WEIGHTED INTEGRAL INEQUALITIES IN TWO DIMENSIONS

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Abstract. Weighted (or product) double integral inequalities are developed and extended to produce weighted cubature rules. The error bounds are of first and second order and rely on the first few moments of the weight. Various properties of the weight and weight null-spaces are considered. Minimization of the bound produces coupled non-linear equations whose solution furnish optimal weighted cubature grids. These grids are evaluated for some of the more popular weight functions.

1. Introduction

Milovanović [3] (see also [4]), Barnett and Dragomir [1] and Hanna et al. [2] developed two dimensional integral inequalities whose error bounds were expressed in Lebesgue norms of the first partial derivatives of the integrand. In other work, Roumeliotis [7] developed and reviewed weighted one dimensional Ostrowski type inequalities with a particular emphasis of identifying optimal quadrature grids. These grids, influenced by the first few moments of the weight function, were evaluated via minimization of the Ostrowski type error bound. In this paper we combine and extend these results to develop weighted first and second order double integral inequalities. Particular attention is paid to the influence of the two dimensional weight function on the error bound and we explore this influence for different weights and weight null-spaces. Furthermore, weighted second order cubature rules are developed and we devise a method for calculating cubature grids that rely only on the first two moments of the weight. A method for calculating *a priori* cubature grids is given.

The work in this paper is presented in the following order. In Section 2, a two variable Taylor expansion is employed to develop weighted two dimensional integral inequalities. Milovanović [3] used this method to extend Ostrowski's inequality to multiple dimensions. Here we will content ourselves with two dimensions, but extend the order of the rule to two. We undertake an examination of the error bound and identify parameters that will minimize the bound. In Section 3, we present a Peano kernel method, based on analogous results in [1], to derive a second order weighted double integral inequality. Error bounds are expressed in terms of the L_1 and L_∞ norms of the first mixed partial derivative of the integrand. Particular attention is paid to minimizing this integrand for different weights and null-spaces. Finally, the results of this section are extended in Section 5 to develop a weighted cubature formula. Minimizing the error bound furnishes a set of non-linear coupled equations in the first two moments of the weight whose solution produces a cubature grid influenced by the weight function. Plots of the grid for various weights are given.

2. Taylor's Formula

In 1975, Milovanović [3] generalised the Ostrowski inequality to multiple dimensions using the multiple variable Taylor formula. As per the Ostrowski result, the inequality was expressed in terms of the first partial derivatives of the integrand. We state the two dimensional formula below.

Following [3], let $D = \{(x_1, x_2) | a_i < x_i < b_i (i = 1, 2)\}$ and let $\bar{D}$ be the closure of D.

Theorem 1. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a differentiable function defined on $\bar{D}$ and let $\left|\frac{\partial f}{\partial t_i}\right| \le M_i$ $(M_i > 0;\ i = 1, 2)$ in D. Then, for every $X = (x_1, x_2) \in \bar{D}$,*

$$\left|\frac{1}{(b_1-a_1)(b_2-a_2)}\int_{a_1}^{b_1}\int_{a_2}^{b_2} f(t_1,t_2)\,dt_2dt_1 - f(x_1,x_2)\right| \le M_1(b_1-a_1)\left(\frac{\left(x_1-\frac{a_1+b_1}{2}\right)^2}{(b_1-a_1)^2}+\frac{1}{4}\right) + M_2(b_2-a_2)\left(\frac{\left(x_2-\frac{a_2+b_2}{2}\right)^2}{(b_2-a_2)^2}+\frac{1}{4}\right). \tag{1}$$

The weighted version of Theorem 1 appears below.

Theorem 2. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a differentiable function defined on $\bar{D}$ and let $\left|\frac{\partial f}{\partial t_i}\right| \le M_i$ $(M_i > 0;\ i = 1, 2)$ in D. Furthermore, let the function $X \mapsto w(X)$ be defined, integrable and $w(X) > 0$ for every $X \in \bar{D}$. Then*

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for every $X \in \bar{D}$,

$$\left| \frac{\int_{a_1}^{b_1}\int_{a_2}^{b_2} w(t_1,t_2) f(t_1,t_2)\, dt_2 dt_1}{\int_{a_1}^{b_1}\int_{a_2}^{b_2} w(t_1,t_2)\, dt_2 dt_1} - f(x_1,x_2) \right| \le \frac{1}{\int_{a_1}^{b_1}\int_{a_2}^{b_2} w(t_1,t_2)\, dt_2 dt_1} \left(M_1 \int_{a_1}^{b_1}\int_{a_2}^{b_2} w(t_1,t_2)|x_1 - t_1|\, dt_2 dt_1 + M_2 \int_{a_1}^{b_1}\int_{a_2}^{b_2} w(t_1,t_2)|x_2 - t_2|\, dt_2 dt_1 \right). \tag{2}$$

Theorem 2 can be extended to higher orders and below we provide such an extension to second order.

Theorem 3. *Let* $f : [a_1, b_1] \times [a_2, b_2] \to \mathbb{R}$ *be such that all its partial derivatives up to order 2 exist and be continuous, i.e.* $\frac{\partial^i f}{\partial t_1^j \partial t_2^k} < \infty, i = 1, 2; j = 0, \ldots, i; k = i - j$. *Furthermore, let* $w : (a_1, b_1) \times (a_2, b_2) \to (0, \infty)$ *be integrable (i.e.* $\iint w\, dA < \infty$*). Then for all* $(x_1, x_2) \in [a_1, b_1] \times [a_2, b_2]$ *the following second order product double integral inequality holds*

$$\begin{aligned} &\Bigg| \int_{a_1}^{b_1}\int_{a_2}^{b_2} w(t_1,t_2) f(t_1,t_2)\, dt_2 dt_1 - f(x_1,x_2) \int_{a_1}^{b_1}\int_{a_2}^{b_2} w(t_1,t_2)\, dt_2 dt_1 \\ &+ \frac{\partial f}{\partial t_1}(x_1,x_2) \int_{a_1}^{b_1}\int_{a_2}^{b_2} w(t_1,t_2)(x_1 - t_1)\, dt_2 dt_1 + \frac{\partial f}{\partial t_2}(x_1,x_2) \int_{a_1}^{b_1}\int_{a_2}^{b_2} w(t_1,t_2)(x_2 - t_2)\, dt_2 dt_1 \Bigg| \\ &\le \frac{\|\frac{\partial^2 f}{\partial t_1^2}\|_\infty}{2} \int_{a_1}^{b_1}\int_{a_2}^{b_2} w(t_1,t_2)(x_1 - t_1)^2\, dt_2 dt_1 + \frac{\|\frac{\partial^2 f}{\partial t_2^2}\|_\infty}{2} \int_{a_1}^{b_1}\int_{a_2}^{b_2} w(t_1,t_2)(x_2 - t_2)^2\, dt_2 dt_1 \\ &\quad + \left\| \frac{\partial^2 f}{\partial t_1 \partial t_2} \right\|_\infty \int_{a_1}^{b_1}\int_{a_2}^{b_2} w(t_1,t_2)|x_1 - t_1||x_2 - t_2|\, dt_2 dt_1. \end{aligned} \tag{3}$$

Proof. The two-variable Taylor formula states that

$$\begin{aligned} f(t_1,t_2) &= f(x_1,x_2) + (t_1 - x_1)\frac{\partial f}{\partial t_1}(x_1,x_2) + (t_2 - x_2)\frac{\partial f}{\partial t_2}(x_1,x_2) \\ &+ \frac{(t_1 - x_1)^2}{2}\frac{\partial^2 f}{\partial t_1^2}(\xi_1,\xi_2) + (t_1 - x_1)(t_2 - x_2)\frac{\partial^2 f}{\partial t_1 \partial t_2}(\xi_1,\xi_2) + \frac{(t_2 - x_2)^2}{2}\frac{\partial^2 f}{\partial t_2^2}(\xi_1,\xi_2), \end{aligned} \tag{4}$$

where $\xi_i = t_i + \theta(x_i - t_i)$, $i = 1, 2$, $0 < \theta < 1$. Multiplying (4) by w and integrating produces the identity

$$\begin{aligned} &\int_{a_1}^{b_1}\int_{a_2}^{b_2} w(t_1,t_2) f(t_1,t_2)\, dt_2 dt_1 - f(x_1,x_2) \int_{a_1}^{b_1}\int_{a_2}^{b_2} w(t_1,t_2)\, dt_2 dt_1 \\ &+ \frac{\partial f}{\partial t_1}(x_1,x_2) \int_{a_1}^{b_1}\int_{a_2}^{b_2} w(t_1,t_2)(x_1 - t_1)\, dt_2 dt_1 \\ &+ \frac{\partial f}{\partial t_2}(x_1,x_2) \int_{a_1}^{b_1}\int_{a_2}^{b_2} w(t_1,t_2)(x_2 - t_2)\, dt_2 dt_1 \\ &= \int_{a_1}^{b_1}\int_{a_2}^{b_2} w(t_1,t_2) \frac{(t_1 - x_1)^2}{2}\frac{\partial^2 f}{\partial t_1^2}(\xi_1,\xi_2)\, dt_2 dt_1 \\ &+ \int_{a_1}^{b_1}\int_{a_2}^{b_2} w(t_1,t_2)(t_1 - x_1)(t_2 - x_2)\frac{\partial^2 f}{\partial t_1 \partial t_2}(\xi_1,\xi_2)\, dt_2 dt_1 \\ &+ \int_{a_1}^{b_1}\int_{a_2}^{b_2} w(t_1,t_2) \frac{(t_2 - x_2)^2}{2}\frac{\partial^2 f}{\partial t_2^2}(\xi_1,\xi_2),\, dt_2 dt_1. \end{aligned} \tag{5}$$

Taking the modulus of both sides of (5), applying the triangle inequality and then Hölder's inequality on the right hand side gives (3). □

Corollary 4. *Let the conditions for f be as in Theorem 3. Then the following double integral inequality holds*

$$\begin{aligned}(6)\quad &\left|\frac{1}{(b_1-a_1)(b_2-a_2)}\int_{a_1}^{b_1}\int_{a_2}^{b_2} f(t_1,t_2)\,dt_2dt_1 - f(x_1,x_2) + \frac{\partial f}{\partial t_1}(x_1,x_2)\left(x_1-\frac{a_1+b_1}{2}\right)\right.\\ &\left.+\frac{\partial f}{\partial t_2}(x_1,x_2)\left(x_1-\frac{a_1+b_1}{2}\right)\right| \le \left\|\frac{\partial^2 f}{\partial t_1^2}\right\|_\infty \frac{(b_1-a_1)^2}{2}\left(\frac{\left(x_1-\frac{a_1+b_1}{2}\right)^2}{(b_1-a_1)^2}+\frac{1}{12}\right)\\ &+\left\|\frac{\partial^2 f}{\partial t_1\partial t_2}\right\|_\infty (b_1-a_1)(b_2-a_2)\left(\frac{\left(x_1-\frac{a_1+b_1}{2}\right)^2}{(b_1-a_1)^2}+\frac{1}{4}\right)\left(\frac{\left(x_2-\frac{a_2+b_2}{2}\right)^2}{(b_2-a_2)^2}+\frac{1}{4}\right)\\ &+\left\|\frac{\partial^2 f}{\partial t_2^2}\right\|_\infty \frac{(b_2-a_2)^2}{2}\left(\frac{\left(x_2-\frac{a_2+b_2}{2}\right)^2}{(b_2-a_2)^2}+\frac{1}{12}\right).\end{aligned}$$

Proof. Substituting $w(t_1,t_2)=1$ into (3) and simplifying produces the desired result. □

The point (x_1,x_2), the sample point of the integration rule, is free to be chosen. Often, such points are chosen to simplify the rule. For example, in (3) if we choose the *weight mean*

$$x_i = \frac{\int_{a_1}^{b_1}\int_{a_2}^{b_2} t_i w(t_1,t_2)\,dt_2dt_1}{\int_{a_1}^{b_1}\int_{a_2}^{b_2} w(t_1,t_2)\,dt_2dt_1},\qquad i=1,2$$

then the partial derivative terms vanish. Fortuitously, in this case, this point also minimizes the bound. In the following sub-section, and indeed this paper, we will not be concerned with simplifying the integration rule, but instead attempt to determine such parameters (for eg. x_1 and x_2) in order for the error bound to be minimized.

2.1. Minimizing the upper bound.

Corollary 5. *The bound in equation (2) is minimized at the median point (x_1,x_2) satisfying*

$$(7)\qquad \int_{a_1}^{x_1}\int_{a_2}^{b_2} w(t_1,t_2)\,dt_2dt_1 = \int_{x_1}^{b_1}\int_{a_2}^{b_2} w(t_1,t_2)\,dt_2dt_1$$

and

$$(8)\qquad \int_{a_2}^{x_2}\int_{a_1}^{b_1} w(t_1,t_2)\,dt_1dt_2 = \int_{x_2}^{b_2}\int_{a_1}^{b_1} w(t_1,t_2)\,dt_1dt_2.$$

Proof. It is a simple matter to show that

$$I(x_1,x_2) = M_1\int_{a_1}^{b_1}\int_{a_2}^{b_2} w(t_1,t_2)|x_1-t_1|\,dt_2dt_1 + M_2\int_{a_1}^{b_1}\int_{a_2}^{b_2} w(t_1,t_2)|x_2-t_2|\,dt_2dt_1$$

is a convex function. Hence the upper bound in (2) is minimized at the stationary point of I. Evaluating the first partial derivatives of I produces equations (7) and (8). □

That is, the minimum point is the *median* of the weight in each direction. This is consistent with first order rules reported in [7].

Minimization of the second order bound in Theorem 3 is not as simple. It is quite difficult to identify a minimum point for the upper bound of (3). This bound is comprised of three components; the first and last are minimized at the mean (in each direction)

$$(9)\qquad x_1 = \frac{\int_{a_1}^{b_1}\int_{a_2}^{b_2} t_1 w(t_1,t_2)\,dt_2dt_1}{\int_{a_1}^{b_1}\int_{a_2}^{b_2} w(t_1,t_2)\,dt_2dt_1},\qquad x_2 = \frac{\int_{a_1}^{b_1}\int_{a_2}^{b_2} t_2 w(t_1,t_2)\,dt_2dt_1}{\int_{a_1}^{b_1}\int_{a_2}^{b_2} w(t_1,t_2)\,dt_2dt_1},$$

while the second is minimized at the root of a median-type expression

$$(10)\qquad \int_{a_1}^{x_1}\int_{a_2}^{b_2} |x_2-t_2|w(t_1,t_2)\,dt_2dt_1 = \int_{x_1}^{b_1}\int_{a_2}^{b_2} |x_2-t_2|w(t_1,t_2)\,dt_2dt_1$$

and

$$(11)\qquad \int_{a_2}^{x_2}\int_{a_1}^{b_1} |x_1-t_1|w(t_1,t_2)\,dt_1dt_2 = \int_{x_2}^{b_2}\int_{a_1}^{b_1} |x_1-t_1|w(t_1,t_2)\,dt_1dt_2.$$

Of course, for weights in which the solutions of (9) are identical to those of (10) and (11) then identification of the minimum point presents little challenge. For example if w is a product weight and symmetric about the midpoint $(\frac{a_1+b_1}{2},\frac{a_2+b_2}{2})$ then the minimum point is the midpoint. That is, if $w(t_1,t_2)=w_1(t_1)w_2(t_2)$

and $w_i((a+b)/2-t) = w_i((a+b)/2+t)$ $(i=1,2)$, then it can be shown that the solution of (9)–(11) is the mid-point. This is the case when $w=1$ and Corollary 4 shows that the upper bound is minimized at $x_i = (a_i+b_i)/2$, $i=1,2$.

The major difficulty with (3) is that the upper bound is comprised of a linear combination of three terms involving norms of the partial derivative of the integrand. Hence it would be near impossible to find a global minimum that depends only on the weight and not f. To obtain a global minimum for a general second order rule will require either simplification of (3) or the derivation of another expression for the bound. The first point is dealt with in the corollary below, while the second is taken up in the next section.

Corollary 6. *Let f and w be as given in Theorem 3. Then for all $(x_1,x_2) \in [a_1,b_1]\times[a_2,b_2]$ the following second order product double integral inequality holds*

$$\begin{aligned}
&\Bigg| \int_{a_1}^{b_1}\int_{a_2}^{b_2} w(t_1,t_2) f(t_1,t_2)\,dt_2dt_1 - f(x_1,x_2)\int_{a_1}^{b_1}\int_{a_2}^{b_2} w(t_1,t_2)\,dt_2dt_1 \\
&\quad + \frac{\partial f}{\partial t_1}(x_1,x_2)\int_{a_1}^{b_1}\int_{a_2}^{b_2} w(t_1,t_2)(x_1-t_1)\,dt_2dt_1 + \frac{\partial f}{\partial t_2}(x_1,x_2)\int_{a_1}^{b_1}\int_{a_2}^{b_2} w(t_1,t_2)(x_2-t_2)\,dt_2dt_1 \Bigg| \\
&\le \left\| \frac{\partial^2 f}{\partial t_1^2}\right\|_\infty \frac{\|w\|_1}{2}\left[\left|x_1-\frac{a_1+b_1}{2}\right| + \frac{b_1-a_1}{2}\right]^2 + \left\| \frac{\partial^2 f}{\partial t_2^2}\right\|_\infty \frac{\|w\|_1}{2}\left[\left|x_2-\frac{a_2+b_2}{2}\right| + \frac{b_2-a_2}{2}\right]^2 \\
&\quad + \left\| \frac{\partial^2 f}{\partial t_1 \partial t_2}\right\|_\infty \|w\|_1 \left[\left|x_1-\frac{a_1+b_1}{2}\right| + \frac{b_1-a_1}{2}\right]\left[\left|x_2-\frac{a_2+b_2}{2}\right| + \frac{b_2-a_2}{2}\right],
\end{aligned} \tag{12}$$

where $\|w\|_1 = \int_{a_1}^{b_1}\int_{a_2}^{b_2} w(t_1,t_2)\,dt_2dt_1$ is the zero-th moment of the weight.

Proof. The proof involves taking an upper bound of (3) using Hölder's inequality. Thus, consider

$$\begin{aligned}
\int_{a_1}^{b_1}\int_{a_2}^{b_2} w(t_1,t_2)(x_1-t_1)^2\,dt_2dt_1 &\le \sup_{t_1\in[a_1,b_1]} (x_1-t_1)^2 \int_{a_1}^{b_1}\int_{a_2}^{b_2} w(t_1,t_2)\,dt_2dt_1 \\
&= \max\{(x_1-a_1)^2,(x_1-b_1)^2\}\|w\|_1 \\
&= \left[\left|x_1-\frac{a_1+b_1}{2}\right| + \frac{b_1-a_1}{2}\right]^2\|w\|_1.
\end{aligned} \tag{13}$$

Similarly

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2} w(t_1,t_2)(x_2-t_2)^2\,dt_2dt_1 \le \left[\left|x_2-\frac{a_2+b_2}{2}\right| + \frac{b_2-a_2}{2}\right]^2\|w\|_1. \tag{14}$$

Finally,

$$\begin{aligned}
&\int_{a_1}^{b_1}\int_{a_2}^{b_2} w(t_1,t_2)|x_1-t_1||x_2-t_2|\,dt_2dt_1 \\
&\quad\le \sup_{(t_1,t_2)\in[a_1,b_1]\times[a_2,b_2]} |x_1-t_1||x_2-t_2|\|w\|_1 \\
&\quad= \max\{x_1-a_1,b_1-x_1\}\max\{x_2-a_2,b_2-x_2\}\|w\|_1 \\
&\quad= \left[\left|x_1-\frac{a_1+b_1}{2}\right| + \frac{b_1-a_1}{2}\right]\left[\left|x_2-\frac{a_2+b_2}{2}\right| + \frac{b_2-a_2}{2}\right]\|w\|_1.
\end{aligned} \tag{15}$$

Making use of (13), (14) and (15) gives (3). □

It is clear that the bound in (12) is minimized at the mid-point of the rectangular region. Unfortunately, the weight does not influence this minimum point.

Taylor's theorem is a popular vehicle for developing cubature and higher dimension rules. Stroud [8] uses Taylor's expansion to develop cubature rules and recently Qi [5], used this technique to derive weighted Iyengar-type multiple integrals. The drawback is in the size of the error bound. For two dimensions, an n-th order rule has a Taylor remainder of $n+1$ terms. Minimizing any rule with order greater than one would be extremely difficult. Thus, in the next section, we turn to the Peano kernel and use the results of [1, 2] to derive a second order weighted double integral inequality that contains only one term in the upper bound.

3. Main Results

Lemma 7. *Let $f : [a_1,b_1]\times[a_2,b_2] \to \mathbb{R}$ be bounded and integrable and whose first partial derivatives exist and are also bounded and integrable. Furthermore, let $w : (a_1,b_1)\times(a_2,b_2) \to (0,\infty)$ be integrable. The following* *identity holds*

$$I=\int_{a_1}^{b_1}\int_{a_2}^{b_2}\left[f\left(x_1,x_2\right)-f\left(x_1,t_2\right)-f\left(t_1,x_2\right)+f\left(t_1,t_2\right)\right]w\left(t_1,t_2\right)\,dt_2dt_1 = \int_{a_1}^{b_1}\int_{a_2}^{b_2}P(t_1,t_2)\frac{\partial^2 f}{\partial t_1\partial t_2}\,dt_2dt_1 \tag{16}$$

where $x_1\in[a_1,b_1]$, $x_2\in[a_2,b_2]$ *and*

$$P(t_1,t_2)=\begin{cases}\displaystyle\int_{a_2}^{t_2}p\left(t_1,u_2\right)du_2, & a_2\le t_2\le x_2,\\ \displaystyle\int_{b_2}^{t_2}p\left(t_1,u_2\right)du_2, & x_2<t_2\le b_2,\end{cases} \tag{17}$$

$$p(t_1,t_2)=\begin{cases}\displaystyle\int_{a_1}^{t_1}w\left(u_1,t_2\right)du_1, & a_1\le t_1\le x_1,\\ \displaystyle\int_{b_1}^{t_1}w\left(u_1,t_2\right)du_1, & x_1<t_1\le b_1.\end{cases} \tag{18}$$

Proof. To begin, let $I=\int_{a_1}^{b_1}I_2\,dt_1$ and consider I_2 where

$$\begin{aligned}I_2&=\int_{a_2}^{b_2}P(t_1,t_2)\frac{\partial^2 f\left(t_1,t_2\right)}{\partial t_1\partial t_2}\,dt_2\\&=\int_{a_2}^{x_2}P(t_1,t_2)\frac{\partial^2 f\left(t_1,t_2\right)}{\partial t_1\partial t_2}\,dt_2+\int_{x_2}^{b_2}P(t_1,t_2)\frac{\partial^2 f\left(t_1,t_2\right)}{\partial t_1\partial t_2}\,dt_2\\&=\int_{a_2}^{x_2}\left(\int_{a_2}^{t_2}p(t_1,u_2)du_2\right)\frac{\partial^2 f\left(t_1,t_2\right)}{\partial t_1\partial t_2}\,dt_2+\int_{x_2}^{b_2}\left(\int_{b_2}^{t_2}p(t_1,u_2)du_2\right)\frac{\partial^2 f\left(t_1,t_2\right)}{\partial t_1\partial t_2}\,dt_2\\&=I_{21}+I_{22}.\end{aligned}$$

Using integration by parts, we find that

$$\begin{aligned}I_{21}&=\int_{a_2}^{t_2}p(t_1,u_2)du_2\;\frac{\partial f\left(t_1,t_2\right)}{\partial t_1}\bigg|_{a_2}^{x_2}-\int_{a_2}^{x_2}\frac{\partial f\left(t_1,t_2\right)}{\partial t_1}p(t_1,t_2)\,dt_2\\&=\int_{a_2}^{x_2}p(t_1,u_2)du_2\frac{\partial f\left(t_1,x_2\right)}{\partial t_1}-\int_{a_2}^{x_2}\frac{\partial f\left(t_1,t_2\right)}{\partial t_1}p(t_1,t_2)\,dt_2\\&=\int_{a_2}^{x_2}p(t_1,t_2)\left(\frac{\partial f\left(t_1,x_2\right)}{\partial t_1}-\frac{\partial f\left(t_1,t_2\right)}{\partial t_1}\right)\,dt_2.\end{aligned}$$

Similarly

$$I_{22}=\int_{x_2}^{b_2}p(t_1,t_2)\left(\frac{\partial f\left(t_1,x_2\right)}{\partial t_1}-\frac{\partial f\left(t_1,t_2\right)}{\partial t_1}\right)\,dt_2.$$

Thus I_2 becomes

$$I_2=\int_{a_2}^{b_2}p(t_1,t_2)\left(\frac{\partial f\left(t_1,x_2\right)}{\partial t_1}-\frac{\partial f\left(t_1,t_2\right)}{\partial t_1}\right)\,dt_2$$

and substituting into I gives

$$\begin{aligned}I=\int_{a_1}^{b_1}\int_{a_2}^{b_2}P(t_1,t_2)\frac{\partial^2 f}{\partial t_1\partial t_2}\,dt_2dt_1&=\int_{a_1}^{b_1}\int_{a_2}^{b_2}p(t_1,t_2)\left(\frac{\partial f\left(t_1,x_2\right)}{\partial t_1}-\frac{\partial f\left(t_1,t_2\right)}{\partial t_1}\right)\,dt_2dt_1\\&=\int_{a_2}^{b_2}\int_{a_1}^{b_1}p(t_1,t_2)\left(\frac{\partial f\left(t_1,x_2\right)}{\partial t_1}-\frac{\partial f\left(t_1,t_2\right)}{\partial t_1}\right)\,dt_1dt_2\\&=\int_{a_2}^{b_2}I_3\,dt_2, \qquad (19)\end{aligned}$$

where

$$I_3=\int_{a_1}^{b_1}p(t_1,t_2)\left(\frac{\partial f\left(t_1,x_2\right)}{\partial t_1}-\frac{\partial f\left(t_1,t_2\right)}{\partial t_1}\right)\,dt_1.$$

Applying the same treatment to I_3 as for I_2 gives

$$I_3=\int_{a_1}^{b_1}w(t_1,t_2)[f(x_1,x_2)-f(t_1,x_2)-f(x_1,t_2)+f(t_1,t_2)]\,dt_1.$$

Substituting I_3 into (19) we find that the identity (16) is thus proved. $\square$

The upper bound of the integration rule will depend on P. Below, we detail some properties of P that will be subsequently used in analysis of the bound.

Lemma 8. *The kernel $P : [a_1, b_1] \times [a_2, b_2] \to \mathbb{R}$ as defined in Lemma 7 has the following properties:*

(1) *P vanishes on the boundary of the rectangle $[a_1, b_1] \times [a_2, b_2]$,*
(2) *$P(t_1, \cdot) : (a_2, b_2) \to \mathbb{R}$ is monotonic increasing for all $t_1 \in (a_1, x_1)$,*
(3) *$P(t_1, \cdot) : (a_2, b_2) \to \mathbb{R}$ is monotonic decreasing for all $t_1 \in (x_1, b_1)$,*
(4) *P is positive on $(a_1, x_1) \times (a_2, x_2)$ and $(x_1, b_1) \times (x_2, b_2)$,*
(5) *P is negative on $(a_1, x_1) \times (x_2, b_2)$ and $(x_1, b_1) \times (a_2, x_2)$,*

for all $(x_1, x_2) \in (a_1, b_1) \times (a_2, b_2)$.

Proof. These properties are quite simple to prove via inspection of the first partial derivatives of P. $\square$

In Figure 1, we plot the surface and contours of (17) for two different weights. The plots exhibit the properties discussed in Lemma 8. It is obvious that the kernel achieves its maximum deviation on of its branches at the discontinuous point (x_1, x_2).

In the following theorem we state the main result by employing the identity in Lemma 7 to produce second order weighted double integral inequalities. In contrast with the inequalities of the previous section, the upper bound here is comprised of just one term.

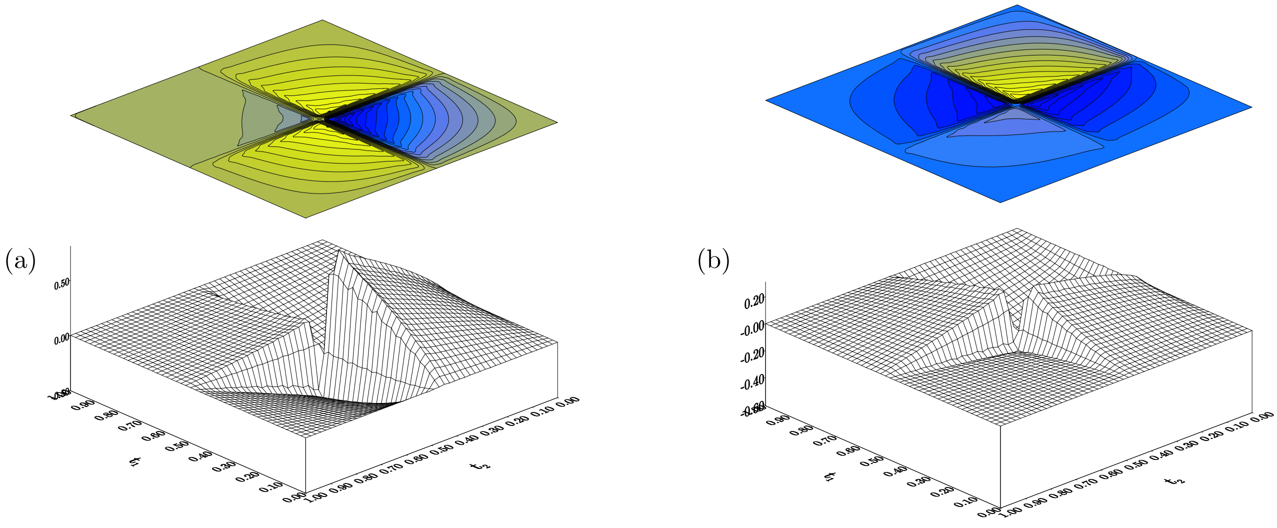

FIGURE 1. Surface and contour plots of the Peano type kernels P defined in (17) for different weights. (a) $w(t_1, t_2) = -\ln(t_1 t_2)$ over the unit square and $x_1 = x_2 = 0.5$, (b) $w(t_1, t_2) = \sqrt{t_1/t_2}$ over the unit square and $x_1 = x_2 = 0.5$.

Theorem 9. *Let the conditions of Lemma 7 hold. The following double integral inequalities involving the usual Lebesgue norms of the first mixed partial derivative of f hold,*

$$|I| \le \left\| \frac{\partial^2 f}{\partial t_1 \partial t_2} \right\|_\infty \int_{a_1}^{b_1} \int_{a_2}^{b_2} |x_1 - t_1| \, |x_2 - t_2| \, w(t_1, t_2) \, dt_1 dt_2, \tag{20}$$

if $\frac{\partial^2 f}{\partial t_1 \partial t_2} \in L_\infty[a_1, b_1] \times [a_2, b_2]$ and

$$\begin{aligned} |I| \le \left\| \frac{\partial^2 f}{\partial t_1 \partial t_2} \right\|_1 \max\Bigg\{ & \int_{a_1}^{x_1} \int_{a_2}^{x_2} w(t_1, t_2) \, dt_2 dt_1, \int_{a_1}^{x_1} \int_{x_2}^{b_2} w(t_1, t_2) \, dt_2 dt_1, \\ & \int_{x_1}^{b_1} \int_{a_2}^{x_2} w(t_1, t_2) \, dt_2 dt_1, \int_{x_1}^{b_1} \int_{x_2}^{b_2} w(t_1, t_2) \, dt_2 dt_1 \Bigg\} \end{aligned} \tag{21}$$

if $\frac{\partial^2 f}{\partial t_1 \partial t_2} \in L_1[a_1, b_1] \times [a_2, b_2]$, where I is defined in equation (16).

Proof. To prove (20) we begin with Hölder's inequality and then simplify using Lemma 8

$$
\begin{aligned}
|I| &= \left| \int_{a_1}^{b_1} \int_{a_2}^{b_2} P(t_1,t_2) \frac{\partial^2 f(t_1,t_2)}{\partial t_1 \partial t_2} dt_1 dt_2 \right| \\
&\le \left\| \frac{\partial^2 f}{\partial t_1 \partial t_2} \right\|_\infty \int_{a_1}^{b_1} \int_{a_2}^{b_2} |P(t_1,t_2)|\, dt_2 dt_1 \\
&= \left\| \frac{\partial^2 f}{\partial t_1 \partial t_2} \right\|_\infty \Big(\int_{a_1}^{x_1} \int_{a_2}^{x_2} P(t_1,t_2)\, dt_2 dt_1 - \int_{a_1}^{x_1} \int_{x_2}^{b_2} P(t_1,t_2)\, dt_2 dt_1 \\
&\qquad - \int_{x_1}^{b_1} \int_{a_2}^{x_2} P(t_1,t_2)\, dt_2 dt_1 + \int_{x_1}^{b_1} \int_{x_2}^{b_2} P(t_1,t_2)\, dt_2 dt_1 \Big). \qquad (22)
\end{aligned}
$$

Now each of the terms in (22) can be evaluated via partial integration and simplified using Lemma 7 and equations (17) and (18). For the first term

$$
\begin{aligned}
\int_{a_1}^{x_1} \int_{a_2}^{x_2} P(t_1,t_2)\, dt_2 dt_1 &= \int_{a_1}^{x_1} \left\{ (t_2 - x_2) P \Big|_{a_2}^{x_2} - \int_{a_2}^{x_2} (t_2 - x_2) p \, dt_2 \right\} dt_1 \\
&= - \int_{a_2}^{x_2} \int_{a_1}^{x_1} (t_2 - x_2) p \, dt_1 dt_2 \\
&= - \int_{a_2}^{x_2} (t_2 - x_2) \left\{ (t_1 - x_1) p \Big|_{a_1}^{x_1} - \int_{a_1}^{x_1} (t_1 - x_1) w \, dt_1 \right\} dt_2 \\
&= \int_{a_2}^{x_2} \int_{a_1}^{x_1} (x_2 - t_2)(x_1 - t_1) w(t_1,t_2)\, dt_1 dt_2. \qquad (23)
\end{aligned}
$$

Employing the same procedure for the other terms we find

$$
\int_{a_1}^{x_1} \int_{x_2}^{b_2} P(t_1,t_2)\, dt_2 dt_1 = \int_{a_1}^{x_1} \int_{x_2}^{b_2} (x_2 - t_2)(x_1 - t_1) w(t_1,t_2)\, dt_1 dt_2, \qquad (24)
$$

$$
\int_{x_1}^{b_1} \int_{a_2}^{x_2} P(t_1,t_2)\, dt_2 dt_1 = \int_{x_1}^{b_1} \int_{a_2}^{x_2} (x_2 - t_2)(x_1 - t_1) w(t_1,t_2)\, dt_1 dt_2, \qquad (25)
$$

$$
\int_{x_1}^{b_1} \int_{x_2}^{b_2} P(t_1,t_2)\, dt_2 dt_1 = \int_{x_1}^{b_1} \int_{x_2}^{b_2} (x_2 - t_2)(x_1 - t_1) w(t_1,t_2)\, dt_1 dt_2. \qquad (26)
$$

Substituting (23)–(26) into (22) gives (20). To prove (21) we again begin with Hölder's inequality

$$
\begin{aligned}
|I| &= \left| \int_{a_1}^{b_1} \int_{a_2}^{b_2} P(t_1,t_2) \frac{\partial^2 f(t_1,t_2)}{\partial t_1 \partial t_2} dt_1 dt_2 \right| \\
&\le \left\| \frac{\partial^2 f}{\partial t_1 \partial t_2} \right\|_1 \sup_{(t_1,t_2) \in [a_1,b_1] \times [a_2,b_2]} |P(t_1,t_2)| \\
&= \left\| \frac{\partial^2 f}{\partial t_1 \partial t_2} \right\|_1 \max \Big\{ \int_{a_1}^{x_1} \int_{a_2}^{x_2} w(t_1,t_2)\, dt_2 dt_1, \int_{a_1}^{x_1} \int_{x_2}^{b_2} w(t_1,t_2)\, dt_2 dt_1, \\
&\qquad \int_{x_1}^{b_1} \int_{a_2}^{x_2} w(t_1,t_2)\, dt_2 dt_1, \int_{x_1}^{b_1} \int_{x_2}^{b_2} w(t_1,t_2)\, dt_2 dt_1 \Big\}. \qquad (27)
\end{aligned}
$$

The last line being computed by appealing to the properties of P as listed in Lemma 8. Thus the theorem is proved. □

If the first moments of the weight w are known, as well as the *one dimensional* integrals

$$
\int_{a_1}^{b_1} f(t_1, x_2) \left(\int_{a_2}^{b_2} w(t_1,t_2)\, dt_2 \right) dt_1 \quad \text{and} \quad \int_{a_2}^{b_2} f(x_1, t_2) \left(\int_{a_1}^{b_1} w(t_1,t_2)\, dt_1 \right) dt_2 \qquad (28)
$$

then (20) can form the basis of a cubature formula for the evaluation of the weighted double integral $\iint_D f(t_1,t_2) w(t_1,t_2)\, dA$ over a rectangular region D. A major drawback is that in most cases the integrals (28) are unknown. These can be eliminated using the one-dimensional weighted results in [6]. Roumeliotis *et al.* [6] showed that for mappings f with bounded second derivative that

$$
\left| \int_a^b w(t) f(t)\, dt - f(x) \int_a^b w(t)\, dt + f'(x) \int_a^b (x-t) w(t)\, dt \right| \le \frac{\|f''\|_\infty}{2} \int_a^b (x-t)^2 w(t)\, dt, \qquad (29)
$$

where $x \in (a,b)$ and w is a weight function. Thus making use of (29), the following inequalities hold

$$\begin{aligned}(30)\quad &\left|\int_{a_1}^{b_1} f(t_1,x_2)\left(\int_{a_2}^{b_2} w(t_1,t_2)\,dt_2\right)dt_1 - f(x_1,x_2)\int_{a_1}^{b_1}\int_{a_2}^{b_2} w(t_1,t_2)\,dt_2\,dt_1\right.\\&\left.+\frac{\partial f}{\partial t_1}(x_1,x_2)\int_{a_1}^{b_1}\int_{a_2}^{b_2}(x_1-t_1)w(t_1,t_2)\,dt_2\,dt_1\right| \le \left\|\frac{\partial^2 f}{\partial t_1^2}\right\|_\infty \int_{a_1}^{b_1}\int_{a_2}^{b_2} w(t_1,t_2)\frac{(x_1-t_1)^2}{2}\,dt_2dt_1\end{aligned}$$

and

$$\begin{aligned}(31)\quad &\left|\int_{a_2}^{b_2} f(x_1,t_2)\left(\int_{a_1}^{b_1} w(t_1,t_2)\,dt_1\right)dt_2 - f(x_1,x_2)\int_{a_1}^{b_1}\int_{a_2}^{b_2} w(t_1,t_2)\,dt_2\,dt_1\right.\\&\left.+\frac{\partial f}{\partial t_2}(x_1,x_2)\int_{a_1}^{b_1}\int_{a_2}^{b_2}(x_2-t_2)w(t_1,t_2)\,dt_2\,dt_1\right| \le \left\|\frac{\partial^2 f}{\partial t_2^2}\right\|_\infty \int_{a_1}^{b_1}\int_{a_2}^{b_2} w(t_1,t_2)\frac{(x_2-t_2)^2}{2}\,dt_2dt_1.\end{aligned}$$

It is of interest to note that combining (20), (30) and (31) will produce (3). Thus, in one sense, (20) is more general than (3) since it is not obvious how one may derive (20) from (3).

One advantage of (20) over (3) is that the upper bound involves one term instead of three. Thus, with (20) we can find points (x_1,x_2) that will minimize upper bound in terms of the weight and independent of the integrand. In the following corollary we will identify points (x_1,x_2) to minimize the bound

$$(32)\qquad \mathcal{J}(x_1,x_2) = \int_{a_1}^{b_1}\int_{a_2}^{b_2} |x_1-t_1|\,|x_2-t_2|\,w\,(t_1,t_2)\,dt_2dt_1.$$

Corollary 10. *$\mathcal{J}(x_1,x_2)$ as defined in (32) is minimized at (x_1^*,x_2^*) where x_1^* and x_2^* satisfy the equations*

$$(33)\qquad \int_{a_1}^{x_1^*}\int_{a_2}^{b_2} |x_2^*-t_2|w(t_1,t_2)\;dt_2dt_1 = \int_{x_1^*}^{b_1}\int_{a_2}^{b_2} |x_2^*-t_2|w(t_1,t_2)\;dt_2dt_1$$

and

$$(34)\qquad \int_{a_2}^{x_2^*}\int_{a_1}^{b_1} |x_1^*-t_1|w(t_1,t_2)\;dt_2dt_1 = \int_{x_2^*}^{b_2}\int_{a_1}^{b_1} |x_1^*-t_1|w(t_1,t_2)\;dt_2dt_1.$$

Proof. Evaluating the partial derivatives of $\mathcal{J}$ gives

$$(35)\qquad \mathcal{J}^{(1)} = \frac{\partial \mathcal{J}}{\partial x_1}(x_1,x_2) = \int_{a_1}^{x_1}\int_{a_2}^{b_2}|x_2-t_2|w(t_1,t_2)\,dt_2dt_1 - \int_{x_1}^{b_1}\int_{a_2}^{b_2}|x_2-t_2|w(t_1,t_2)\,dt_2dt_1,$$

$$(36)\qquad \mathcal{J}^{(2)} = \frac{\partial \mathcal{J}}{\partial x_2}(x_1,x_2) = \int_{a_2}^{x_2}\int_{a_1}^{b_1}|x_1-t_1|w(t_1,t_2)\,dt_1dt_2 - \int_{x_2}^{b_2}\int_{a_1}^{b_1}|x_1-t_1|w(t_1,t_2)\,dt_1dt_2.$$

Inspection of (35) reveals that, for fixed x_2, $\mathcal{J}^{(1)}$ is monotonic increasing and $\mathcal{J}^{(1)}(a_1,x_2) = -\mathcal{J}^{(2)}(b_1,x_2) \le 0$. $\mathcal{J}^{(2)}$ also exhibits similar properties and hence there exists a unique point (x_1^*,x_2^*) that is the zero of (35) and (36) and minimizes $\mathcal{J}$. □

The behaviour of (32) is very dependant on the behaviour of the weight. In Figure 2 contours of $\mathcal{J}$ are plotted for different weight functions. In each case, the minimum point is readily observed and its location depends on the weight and weight null-space.

In the following section, properties of the minimum point of $\mathcal{J}$ are identified for various conditions on w.

4. Minimizing the bound

Solution of equations (33) and (34) provide the point that minimizes the bound (32). The equations are non-linear and two dimensional, thus, in most cases, require numerical treatment. In this section we identify solutions or simplifications to (33) and (34) for specific weight types. Some of these weights are of importance since they appear in the important areas of integral transforms and integral equations.

With functions of two or more variables it is common that an identifiable relationship between the variables is observed. That is, $w(t_1,t_2) = w(\phi(t_1,t_2))$ for some ϕ. For singular weights, the null-space of ϕ, $\{(t_1,t_2) : \phi(t_1,t_2) = 0\}$, may be of interest since this may furnish the singularity structure of the integral. Below, we explore the properties of $\mathcal{J}$ for ϕ being the difference mapping on a square and generalise to more general null-spaces in other corollaries.

Corollary 11 (Difference weight)**.** *Let $w : (a,b) \to (0,\infty)$ be integrable and let $a < x_1, x_2 < b$. Then the bound*

$$\mathcal{J}(x_1,x_2) = \int_a^b\int_a^b |x_1-t_1|\,|x_2-t_2|\,w\,|t_1-t_2|\,dt_2dt_1.$$

is minimized at the midpoint $x_1 = x_2 = \frac{a+b}{2}$.

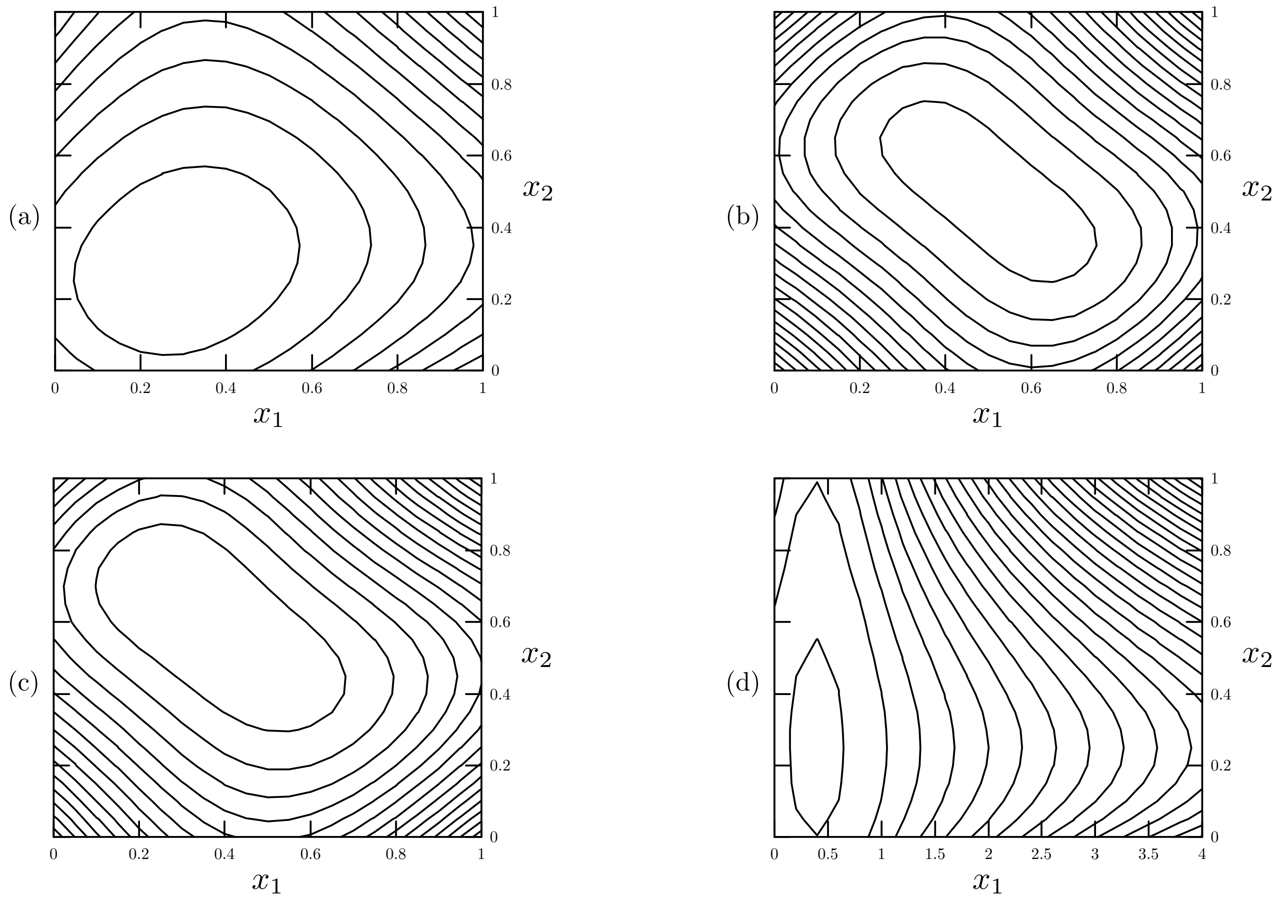

FIGURE 2. Contour plots of the $\mathcal{J}(x_1, x_2)$ given by (32) for various weight functions. (a) $w(t_1, t_2) = -\ln(t_1 t_2)$, $(t_1, t_2) \in (0, 1) \times (0, 1)$, (b) $w(t_1, t_2) = -\ln|t_1 - t_2|$, $(t_1, t_2) \in (0, 1) \times (0, 1)$, (c) $w(t_1, t_2) = -\ln|t_1 - t_2^2|$, $(t_1, t_2) \in (0, 1) \times (0, 1)$ and (d) $w(t_1, t_2) = e^{-t_1}/\sqrt{t_2}$, $(t_1, t_2) \in (0, 4) \times (0, 1)$.

Proof. As stated in Corollary 10, $\mathcal{J}$ is minimized at the root of equations (33) and (34). Substituting the midpoint in (33) gives

$$\begin{aligned}
&\int_a^{(a+b)/2} \int_a^b \left|\frac{a+b}{2} - t_2\right| w|t_1 - t_2|\, dt_2 dt_1 - \int_{(a+b)/2}^b \int_a^b \left|\frac{a+b}{2} - t_2\right| w|t_1 - t_2|\, dt_2 dt_1 \\
&= \int_{(a+b)/2}^b \int_a^b \left|\frac{a+b}{2} - v\right| w|u - v|\, dv du - \int_{(a+b)/2}^b \int_a^b \left|\frac{a+b}{2} - t_2\right| w|t_1 - t_2|\, dt_2 dt_1 \\
&= 0,
\end{aligned}$$

where $u = a + b - t_1$ and $v = a + b - t_2$ are integral substitutions. The same treatment on (34) shows that the midpoint minimizes the bound □

The following two corollaries show that the simultaneous equations (33) and (34) may be decoupled under certain conditions for the weight.

Corollary 12 (Separable weight)**.** *Let the conditions in Corollary 10 hold. Furthermore, let w be separable, that is $w(t_1, t_2) = w_1(t_1)w_2(t_2)$, where w_i are themselves weight functions defined on $[a_i, b_i]$, i=1,2. Then $\mathcal{J}$ is minimized at the median of each weight*

$$\int_{a_i}^{x_i} w_i(t_i)\, dt_i = \int_{x_i}^{b_i} w_i(t_i)\, dt_i, \qquad i = 1, 2.$$

Proof. Substituting $w(t_1, t_2) = w_1(t_1)w_2(t_2)$ into (33) and (34) and simplifying produces the result. □

Corollary 13 (Symmetric weight)**.** *Let the conditions in Corollary 10 hold and let $w : (a, b) \times (a, b) \to \mathbb{R}$ be symmetric, that is, $w(t_1, t_2) = w(t_2, t_1)$. Then the minimum point is at $x_1 = x_2$.*

Proof. With the above conditions, the two equations in Corollary 10 are

$$\frac{\partial \mathcal{J}}{\partial x_1}(x_1,x_2) = \int_a^{x_1}\int_a^b |x_2-t_2| w(t_1,t_2)\,dt_2dt_1 - \int_{x_1}^b\int_a^b |x_2-t_2| w(t_1,t_2)\,dt_2dt_1, \tag{37}$$

$$\frac{\partial \mathcal{J}}{\partial x_2}(x_1,x_2) = \int_a^{x_2}\int_a^b |x_1-t_1| w(t_1,t_2)\,dt_1dt_2 - \int_{x_2}^b\int_a^b |x_1-t_1| w(t_1,t_2)\,dt_1dt_2. \tag{38}$$

Beginning with (37) we have

$$\begin{aligned}\frac{\partial \mathcal{J}}{\partial x_1}(x_1,x_2) &= \int_a^{x_1}\int_a^b |x_2-t_2| w(t_1,t_2)\,dt_2dt_1 - \int_{x_1}^b\int_a^b |x_2-t_2| w(t_1,t_2)\,dt_2dt_1\\ &= \int_a^{x_1}\int_a^b |x_2-t_1| w(t_2,t_1)\,dt_1dt_2 - \int_{x_1}^b\int_a^b |x_2-t_1| w(t_2,t_1)\,dt_1dt_2\\ &= \int_a^{x_1}\int_a^b |x_2-t_1| w(t_1,t_2)\,dt_1dt_2 - \int_{x_1}^b\int_a^b |x_2-t_1| w(t_1,t_2)\,dt_1dt_2\\ &= \frac{\partial \mathcal{J}}{\partial x_2}(x_2,x_1).\end{aligned}$$

Thus the solution of

$$\frac{\partial \mathcal{J}}{\partial x_1}(x_1,x_2)=0 \qquad \text{and} \qquad \frac{\partial \mathcal{J}}{\partial x_2}(x_1,x_2)=0,$$

is identical to

$$\frac{\partial \mathcal{J}}{\partial x_2}(x_2,x_1)=0 \qquad \text{and} \qquad \frac{\partial \mathcal{J}}{\partial x_2}(x_1,x_2)=0,$$

and hence the solution occurs at $x_1 = x_2$. □

In Corollary 11 we showed that if a weight has a "difference" null-space on a square then the bound (32) is minimized at the centre of the square. The following corollary will generalise this result and we will consider a null space of the form $t_1 = \phi(t_2)$ where ϕ is anti-symmetric on a rectangle.

Corollary 14. *Let $w : (-a,a)\times(-A,A) \to (0,\infty)$ be a weight function of the form $w(t_1,t_2) = w|t_1-\phi(t_2)|$, where $\phi : (-A,A) \to (-a,a)$ is surjective and odd, for some $a, A > 0$, that is $\phi(-t) = -\phi(t)$. Then $\mathcal{J}$ as defined in (32) is minimized at the origin.*

Proof. We need to show that

$$\int_{-a}^0\int_{-A}^A |t_2| w|t_1-\phi(t_2)|\,dt_2dt_1 = \int_0^a\int_{-A}^A |t_2| w|t_1-\phi(t_2)|\,dt_2dt_1 \tag{39}$$

and

$$\int_{-A}^0\int_{-a}^a |t_1| w|t_1-\phi(t_2)|\,dt_1dt_2 = \int_0^A\int_{-a}^a |t_1| w|t_1-\phi(t_2)|\,dt_1dt_2. \tag{40}$$

Making the substitution $t_1 = -u$ and $t_2 = -v$ in the first integral of (39) we have

$$\int_{-a}^0\int_{-A}^A |t_2| w|t_1-\phi(t_2)|\,dt_2dt_1 = \int_0^a\int_{-A}^A |v| w|u-\phi(v)|\,dvdu.$$

Similarly

$$\int_{-A}^0\int_{-a}^a |t_1| w|t_1-\phi(t_2)|\,dt_1dt_2 = \int_0^A\int_{-a}^a |u| w|u-\phi(v)|\,dudv.$$

Hence, the corollary is proved. □

5. Cubature and grid generation

Theorem 9 can form the basis of a cubature formula for weighted double integrals. That is, we can form a mesh and apply equation (20) to each grid rectangle. The minimum point of each rectangle would be given by (33) and (34). The question that would remain is how would such a grid be "optimally" constructed? For example, for four grid rectangles, as shown in Figure 3, how would ξ_1 and ξ_2 be chosen?

Let us consider a partition $a_i \le \xi_i \le b_i$ of the interval $[a_i, b_i]$, with $x_{i,1} \in [a_i, \xi_i]$ and $x_{i,2} \in [\xi_i, b_i]$, for $i = 1, 2$. In addition, define D to be the rectangular region $[a_1, b_1] \times [a_2, b_2]$ and define the sub-regions $D_{1,1} = [a_1, \xi_1] \times [a_2, \xi_2]$, $D_{1,2} = [\xi_1, b_1] \times [a_2, \xi_2]$, $D_{2,1} = [a_1, \xi_1] \times [\xi_2, b_2]$ and $D_{2,2} = [\xi_1, b_1] \times [\xi_2, b_2]$. A sketch of this partition is shown in Figure 3.

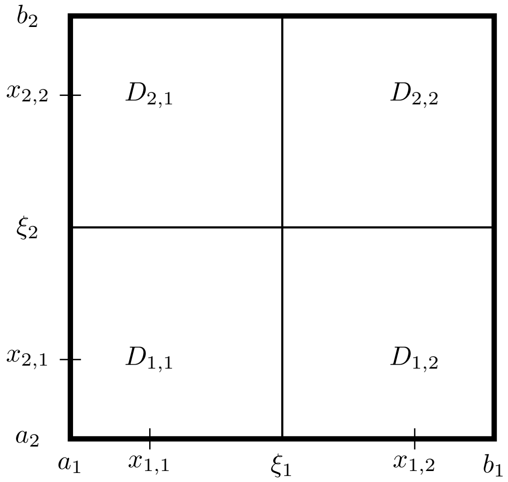

FIGURE 3. A partition of the rectangular region $D = [a_1, b_1] \times [a_2, b_2]$ showing the sub-regions $D_{i,j}$, $i, j = 1, 2$.

Theorem 15. *Let the conditions in Theorem 9 hold. Given the partition defined above, the following double integral inequality holds*

$$\begin{aligned}(41)\quad &\Bigg| \iint_D f(t_1,t_2)w(t_1,t_2)\,dt_1dt_2 - \sum_{i=1}^{2}\Bigg(\iint_{D_{1,i}+D_{2,i}} f(x_{1,i},t_2)w(t_1,t_2)\,dt_1dt_2 \\ &- \iint_{D_{i,1}+D_{i,2}} f(t_1,x_{2,i})w(t_1,t_2)\,dt_1dt_2\Bigg) + \sum_{i=1}^{2}\sum_{j=1}^{2} f(x_{1,j},x_{2,i})\iint_{D_{i,j}} w(t_1,t_2)\,dt_1dt_2\Bigg| \\ &\le \left\| \frac{\partial^2 f}{\partial t_1 \partial t_2}\right\|_\infty \sum_{i=1}^{2}\sum_{j=1}^{2}\iint_{D_{i,j}} |x_{1,i}-t_1||x_{2,j}-t_2|w(t_1,t_2)\,dt_1dt_2.\end{aligned}$$

The bound is minimized at the points $x_{i,j}, \xi_i$, $(i, j = 1, 2)$ satisfying

$$\begin{aligned}(42)\quad &\int_{a_1}^{x_{1,1}}\int_{a_2}^{\xi_2} |x_{2,1}-t_2|w(t_1,t_2)\,dt_2dt_1 + \int_{a_1}^{x_{1,1}}\int_{\xi_2}^{b_2} |x_{2,2}-t_2|w(t_1,t_2)\,dt_2dt_1 \\ &= \int_{x_{1,1}}^{\xi_1}\int_{a_2}^{\xi_2} |x_{2,1}-t_2|w(t_1,t_2)\,dt_2dt_1 + \int_{x_{1,1}}^{\xi_1}\int_{\xi_2}^{b_2} |x_{2,2}-t_2|w(t_1,t_2)\,dt_2dt_1,\end{aligned}$$

$$\begin{aligned}(43)\quad &\int_{\xi_1}^{x_{1,2}}\int_{a_2}^{\xi_2} |x_{2,1}-t_2|w(t_1,t_2)\,dt_2dt_1 + \int_{\xi_1}^{x_{1,2}}\int_{\xi_2}^{b_2} |x_{2,2}-t_2|w(t_1,t_2)\,dt_2dt_1 \\ &= \int_{x_{1,2}}^{b_1}\int_{a_2}^{\xi_2} |x_{2,1}-t_2|w(t_1,t_2)\,dt_2dt_1 + \int_{x_{1,2}}^{b_1}\int_{\xi_2}^{b_2} |x_{2,2}-t_2|w(t_1,t_2)\,dt_2dt_1,\end{aligned}$$

$$\begin{aligned}(44)\quad &\int_{a_2}^{x_{2,1}}\int_{a_1}^{\xi_1} |x_{1,1}-t_1|w(t_1,t_2)\,dt_1dt_2 + \int_{a_2}^{x_{2,1}}\int_{\xi_1}^{b_1} |x_{1,2}-t_1|w(t_1,t_2)\,dt_1dt_2 \\ &= \int_{x_{2,1}}^{\xi_2}\int_{a_1}^{\xi_1} |x_{1,1}-t_1|w(t_1,t_2)\,dt_1dt_2 + \int_{x_{2,1}}^{\xi_2}\int_{\xi_1}^{b_1} |x_{1,2}-t_1|w(t_1,t_2)\,dt_1dt_2,\end{aligned}$$

$$\begin{aligned}(45)\quad &\int_{\xi_2}^{x_{2,2}}\int_{a_1}^{\xi_1} |x_{1,1}-t_1|w(t_1,t_2)\,dt_1dt_2 + \int_{\xi_2}^{x_{2,2}}\int_{\xi_1}^{b_1} |x_{1,2}-t_1|w(t_1,t_2)\,dt_1dt_2 \\ &= \int_{x_{2,2}}^{b_2}\int_{a_1}^{\xi_1} |x_{1,1}-t_1|w(t_1,t_2)\,dt_1dt_2 + \int_{x_{2,2}}^{b_2}\int_{\xi_1}^{b_1} |x_{1,2}-t_1|w(t_1,t_2)\,dt_1dt_2,\end{aligned}$$

$$(46)\quad \xi_1 = \frac{x_{1,1}+x_{1,2}}{2} \quad \text{and} \quad \xi_2 = \frac{x_{2,1}+x_{2,2}}{2}.$$

Proof. To obtain (41), it is a simple matter of applying equation (20) of Theorem 9 to each region $D_{i,j}$ $(i, j = 1, 2)$, summing and finally employing the triangle inequality.

To show equations (42)–(46), we calculate the stationary point of the bound

$$\mathcal{J} = \sum_{i=1}^{2}\sum_{j=1}^{2}\iint_{D_{i,j}} |x_{1,i}-t_1||x_{2,j}-t_2|w(t_1,t_2)\,dt_1dt_2. \tag{47}$$

For $x_{1,1}$,

$$\begin{aligned}\frac{\partial \mathcal{J}}{\partial x_{1,1}} &= \frac{\partial}{\partial x_{1,1}}\Big\{\sum_{j=1}^{2}\iint_{D_{1,j}} |x_{1,1}-t_1||x_{2,j}-t_2|w(t_1,t_2)\,dt_1dt_2\Big\}\\
&= \frac{\partial}{\partial x_{1,1}}\Big\{\int_{a_1}^{x_{1,1}}\int_{a_2}^{\xi_2}(x_{1,1}-t_1)|x_{2,1}-t_2|w(t_1,t_2)\,dt_2dt_1\\
&\qquad + \int_{x_{1,1}}^{\xi_1}\int_{a_2}^{\xi_2}(t_1-x_{1,1})|x_{2,1}-t_2|w(t_1,t_2)\,dt_2dt_1\\
&\qquad + \int_{a_1}^{x_{1,1}}\int_{\xi_2}^{b_2}(x_{1,1}-t_1)|x_{2,2}-t_2|w(t_1,t_2)\,dt_2dt_1\\
&\qquad + \int_{x_{1,1}}^{\xi_1}\int_{\xi_2}^{b_2}(t_1-x_{1,1})|x_{2,2}-t_2|w(t_1,t_2)\,dt_2dt_1\Big\}\\
&= \int_{a_1}^{x_{1,1}}\int_{a_2}^{\xi_2}|x_{2,1}-t_2|w(t_1,t_2)\,dt_2dt_1 - \int_{x_{1,1}}^{\xi_1}\int_{a_2}^{\xi_2}|x_{2,1}-t_2|w(t_1,t_2)\,dt_2dt_1\\
&\quad + \int_{a_1}^{x_{1,1}}\int_{\xi_2}^{b_2}|x_{2,2}-t_2|w(t_1,t_2)\,dt_2dt_1 - \int_{x_{1,1}}^{\xi_1}\int_{\xi_2}^{b_2}|x_{2,2}-t_2|w(t_1,t_2)\,dt_2dt_1.\end{aligned}$$

Setting the last expression to zero gives (42) and the same process can be used to show equations (43)–(45).

To show (46), observe that

$$\begin{aligned}\frac{\partial \mathcal{J}}{\partial \xi_1} &= \int_{a_2}^{\xi_2}(\xi_1-x_{1,1})|x_{2,1}-t_2|w(\xi_1,t_2)\,dt_2 - \int_{a_2}^{\xi_2}(x_{1,2}-\xi_1)|x_{2,1}-t_2|w(\xi_1,t_2)\,dt_2\\
&\quad + \int_{\xi_2}^{b_2}(\xi_1-x_{1,1})|x_{2,2}-t_2|w(\xi_1,t_2)\,dt_2 - \int_{\xi_2}^{b_2}(x_{1,2}-\xi_1)|x_{2,1}-t_2|w(\xi_1,t_2)\,dt_2\\
&= 2\int_{a_2}^{\xi_2}\left(\xi_1-\frac{x_{1,1}+x_{1,2}}{2}\right)|x_{2,1}-t_2|w(\xi_1,t_2)\,dt_2\\
&\quad + 2\int_{\xi_2}^{b_2}\left(\xi_1-\frac{x_{1,1}+x_{1,2}}{2}\right)|x_{2,2}-t_2|w(\xi_1,t_2)\,dt_2,\end{aligned}$$

which obviously has a root at $(46)_1$. Similarly, we can show $(46)_2$. □

We now proceed to a full weighted cubature formulae.

Define the following partitions of the intervals $[a_i,b_i]$

$$I_i : a_i = \xi_{i,0} \le \xi_{i,1} \le \cdots \le \xi_{i,n} = b_i,$$

and let $x_{i,j}\in[\xi_{i,j-1},\xi_{i,j}]$ for $i=1,2$ and $j=1,2,\dots,n$. Furthermore, let $D_{i,j}=[\xi_{1,i-1},\xi_{1,i}]\times[\xi_{2,j-1},\xi_{2,j}]$, $D_i^{(1)}=\bigcup_{k=1}^n D_{i,k}$ and $D_i^{(2)}=\bigcup_{k=1}^n D_{k,i}$, for $i,j=1,2,\dots,n$.

Consider the weighted cubature formula

$$\begin{aligned}&A(f,w,I_1,I_2,\boldsymbol{\xi},\boldsymbol{x})\\
&\quad=\sum_{i=1}^{n}\Big(\iint_{D_i^{(1)}} f(x_{1,i},t_2)w(t_1,t_2)\,dt_1dt_2 + \iint_{D_i^{(2)}} f(t_1,x_{2,i})w(t_1,t_2)\,dt_1dt_2\Big)\\
&\qquad\qquad - \sum_{i=1}^{n}\sum_{j=1}^{n} f(x_{1,i},x_{2,j})\iint_{D_{i,j}} w(t_1,t_2)\,dt_1dt_2.\end{aligned} \tag{48}$$

Using the above assumptions, we can write the following theorem.

Theorem 16. *Let $f:[a_1,b_1]\times[a_2,b_2]\to\mathbb{R}$ and $w:(a_1,b_1)\times(a_2,b_2)\to(0,\infty)$ be as in Theorem 9 and $I_1,I_2,\boldsymbol{\xi},\boldsymbol{x}$ be given above. The following weighted cubature formula holds*

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2} f(t_1,t_2)w(t_1,t_2)\,dt_2dt_1 = A(f,w,I_1,I_2,\boldsymbol{\xi},\boldsymbol{x}) + R(f,w,I_1,I_2,\boldsymbol{\xi},\boldsymbol{x}), \tag{49}$$

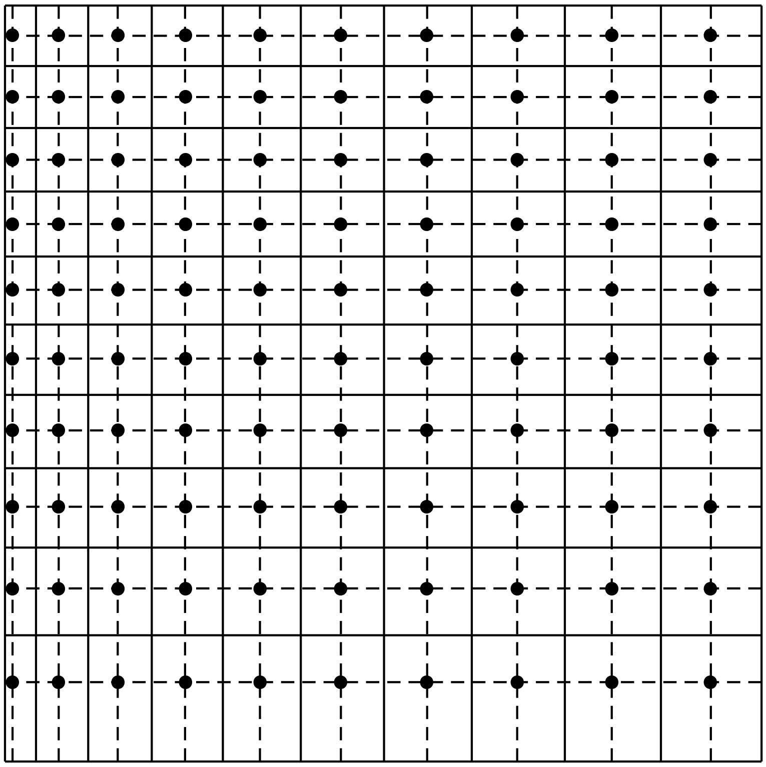

FIGURE 4. Grid generated from the solution of equations (51)– (53) for the weight $w(t_1,t_2) = \sqrt{t_2/t_1}$ over $[0,1]\times[0,1]$ and $n = 10$. The solid lines indicate the composite grid; in each grid square there is one function evaluation (dot) and two single integral evaluations (dashed lines).

where

$$|R(f,w,I_1,I_2,\boldsymbol{\xi},\boldsymbol{x})| \le \left\|\frac{\partial^2 f}{\partial t_1 \partial t_2}\right\|_\infty \sum_{i=1}^n \sum_{j=1}^n \iint_{D_{i,j}} |x_{1,i}-t_1||x_{2,j}-t_2| w(t_1,t_2)\, dt_1 dt_2. \tag{50}$$

The bound (50) is minimized when $\boldsymbol{x}$ and $\boldsymbol{\xi}$ satisfy

$$\sum_{j=1}^n \int_{\xi_{1,i-1}}^{x_{1,i}} \int_{\xi_{2,j-1}}^{\xi_{2,j}} |x_{2,j}-t_2| w(t_1,t_2)\, dt_2 dt_1 = \sum_{j=1}^n \int_{x_{1,i}}^{\xi_{1,i}} \int_{\xi_{2,j-1}}^{\xi_{2,j}} |x_{2,j}-t_2| w(t_1,t_2)\, dt_2 dt_1 \tag{51}$$

$$\sum_{j=1}^n \int_{\xi_{2,i-1}}^{x_{2,i}} \int_{\xi_{1,j-1}}^{\xi_{1,j}} |x_{1,j}-t_1| w(t_1,t_2)\, dt_1 dt_2 = \sum_{j=1}^n \int_{x_{2,i}}^{\xi_{2,i}} \int_{\xi_{1,j-1}}^{\xi_{1,j}} |x_{1,j}-t_1| w(t_1,t_2)\, dt_1 dt_2 \tag{52}$$

$$\xi_{k,l} = \frac{x_{k,l}+x_{k,l+1}}{2}, \qquad \textit{for } i=1,\dots,n,\ l=1,\dots,n-1,\ k=1,2. \tag{53}$$

Proof. The proof follows that of Theorem 16. □

To find the $4n-2$ unknowns

$$x_{i,1} \le \xi_{i,1} \le x_{i,2} \le \cdots \le \xi_{i,n-1} \le x_{i,n},$$

for $i = 1,2$, we need to solve the $4n-2$ coupled non-linear equations (51), (52) and (53). These equations are easily solved iteratively with a uniform grid as the starting point. With this method of solution all variables are fixed apart from the parameter of interest. Thus for example if $k = 1$ and we fix i, then equation (51) may be considered as a function of $x_{1,i}$ only; say $F(x_{1,i})$. It is easy to see that

$$F'(x_{1,i}) = 2\sum_{j=1}^n \int_{\xi_{2,j-1}}^{\xi_{2,j}} |x_{2,j}-t_2| w(x_{1,i},t_2)\, dt_2 \ge 0$$

and $F(\xi_{1,i-1}) \le 0$, $F(\xi_{1,i}) \ge 0$. Thus F has a unique root and the bisection algorithm would be an appropriate numerical technique to produce the solution.

In Figures 4, 5, 6 and 7, the grid obtained via numerical solution of (51)–(53) is plotted for various weight functions and n. We can see that the grid clustering reflects the weight behaviour.

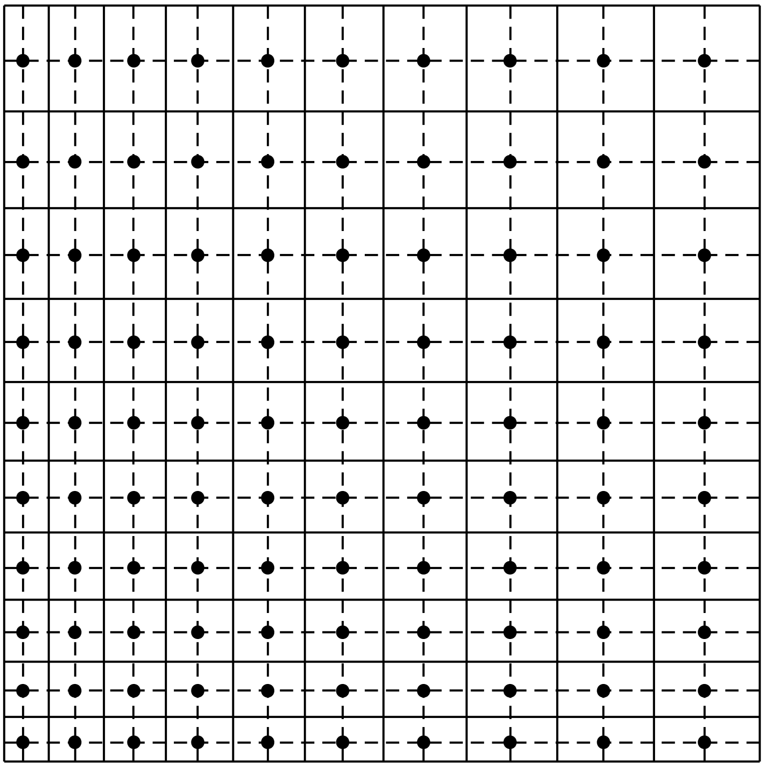

FIGURE 5. Grid generated from the solution of equations (51)– (53) for the weight $w(t_1, t_2) = -\ln(t_1 t_2)$ over $[0,1] \times [0,1]$ and $n = 10$. The solid lines indicate the composite grid; in each grid square there is one function evaluation (dot) and two single integral evaluations (dashed lines).

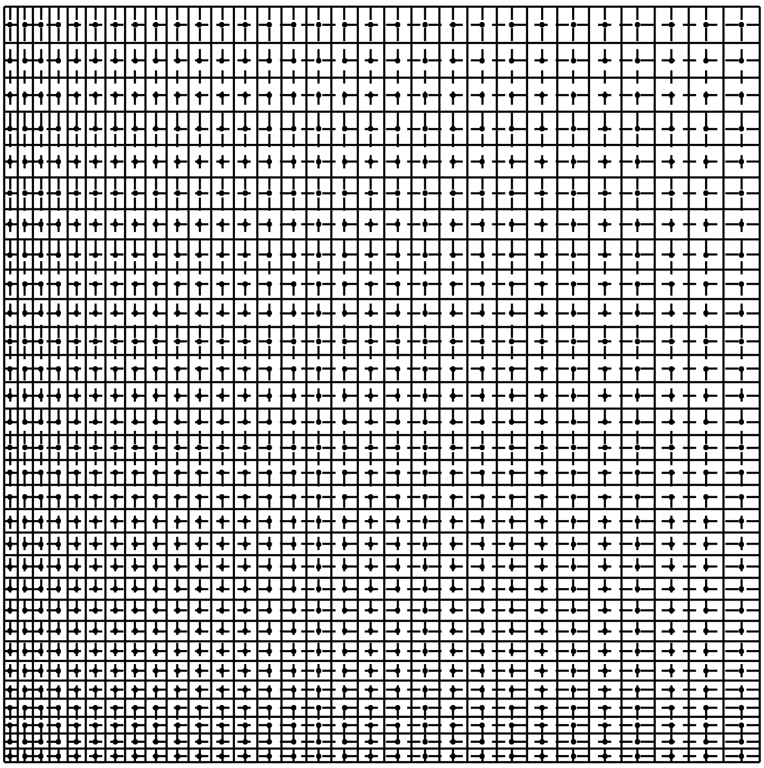

FIGURE 6. Grid generated from the solution of equations (51)– (53) for the weight $w(t_1, t_2) = -\ln(t_1 t_2)$ over $[0,1] \times [0,1]$ and $n = 30$. The solid lines indicate the composite grid; in each grid square there is one function evaluation (dot) and two single integral evaluations (dashed lines).

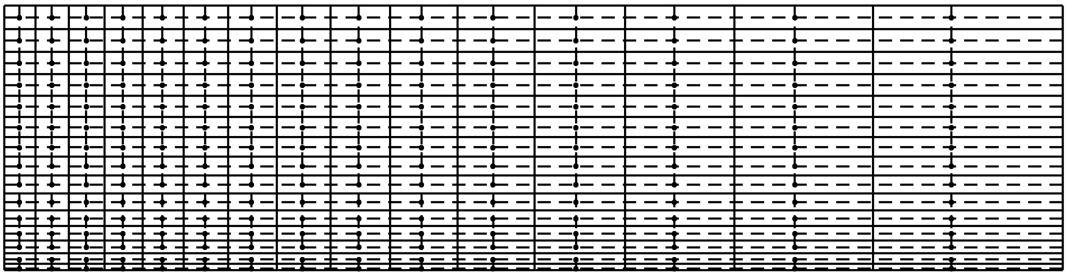

FIGURE 7. Grid generated from the solution of equations (51)– (53) for the weight $w(t_1, t_2) = e^{-t_1}/\sqrt{t_2}$ over $[0, 4] \times [0, 1]$ and $n = 15$. The solid lines indicate the composite grid; in each grid square there is one function evaluation (dot) and two single integral evaluations (dashed lines).

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