GENERALISATION OF BERNOULLI POLYNOMIALS

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Abstract. In this article, the Bernoulli polynomials are generalised and some properties of the resulting generalisations are presented.

1. Introduction

It is well-known that the Bernoulli numbers B_n can be defined [1, 2, 14] as

$$\phi(x) \triangleq \frac{x}{\mathrm{e}^x - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} \cdot x^n, \quad |x| < 2\pi. \tag{1}$$

The Bernoulli polynomials $B_n(x)$ can be defined [1, 2, 14] by

$$\phi(z, x) \triangleq \frac{z\,\mathrm{e}^{xz}}{\mathrm{e}^z - 1} = \sum_{n=0}^{\infty} \frac{B_n(x)}{n!} \cdot z^n, \quad |z| < 2\pi, \tag{2}$$

and write $B_n = B_n(0)$ for the Bernoulli numbers.

The usual definition of the generalised Bernoulli polynomials is

$$\frac{t^\sigma\,\mathrm{e}^{ut}}{(\mathrm{e}^t - 1)^\sigma} = \sum_{n=0}^{\infty} B_n^\sigma(u) \cdot \frac{t^n}{n!}, \quad |t| < 2\pi. \tag{3}$$

For more information about Bernoulli numbers and Bernoulli polynomials, please refer to [6, 15, 16].

Many approaches for calculating Bernoulli numbers are presented in [1, 2, 5, 14].

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Now we introduce a new function $B_n(a,b)$ for $b>a>0$ which is defined as

$$\phi(x;a,b)\triangleq\frac{x}{b^x-a^x}=\sum_{n=0}^\infty B_n(a,b)\cdot\frac{x^n}{n!},\quad |x|<\frac{2\pi}{\ln b-\ln a}. \tag{4}$$

In this article, we will give some relations between B_n, $B_n(x)$ and $B_n(a,b)$, and many properties of the function $B_n(a,b)$.

2. Relationships between B_n, $B_n(x)$ and $B_n(a,b)$

It is clear that

$$B_0(a,b)=\frac{1}{\ln b-\ln a}\quad\text{and}\quad B_n(1,e)=B_n. \tag{5}$$

Since

$$\begin{aligned}\frac{x}{b^x-a^x}&=\frac{1}{a^x}\cdot\frac{x}{\mathrm{e}^{x(\ln b-\ln a)}-1}\\&=\left(\sum_{n=0}^\infty\frac{(\ln b-\ln a)^{n-1}}{n!}B_nx^n\right)\left(\sum_{k=0}^\infty\frac{(\ln a)^k}{k!}(-1)^kx^k\right)\\&=\sum_{j=0}^\infty\left(\sum_{i=0}^j(-1)^{j-i}B_i\cdot\frac{(\ln b-\ln a)^{i-1}(\ln a)^{j-i}}{i!(j-i)!}\right)x_j,\end{aligned}$$

hence

$$B_j(a,b)=\sum_{i=0}^j(-1)^{j-i}(\ln b-\ln a)^{i-1}(\ln a)^{j-i}\binom{j}{i}B_i. \tag{6}$$

Further, because

$$\begin{aligned}\frac{x}{b^x-a^x}&=\frac{x\,\mathrm{e}^{-x\ln a}}{\mathrm{e}^{x(\ln b-\ln a)}-1}\\&=\frac{1}{\ln b-\ln a}\sum_{n=0}^\infty\frac{(\ln b-\ln a)^n}{n!}\cdot B_n\left(\frac{\ln a}{\ln a-\ln b}\right)\cdot x^n\\&=\sum_{n=0}^\infty\frac{(\ln b-\ln a)^{n-1}}{n!}\cdot B_n\left(\frac{\ln a}{\ln a-\ln b}\right)\cdot x^n,\end{aligned}$$

then we have

$$B_n(a,b)=(\ln b-\ln a)^{n-1}\cdot B_n\left(\frac{\ln a}{\ln a-\ln b}\right). \tag{7}$$

Moreover, since

$$\frac{x\,\mathrm{e}^{tx}}{\mathrm{e}^x-1}=\frac{x}{(\mathrm{e}^{1-t})^x-(\mathrm{e}^{-t})^x},$$

thus

$$B_n(t)=B_n(\mathrm{e}^{-t},\mathrm{e}^{1-t}). \tag{8}$$

3. Some Properties of Generalisation of Bernoulli Polynomials

For real numbers $b > a > 0$ and $x \in \mathbb{R}$, define

$$g(x) = g(x; a, b) = \begin{cases} \dfrac{b^x - a^x}{x}, & x \neq \text{[illegible]} \\ & \end{cases}$$

The Mathieu's series defined in [3] can be expressed as

$$S(r) = \frac{1}{r^2} \int_0^\infty \frac{\sin t}{g(t/r; 1, e)} \, \mathrm{d}t = \frac{1}{r} \int_0^\infty \phi(x) \sin(rt) \, \mathrm{d}t. \tag{17}$$

Recently, some new results of Mathieu's series were obtained in [7].

By mathematical induction on $n \in \mathbb{N}$, we obtain a recursion formula for derivatives of g with respect to x of g as follows

$$(n+1)g^{(n)}(x) + xg^{(n+1)}(x) = (\ln b)^{n+1} b^x - (\ln a)^{n+1} a^x. \tag{18}$$

In particular, if we put $b = e$ and $a = 1$, then

$$(n+1)g^{(n)}(x; 1, e) + xg^{(n+1)}(x; 1, e) = \mathrm{e}^x \, . \tag{19}$$

Note that the function $g(x; 1, e)$ is absolutely monotonic increasing, see [8]–[11].

Since $[g'(x; 1, e)]^2 \geqslant g(x; 1, e) \cdot g''(x; 1, e)$, by standard arguments, we deduce that $\varphi(x)$ is convex and $3\left(\varphi'(x)\right)^2 \leqslant \varphi(x)\varphi''(x)$.

Using the expression (16) of function g, many new Steffensen pairs have been established in [4, 9, 10].

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