎

Proposition 3. *Let $A, B \in \mathcal{L}(X)$ such that $\|A\| \neq \|B\|$. Then*

$$\left\| e^{\frac{1}{2}A}(B-A)e^{\frac{1}{2}B} - \left(e^B - e^A\right)\right\| \leq \frac{1}{2}\|B-A\| \cdot (\|A\|+\|B\|) \cdot \frac{e^{\|B\|}-e^{\|A\|}}{\|B\|-\|A\|}.$$

Proof. Let $f:[0,1] \to \mathcal{L}(X)$ be defined by

$$f(t) = e^{(1-t)A}(B-A)e^{tB}.$$

We have

$$\begin{aligned}\int_0^1 f(t)\,dt &= \int_0^1 e^{(1-t)A}\left(e^{tB}\right)' dt + \int_0^1 \left(e^{(1-t)A}\right)' e^{tB} dt \\ &= 2\left(e^B - e^A\right) - \int_0^1 f(t)\,dt.\end{aligned}$$

Then from Corollary 1 it follows that

$$\begin{aligned}&\left\| e^{\frac{1}{2}A}(B-A)e^{\frac{1}{2}B} - \left(e^B - e^A\right)\right\| \\ \leq\ & \frac{1}{2}\bigvee_0^1 (f) = \frac{1}{2}\int_0^1 \|f'(t)\|\,dt \\ \leq\ & \frac{1}{2}\|B-A\| \cdot \frac{\|A\|+\|B\|}{\|B\|-\|A\|}\int_0^1 e^{(1-t)\|A\|}(\|B\|-\|A\|)e^{t\|B\|}dt \\ =\ & \frac{1}{2}\|B-A\|(\|A\|+\|B\|)\cdot\frac{e^{\|B\|}-e^{\|A\|}}{\|B\|-\|A\|}.\end{aligned}$$

We have used the inequalities

$$\left\|e^{tA}\right\| \leq e^{|t|\cdot\|A\|}, \text{ for all } t \in \mathbb{R}$$

and

$$\|T_1T_2\| \leq \|T_1\|\cdot\|T_2\|, \text{ for all } T_1, T_2 \in \mathcal{L}(X).$$

∎

The above theorem may be used for the numerical approximation of the integral $\int_a^b f(t)\,dt$ in terms of arbitrary Riemann sums.

Let $I_n : a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b$ be a division of $[a,b]$, $h_i := t_{i+1} - t_i$ $\left(i = \overline{0, n-1}\right)$ and $\nu(h) := \max_{i=\overline{0,n-1}}\{h_i\}$. Consider the intermediate points $\xi_i \in [t_i, t_{i+1}]$ $\left(i = \overline{0, n-1}\right)$ and define the Riemann sum

$$R_n(f; I_n, \boldsymbol{\xi}) := \sum_{i=0}^{n-1} h_i f(\xi_i). \tag{2.8}$$

The following result holds.

Theorem 2. *Let $f:[a,b] \to X$ be of bounded semivariation on $[a,b]$. Then we have*

$$\int_a^b f(t)\,dt = R_n(f; I_n, \boldsymbol{\xi}) + V_n(f; I_n, \boldsymbol{\xi}), \tag{2.9}$$

where the quadrature formula $R_n(f; I_n, \boldsymbol{\xi})$ is defined in (2.8) and the remainder $V_n(f; I_n, \boldsymbol{\xi})$ satisfies the estimate:

$$\begin{aligned}
(2.10)\quad & \|V_n(f; I_n, \boldsymbol{\xi})\| \\
\leq\; & \sum_{i=0}^{n-1} (\xi_i - t_i)\left(w - \bigvee_{t_i}^{\xi_i}(f)\right) + \sum_{i=0}^{n-1}(t_{i+1} - \xi_i)\left(w - \bigvee_{\xi_i}^{t_{i+1}}(f)\right) \\
\leq\; & \left[\frac{1}{2}\nu(h) + \max_{i=\overline{0,n-1}}\left|\xi_i - \frac{t_{i+1} - t_i}{2}\right|\right]\left(w - \bigvee_a^b(f)\right) \\
\leq\; & \nu(h)\left(w - \bigvee_a^b(f)\right).
\end{aligned}$$

Proof. If we apply (2.1) on the interval $[x_i, x_{i+1}]$ $\left(i = \overline{0, n-1}\right)$, we may write that

$$\begin{aligned}
(2.11)\quad & \left\|\int_{t_i}^{t_{i+1}} f(t)\,dt - h_i f(\xi_i)\right\| \\
\leq\; & (\xi_i - t_i)\left(w - \bigvee_{t_i}^{\xi_i}(f)\right) + (t_{i+1} - \xi_i)\left(w - \bigvee_{\xi_i}^{t_{i+1}}(f)\right) \\
\leq\; & \left[\frac{1}{2}h_i + \left|\xi_i - \frac{t_{i+1} - t_i}{2}\right|\right]\left(w - \bigvee_{t_i}^{t_{i+1}}(f)\right).
\end{aligned}$$

Summing over i from 0 to $n-1$ and using the generalised triangle inequality, we have:

$$\begin{aligned}
\|V_n(f; I_n, \boldsymbol{\xi})\| \leq\; & \sum_{i=0}^{n-1} (\xi_i - t_i)\left(w - \bigvee_{t_i}^{\xi_i}(f)\right) + \sum_{i=0}^{n-1}(t_{i+1} - \xi_i)\left(w - \bigvee_{\xi_i}^{t_{i+1}}(f)\right) \\
\leq\; & \sum_{i=0}^{n-1}\left[\frac{1}{2}h_i + \left|\xi_i - \frac{t_{i+1} - t_i}{2}\right|\right]\left(w - \bigvee_{t_i}^{t_{i+1}}(f)\right) \\
\leq\; & \left[\frac{1}{2}\nu(h) + \max_{i=\overline{0,n-1}}\left|\xi_i - \frac{t_{i+1} - t_i}{2}\right|\right]\left(w - \bigvee_a^b(f)\right) \\
\leq\; & \nu(h)\left(w - \bigvee_a^b(f)\right).
\end{aligned}$$

∎

If we consider the mid-point rule defined by

$$(2.12)\qquad M_n(f; I_n) := \sum_{i=0}^{n-1} h_i f\left(\frac{t_i + t_{i+1}}{2}\right),$$

then we may state the following corollary.

Corollary 2. *Let $f : [a, b] \to X$ be of bounded semivariation on $[a, b]$. Then we have:*

$$(2.13)\qquad \int_a^b f(t)\,dt = M_n(f; I_n) + Q_n(f; I_n),$$

where $M_n(f;I_n)$ is the mid-point rule defined by (2.12) and the remainder $Q_n(f;I_n)$ satisfies the estimate:

$$\|Q_n(f;I_n)\| \leq \frac{1}{2}\sum_{i=0}^{n-1} h_i \left(w - \bigvee_{t_i}^{t_{i+1}}(f)\right) \leq \frac{1}{2}\nu(h)\left(w - \bigvee_a^b (f)\right). \tag{2.14}$$

In practical applications, it is useful to consider an equidistant partitioning

$$E_n : x_i := a + \frac{i}{n}(b-a), \quad i = \overline{0,n}.$$

Thus, the mid-point rule becomes

$$M_n(f) := \frac{1}{n}\sum_{i=0}^{n-1} f\left[a + \left(i+\frac{1}{2}\right)\cdot\frac{b-a}{n}\right]$$

and we have the representation

$$\int_a^b f(t)\,dt = M_n(f) + Q_n(f), \tag{2.15}$$

where the remainder $Q_n(f)$ satisfies the bounds

$$\|Q_n(f)\| \leq \frac{1}{2n}\left(w - \bigvee_a^b (f)\right). \tag{2.16}$$

If one would like to approximate the integral of a function $f : [a,b] \to X$ of bounded semivariation with a theoretical error less than $\varepsilon > 0$, the required minimal number n_ε in the equidistant partitioning is

$$n_\varepsilon = \left[\frac{1}{2\varepsilon}\left(w - \bigvee_a^b (f)\right)\right] + 1, \tag{2.17}$$

where $[r]$ denotes the integer part of $r \in \mathbb{R}$.

3. Application for Differential Equations in Banach Spaces

Let us consider the Cauchy problem

$$(A,s,x) \qquad \begin{cases} \dot{u}(t) = A(t)\,u(t), & t \in \mathbb{R}; \\ u(s) = x \end{cases}$$

on a Banach space X. Here $A(t)$ is a bounded linear operator on X for each $t \in \mathbb{R}$, the function $t \longmapsto A(t) : \mathbb{R} \to \mathcal{L}(X)$ is continuous and integrally bounded, i.e., there exists a $\delta > 0$ such that

$$\sup_{t\in\mathbb{R}} \int_t^{t+\delta} \|A(u)\|\,du = K_\delta < \infty,$$

and $s \in \mathbb{R}$, $x \in X$ are given.

It is well-known that the solution of (A,s,x) is given by

$$u(t) = U(t,s)\,x$$

where $U(t,s) := P(t) P^{-1}(s)$ and $P(\cdot)$ is the solution of the operatorial Cauchy problem

$$\begin{cases} \dot{X}(t) = A(t) X(t) \\ \\ X(0) = I \end{cases}.$$

Here I denotes the identity operator on $\mathcal{L}(X)$. Let $f : \mathbb{R} \to X$ be a continuously differentiable function. We also consider the inhomogeneous and nonautonomous Cauchy problem

$$(A, f, s, x) \qquad \begin{cases} \dot{u}(t) = A(t) u(t) + f(t), \quad t \in \mathbb{R}; \\ \\ u(s) = x. \end{cases}$$

The solution of (A, f, s, x) is given by

$$u(t) := U(t,s) x + \int_s^t U(t,\tau) f(\tau) d\tau. \tag{3.1}$$

In the above conditions the family of bounded linear operators $\{U(t,\tau) : t, \tau \in \mathbb{R}\}$ has some properties which will be summarized next.

(1) $U(t,\xi) U(\xi,\tau) = U(t,\tau)$ for all $t, \xi, \tau \in \mathbb{R}$;
(2) $U(t,t) = I$ for each $t \in \mathbb{R}$;
(3) there exist $\omega \in \mathbb{R}$ and $M > 0$ such that

$$\|U(t,\xi)\| \leq M e^{\omega|t-\xi|} \text{ for every } t \in \mathbb{R} \text{ and } \xi \in \mathbb{R}; \tag{3.2}$$

(4) the functions $t \mapsto U(t,\xi_0)$ and $\xi \mapsto U(t_0,\xi)$ are continuously differentiable for each fixed $\xi_0 \in \mathbb{R}$ and $t_0 \in \mathbb{R}$ respectively. Moreover,

$$\frac{d}{dt}[U(t,\xi_0)] = A(t) U(t,\xi_0)$$

and

$$\frac{d}{dt}[U(t_0,\xi)] = -U(t_0,\xi) A(\xi).$$

A proof of these properties can be found in [3].

Theorem 3. *We will preserve all the hypotheses made on the functions* $A(\cdot)$ *and* $f(\cdot)$ *before. The solution* $u(\cdot)$ *of* $(A, f, 0, x)$ *can be represented as*

$$u(t) = U(t,0) x + S_n(\boldsymbol{\lambda}, \boldsymbol{\mu}, t) + Q_n(\boldsymbol{\lambda}, \boldsymbol{\mu}, t), \quad t \geq 0, \tag{3.3}$$

where

$$S_n(\boldsymbol{\lambda}, \boldsymbol{\mu}, t) = t \sum_{i=0}^{n-1} (\lambda_{i+1} - \lambda_i) U(t, \mu_i t) f(\mu_i t),$$

$\boldsymbol{\lambda} : 0 = \lambda_0 < \lambda_1 < \cdots < \lambda_{n-1} < \lambda_n = 1$ *is a partition of the interval* $[0,1]$ *and* $\lambda_i \leq \mu_i \leq \lambda_{i+1}$ *for all positive integers* i *with* $0 \leq i \leq n-1$. *Moreover, the remainder* $Q_n(\boldsymbol{\lambda}, \boldsymbol{\mu}, t)$ *satisfies the estimates:*

$$\begin{aligned} &\|Q_n(\boldsymbol{\lambda}, \boldsymbol{\mu}, t)\| \\ \leq\ & \frac{1}{2}\nu(\boldsymbol{\lambda}) t \cdot M e^{\omega t} \left[|||A(\cdot)|||_{[0,t],\infty} |||f(\cdot)|||_{[0,t],\infty} + |||f'(\cdot)|||_{[0,t],\infty} \right] \end{aligned} \tag{3.4}$$

and

$$\|Q_n(\boldsymbol{\lambda},\boldsymbol{\mu},t)\| \leq \frac{1}{2}\nu(\boldsymbol{\lambda})\cdot Me^{\omega t}\left[K_\delta\left(2+\frac{t}{\delta}\right)|||f(\cdot)|||_{[0,t],\infty}+\bigvee_0^t(f)\right], \tag{3.5}$$

respectively for each $t\in[0,\infty)$, where ω is a positive number such that the estimate (3.2) holds.

Proof. For a fixed $t>0$ consider the function $g(\tau)=U(t,\tau)f(\tau)$ for $\tau\in[0,t]$. Then g is differentiable on $[0,t]$ and

$$g'(\tau)=-U(t,\tau)A(\tau)f(\tau)+U(t,\tau)f(\tau),\quad\text{for all}\quad \tau\in[0,t].$$

We have

$$\begin{aligned}\|g'(\tau)\| &\leq \|U(t,\tau)\|\,\|A(\tau)\|\,\|f(\tau)\|+\|U(t,\tau)\|\,\|f(\tau)\|\\ &\leq Me^{\omega t}\left[|||A(\cdot)|||_{[0,t],\infty}\cdot|||f(\cdot)|||_{[0,t],\infty}+|||f'(\cdot)|||_{[0,t],\infty}\right]\end{aligned}$$

and then

$$\begin{aligned}\bigvee_0^t(g) &= \int_0^t\|g'(\tau)\|\,d\tau\\ &\leq Mte^{\omega t}\left[|||A(\cdot)|||_{[0,t],\infty}\cdot|||f(\cdot)|||_{[0,t],\infty}+|||f'(\cdot)|||_{[0,t],\infty}\right].\end{aligned}$$

Now the estimate from (3.3) easily follows from (2.14).

On the other hand

$$\begin{aligned}\bigvee_0^t(g) &= \int_0^t\|g'(\tau)\|\,d\tau \\ &\leq Me^{\omega t}|||f(\cdot)|||_{[0,t],\infty}\int_0^t\|A(\tau)\|\,d\tau+Me^{\omega t}\bigvee_0^t(f)\\ &= Me^{\omega t}\left[|||f(\cdot)|||_{[0,t],\infty}\left(\sum_{i=0}^{n_t}\int_i^{i+\delta}\|A(\tau)\|\,d\tau+\int_{n_t+\delta}^t\|A(\tau)\|\,d\tau\right)+\bigvee_0^t(f)\right]\\ &\leq Me^{\omega t}\left[|||f(\cdot)|||_{[0,t],\infty}(n_t+2)K_\delta+\bigvee_0^t(f)\right]\\ &\leq Me^{\omega t}\left[|||f(\cdot)|||_{[0,t],\infty}\left(\frac{t}{\delta}+2\right)K_\delta+\bigvee_0^t(f)\right],\end{aligned} \tag{3.6}$$

where n_t is the integer part of $\frac{t}{\delta}$.

Using (3.6) and (2.14), we obtain the estimate (3.5). ∎

If we define the quadrature formula

$$M_n(\boldsymbol{\lambda},t):=t\sum_{i=0}^{n-1}(\lambda_{i+1}-\lambda_i)\,U\left(t,\frac{\lambda_i+\lambda_{i+1}}{2}\cdot t\right)f\left(\frac{\lambda_i+\lambda_{i+1}}{2}\cdot t\right), \tag{3.7}$$

then we may state the following corollary.

Corollary 3. *The solution of $(A, f, 0, x)$ can be represented as*

$$u(t) = U(t,0)\,x + M_n(\boldsymbol{\lambda}, t) + L_n(\boldsymbol{\lambda}, t),$$

where $M_n(\boldsymbol{\lambda}, t)$ is as given in (3.7) and the remainder $L_n(\boldsymbol{\lambda}, t)$ satisfies the estimates

$$\|L_n(\boldsymbol{\lambda}, t)\| \le \frac{1}{2}\nu(\boldsymbol{\lambda})\,t^2 \cdot Me^{\omega t}\left[|||A(\cdot)|||_{[0,t],\infty}\,|||f(\cdot)|||_{[0,t],\infty} + |||f'(\cdot)|||_{[0,t],\infty}\right]$$

and

$$\|L_n(\boldsymbol{\lambda}, t)\| \le \frac{1}{2}\nu(\boldsymbol{\lambda})\,t \cdot Me^{\omega t}\left[K_\delta\left(2 + \frac{t}{\delta}\right)|||f(\cdot)|||_{[0,t],\infty} + \bigvee_0^t(f)\right],$$

respectively for each $t \in [0, \infty)$.

Remark 4. *In practical applications, it is easier to consider a uniform partitioning of $[0, t]$ given by*

$$E_n : x_i := \frac{i}{n} \cdot t, \quad 0 \le i \le n$$

and then (3.7) becomes

$$\tilde{M}_n(t) := \frac{t}{n}\sum_{i=0}^{n-1}(\lambda_{i+1} - \lambda_i)\,U\left(t, \frac{2i+1}{2n} \cdot t\right) f\left(\frac{2i+1}{2n} \cdot t\right)$$

In this, case, we have the representation of $u(\cdot)$ given by

$$u(t) = U(t,0)\,x + \tilde{M}_n(t) + \tilde{L}_n(t)$$

where the remainder $\tilde{L}_n(\cdot)$ satisfies the error bounds

$$\left\|\tilde{L}_n(t)\right\| \le \frac{1}{2n}t^2 \cdot Me^{\omega t}\left[|||A(\cdot)|||_{[0,t],\infty}\,|||f(\cdot)|||_{[0,t],\infty} + |||f'(\cdot)|||_{[0,t],\infty}\right]$$

and

$$\left\|\tilde{L}_n(t)\right\| \le \frac{1}{2n}t \cdot Me^{\omega t}\left[K_\delta\left(2 + \frac{t}{\delta}\right)|||f(\cdot)|||_{[0,t],\infty} + \bigvee_0^t(f)\right]$$

respectively.

4. A Numerical Example

Let $X = \mathbb{R}^2$, $x = (\xi, \eta) \in \mathbb{R}^2$, $\|x\|_2 = \sqrt{\xi^2 + \eta^2}$. We consider the linear, 2-dimensional, non-autonomous and inhomogeneous differential system

$$\begin{cases} \dot{u}_1(t) = \left(-1 - \sin^2 t\right)u_1(t) + (-1 + \sin t\cos t)\,u_2(t) + e^{-t}; \\ \dot{u}_1(t) = (1 + \sin t\cos t)\,u_1(t) + \left(-1 - \cos^2 t\right)u_2(t) + e^{-2t}; \\ u_1(0) = u_2(0) = 0. \end{cases} \tag{4.1}$$

If we denote

$$A(t) := \begin{pmatrix} -1 - \sin^2 t & -1 + \sin t\cos t \\ 1 + \sin t\cos t & -1 - \cos^2 t \end{pmatrix}, \quad f(t) = \left(e^{-t}, e^{-2t}\right), \quad x = (0,0)$$

and we identify (ξ,η) with $\begin{pmatrix} \xi \\ \eta \end{pmatrix}$, then the above system is the Cauchy problem $(A,f,0,x)$. The fundamental matrix associated with $A(t)$ is

$$U(t,s)=P(t)P^{-1}(s),\quad t\in\mathbb{R},\ s\in\mathbb{R}, \tag{4.2}$$

where $P(\cdot)$ is the solution of the following operatorial Cauchy problem

$$\dot{Y}(t)=A(t)Y(t),\quad Y(0)=I_2,\quad t\in\mathbb{R}, \tag{4.3}$$

and I_2 is the 2-dimensional, quadratic real matrix identity.

Let $W-(t):=\begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}$. Then it is easy to see that

$$\begin{aligned} \dot{W}(t)W^{-1}(t) &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\ \text{and}\ \ W^{-1}(t)\begin{pmatrix} -1 & -1 \\ 1 & -2 \end{pmatrix}W(t) &= A(t),\ \text{for all } t\in\mathbb{R}. \end{aligned}$$

Now, let $Z(t):=W-(t)P(t)$. We have

$$\begin{aligned} \dot{Z}(t) &= \dot{W}(t)P(t)+W(t)\dot{P}(t) \\ &= \left[\dot{W}(t)W^{-1}(t)+W(t)A(t)W^{-1}(t)\right]Z(t) \\ &= BZ(t), \end{aligned}$$

where $B=\begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}$. Also, using the fact that $Z(0)=I_2$ it follows that

$$Z(t)=\begin{pmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{pmatrix},\quad t\in\mathbb{R}.$$

Then the solution $P(\cdot)$ of the operatorial Cauchy problem (4.2) is

$$P(t)=\begin{pmatrix} e^{-t}\cos t & e^{-2t}\sin t \\ & \\ -e^{-t}\sin t & e^{-2t}\cos t \end{pmatrix},\quad t\in\mathbb{R}$$

and the exact solution of the system (4.1) is $u(t)=(u_1(t),u_2(t))$, where

$$\begin{cases} u_1(t)=e^{-t}\cos t\cdot E_1(t)+e^{-2t}\sin t\cdot E_2(t) \\ \\ u_2(t)=-e^{-t}\sin t\cdot E_1(t)+e^{-2t}\cos t\cdot E_2(t) \end{cases} \quad t\in\mathbb{R}, \tag{4.4}$$

and

$$\begin{aligned} E_1(t) &= \int_0^t \left(\cos s-e^{-s}\sin s\right)ds \\ &= \sin t+\frac{1}{2}e^{-t}(\cos t+\sin t)-\frac{1}{2}, \\ E_2(t) &= \int_0^t (\cos s+e^{s}\sin s)\,ds \\ &= \sin t+\frac{1}{2}(\sin t-\cos t)\cdot e^{t}+\frac{1}{2}. \end{aligned}$$

Now, if we consider

$$\begin{aligned}\tilde{M}_n(t) &= \frac{t}{n}\left[\left(e^{-t}\cos t\right)\cdot S_1(n)+\left(e^{-2t}\sin t\right)\cdot S_2(n)\right.,\\ &\left.\left(e^{-t}\sin t\right)\cdot S_1(n)+\left(e^{-2t}\cos t\right)\cdot S_2(n)\right],\end{aligned}$$

where

$$S_1(n)=\sum_{i=0}^{n-1}\left[\cos\left(\frac{2i+1}{2n}\cdot t\right)-e^{-\left(\frac{2i+1}{2n}\cdot t\right)}\cdot\sin\left(\frac{2i+1}{2n}\cdot t\right)\right]$$

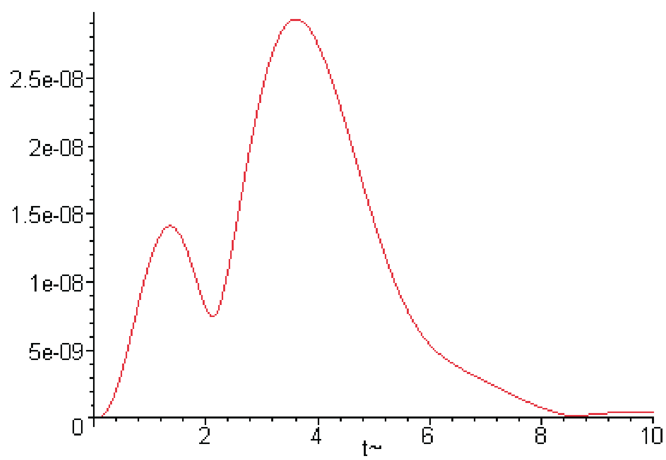
and

$$S_2(n)=\sum_{i=0}^{n-1}\left[\cos\left(\frac{2i+1}{2n}\cdot t\right)+e^{\frac{2i+1}{2n}\cdot t}\cdot\sin\left(\frac{2i+1}{2n}\cdot t\right)\right],$$

then the exact solution given in (4.4) may be represented by

$$u(t)=\tilde{M}_n(t)+\tilde{L}_n(t),\quad t\in\mathbb{R},$$

For $n=10^3$, the plot of the 2−norm of the error $\left\|\tilde{L}_n(\cdot)\right\|_2$ is embodied in Figure 2.



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