ON THE ALGEBRAIC CHARACTER OF BLUNDON'S INEQUALITIES

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Abstract. W. J. Blundon has proved in 1965 that

$$\left|p^2 - \left(2R^2 + 10Rr - r^2\right)\right| \leq 2(R-2r)\sqrt{R(R-2r)},$$

where R, r and p are respectively the radii of the circumcircle, the incircle and the semiperimeter of a triangle. The aim of this paper is to outline the connection of these inequalities with some deep results in real algebraic geometry.

1. Introduction

Let R, r and p be respectively the radii of the circumcircle, the incircle and the semiperimeter of a triangle. W. J. Blundon [2] has proved in 1965 that

$$(B) \qquad \begin{aligned} 2R^2 + 10Rr - r^2 - 2(R-2r)\sqrt{R(R-2r)} &\leq p^2 \leq \\ &\leq 2R^2 + 10Rr - r^2 + 2(R-2r)\sqrt{R(R-2r)}. \end{aligned}$$

The equality occurs in the left-side inequality if and only if the triangle is either equilateral or isosceles, having the basis greater than the congruent sides; the equality occurs in the right-side inequality if and only if the triangle is either equilateral or isosceles, with the basis less than the congruent sides.

Following a frequently used technique to prove geometric inequalities by algebraic means (e.g., see [3]), we shall prove that W. J. Blundon's inequalities are a direct consequence of an algebraic inequality involving elementary symmetric functions:

Theorem A. *Let $x, y, z \in \mathbb{C}$ be such that $x+y+z$, $xy+yz+zx$, $xyz \in \mathbb{R}$. Then $x, y, z \in \mathbb{R}$, if and only if*

$$\begin{aligned} &(x+y+z)^2(xy+yz+zx)^2 + 18(x+y+z)(xy+yz+zx)xyz \geq \\ &\geq 4(x+y+z)^3 xyz + 4(xy+yz+zx)^3 + 27x^2y^2z^2. \end{aligned}$$

Moreover, the above inequality is strict unless $x = y = z$.

In any triangle we have:

$$\begin{aligned} a+b+c &= 2p \\ ab+bc+ca &= p^2 + r^2 + 4Rr \\ abc &= 4Rrp \end{aligned}$$

so that, for

$$x = p-a, \quad y = p-b \text{ and } z = p-c$$

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we have

$$\begin{aligned} x+y+z &= p \\ xy+yz+zx &= r(4R+r) \\ xyz &= pr^2 \end{aligned}$$

Now (after reducing the appropriate terms) we can write the inequality in Theorem A as

$$(*) \qquad p^4 - 2\left(2R^2+10Rr-r^2\right)p^2 + 64rR^3 + 48r^2R^2 + 12r^3R + r^4 \le 0,$$

i.e.,

$$\left(p^2-2R^2-10Rr+r^2\right)^2 \le 4R\left(R-2r\right)^3$$

which implies both Euler's inequality $R \ge 2r$ and W. J. Blundon's inequality.

Theorem A leads also to the inequality (*) in any of the following cases:

$$x=\frac{1}{a}\,,\ y=\frac{1}{b},\ \ z=\frac{1}{c}\,;\quad x=tg\,\frac{A}{2}\,,\ y=tg\,\frac{B}{2},\ \ z=tg\,\frac{C}{2}\,;$$
$$x=r_a\,,\ y=r_b,\ \ z=r_c\,.$$

Inequalities between elementary symmetric functions (the best known being the AM-GM inequality) are treated in details in many books such as those by G. Hardy, J. E. Littlewood and G. Polya [4] and A. W. Marshall and I. Olkin [6]. However we have to mention that the present paper is based on an idea initiated by Newton [7] and his disciple Maclaurin [5] (and accomplished later by J. J. Sylvester [?]), It is the fact that the roots of an algebraic equation are real may be expressed in terms of inequalities for the coefficients of the equations, i.e., in terms of elementary symmetric functions. See Theorem B below.

2. Proof of Theorem A

The well-known result upon establishing the nature of the roots of a second degree trinomial with real coefficients by the sign of the discriminant, can be extended for third degree algebraic equations.

Lemma 1. *Let $a_1, a_2, a_3 \in \mathbb{R}$. The roots x_1, x_2, x_3 of the equation*

$$x^3 - a_1x^2 + a_2x - a_3 = 0$$

are real numbers if and only if the following inequality holds

$$D = (x_1-x_2)^2\,(x_2-x_3)^2\,(x_3-x_1)^2 \ge 0.$$

The quantity D appearing in the statement of Lemma 1 is called the *discriminant* (of the polynomial $x^3 - a_1x^2 + a_2x - a_3$). The connection with Theorem A is immediate, once we see that

$$\begin{aligned} D &= \det\left(\begin{pmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ x_1^2 & x_2^2 & x_3^2 \end{pmatrix} \cdot \begin{pmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{pmatrix}\right) = \\ &= \det\begin{pmatrix} 3 & \sum x_k & \sum x_k^2 \\ \sum x_k & \sum x_k^2 & \sum x_k^3 \\ \sum x_k^2 & \sum x_k^3 & \sum x_k^4 \end{pmatrix} = \\ &= 18a_1a_2a_3 + a_1^2a_2^2 - 4a_1^3a_3 - 4a_2^3 - 27a_3^2 \end{aligned}$$

the last equality being motivated by Viète relations.

Proof of Lemma 1. If the roots x_1, x_2, x_3 are real, it is then clear that $D \geq 0$; moreover, $D = 0$ if and only if $x_1 = x_2 = x_3$.

If the equation has not all roots real, then necessarily two are complex conjugate and the third is real, for instance

$$x_1 = a + bi, \ \ x_2 = a - bi, \ \ x_3 = c$$

where $a, \ b, \ c \in \mathbb{R}$ and $b \neq 0$. In such a case,

$$D = -b^2[(a-c)^2 + b^2] < 0. \ \blacksquare$$

For algebraic equations of degree ≥ 4, the notion of discriminant is no longer sufficient to describe the nature of the roots. See the case of a 4th degree equation with two pairs of complex conjugate roots. The role is taken by the *discriminant families.* In this way, the generalization of Theorem A is actually the following result due to by J. J. Sylvester [11], [12]:

Theorem B. *For each integer $n \geq 1$ there is a set of at most $n-1$ polynomials with integer coefficients*

$$R_{n,1}(x_1, ..., x_n), \ ... \ , \ R_{n,k(n)}(x_1, ..., x_n)$$

having the property that the polynomials with real coefficients,

$$P(x) = x^n - a_1 x^{n-1} + ... + (-1)^n a_n,$$

that have real roots are precisely those for which

$$R_{n,1}(a_1, ..., a_n) \geq 0, \ ... \ , \ R_{n,k(n)}(a_1, ..., a_n) \geq 0 \, .$$

The algorithmic procedure shown by the proof of this result, allows us to choose the $R_{n,j}$ *'s* as determinants extracted from the Sylvester matrix associated to $P(x)$; particularly, $R_1(x_1, ..., x_n)$ is the discriminant of order n. See [1]. Unfortunately the inequalities we get are excessively long because the numbers of terms of a discriminant grows very fast with its order. For instance, the 8th order discriminant has no less than 26095 terms! See [9].

For $n = 4$, a simple proof of Theorem B can be found in [8].

3. The analytic approach of Theorem A. Newton's inequalities.

A well-known consequence of the Rolle theorem (due to C. Maclaurin [5] and mentioned also by the mathematical analysis textbooks) asserts that if a polynomial has only real roots, then its derivative also has only real roots.

Lemma 2. *Let $a_1, a_2, a_3 \in \mathbb{R}$. The necessary and sufficient condition for the roots x_1, x_2, x_3 of the equation*

$$(E) \qquad x^3 - a_1 x^2 + a_2 x - a_3 = 0$$

to be real is that

$$18 a_1 a_2 a_3 + a_1^2 a_2^2 - 4 a_1^3 a_3 - 4 a_2^3 - 27 a_3^2 \geq 0.$$

Proof. The roots of the equation (E) are real if and only if the roots of the reduced equation

$$y^3 - py + q = 0$$

(which is obtained by the change of variable $x = y + a_1/3$), are real. Notice that

$$p = \frac{1}{3} a_1^2 - a_2 \quad \text{and} \quad q = \frac{1}{3} \, a_1 a_2 - a_3 - \frac{2}{27} a_1^3 \, .$$

Then, Rolle's technique shows that the reduced equation has only real roots if and only if

$$\left(\frac{p}{3}\right)^3 \geq \left(\frac{q}{2}\right)^2.$$

Replacing p and q in terms of a_1, a_2, a_3 we get the stated condition. ■

If the polynomial $x^3 - a_1x^2 + a_2x - a_3$ has only real roots, then its derivative $3x^2 - 2a_1x + a_2$ also has this property. Therefore $a_1^2 \geq 3a_2$.

By applying the same procedure to the equation obtained with the change of variable $y = 1/x$, we get the inequality $a_2^2 \geq 3a_1a_3$. The inequalities

$$(N) \qquad a_1^2 \geq 3a_2 \quad \text{and} \quad a_2^2 \geq 3a_1a_3$$

have been noticed for the first time by Newton [7] and bear his name. Maclaurin [5], to whom the above approach is due, has noticed that they yield

$$(a_1/3)^3 \geq a_3$$

equivalently,

$$\left(\frac{x+y+z}{3}\right)^3 \geq xyz,$$

when x, y, $z \geq 0$. This fact represents the $AM-GM$ inequality for families of three real numbers. Following the above idea, he proved the $AM-GM$ inequality in the general case, settling also with the equality case. A century later, A.-L. Cauchy gave his well-known proof by induction for this inequality. See [4].

Example. According to the discussion above, the condition in Lemma 2 implies Newton inequalities (N). Is the converse true ? The answer is negative. For, consider the equation

$$x^3 - 8.9x^2 + 26x - 24 = 0,$$

In this case $a_1 = 8.9$, $a_2 = 26$, $a_3 = 24$. This equation has (approximatively) the roots

$$x_1 = 1.\,8587, \quad x_2 = 3.\,5207 - 0.\,71933i, \quad x_3 = 3.\,5207 + 0.\,71933i.$$

It is interesting to observe that it represents a "small" perturbation of a "well behaved" equation,

$$x^3 - 9x^2 + 26x - 24 = (x-2)(x-3)(x-4) = 0.$$

Newton inequalities still work here because

$$a_1^2 - 3a_2 = (8.9)^2 - 3 \cdot 26 = 1.\,21 \quad \text{and} \quad a_2^2 - 3a_1a_3 = (26)^2 - 3 \cdot 8.9 \cdot 24 = 35.\,2$$

but, due to the presence of the complex roots, the condition of Lemma 2 is no more verified.

References

[1] R. Benedetti and J.-J. Risler, *Real algebraic and semi-algebraic sets*, Hermann, Editeurs des Sciences et des Arts, Paris, 1990.

[2] W. J. Blundon, *Inequalities associated with the triangle,* Canad. Math. Bull. **8** (1965), 615-626.

[3] A. Engel, *Problem-Solving Strategies*, Springer-Verlag, 1998.

[4] G. Hardy, J. E. Littewood and G. Polya, *Inequalities,* Cambridge Mathematical Library, 2nd ed., 1952.

[5] C. Maclaurin, *A second letter to Martin Folkes, Esq.; concerning the roots of equations, with the demonstration of other rules in algebra,* Phil. Transactions **36** (1729), 59-96.

[6] A. W. Marshall and I. Olkin, *Inequalities: Theory of Majorization and its Applications*, Academic Press, 1979.

[7] Sir Isaac Newton, *Arithmetica universalis*: *sive de compositione et resolutione arithmetica liber,* 1707.

[8] C. P. Niculescu, *A New Look at Newton's Inequalities,* J. Inequal. in Pure and Appl. Math. **1** (2000), no. 2

[9] T. Sasaki, Y. Kanada and S. Watanabe, *Calculation of discriminants of higher degree equations*, Tokyo J. Math. **4** (1981), 493-503.

[10] J. Sylvester, *On a theory of syzgetic relations of two rational integral functions comprising an appplication to the theory of Sturm's functions and that of the greatest algebraical common measure*, Phyl. Trans. of the Royal Soc. of London **CXLIII** (1853), 407-548. See also *The Collected Mathematical Papers of James Joseph Sylvester*, vol. I (1837-1853), pp. 429-586, Cambridge Univ. Press, 1904.

[11] J. Sylvester, *On Newton's Rule for the discovery of imaginary roots of equations,* Proc. of the Royal Society of London, xiv (1865), 268-270. See also *The Collected Mathematical Papers of James Joseph Sylvester*, vol. II (1854-1873), pp. 493-494, Cambridge Univ. Press, 1908.

[12] J. Sylvester, *On an elementary proof and generalization of Sir Isaac Newton's hitherto undemonstrated rule for discovery of imaginary roots*, Proc. of the London Math. Soc. vol. **1** (1865-1866), 1-16. See also *The Collected Mathematical Papers of James Joseph Sylvester*, vol. II (1854-1873), pp. 498-513, Cambridge Univ. Press, 1908.

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