ON GENERALISATIONS OF THE HARDY-HILBERT INTEGRAL INEQUALITY

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Abstract. In this paper, by introducing some parameters, we give a new generalisation of the Hardy-Hilbert inequality with a best constant factor which involves the $\beta-$function. We also consider its more extended form.

1. Introduction

If $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $a_n, b_n \geq 0$, and $0 < \sum_{n=1}^{\infty} a_n^p < \infty$, $0 < \sum_{n=1}^{\infty} b_n^q < \infty$, then

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_m b_n}{m+n} < \frac{\pi}{\sin\left(\frac{\pi}{p}\right)} \left(\sum_{n=1}^{\infty} a_n^p\right)^{\frac{1}{p}} \left(\sum_{n=1}^{\infty} b_n^q\right)^{\frac{1}{q}}, \tag{1.1}$$

where the constant $\frac{\pi}{\sin\left(\frac{\pi}{p}\right)}$ is the best possible. (1.1) is well knowk as the Hard-Hilbert inequality. Its integral form is:

If $f(t), g(t) \geq 0$, and

$$0 < \int_0^{\infty} f^p(t)\, dt < \infty, \quad 0 < \int_0^{\infty} g^q(t)\, dt < \infty,$$

then

$$\int_0^{\infty} \int_0^{\infty} \frac{f(x) g(y)}{x+y} dxdy < \frac{\pi}{\sin\left(\frac{\pi}{p}\right)} \left(\int_0^{\infty} f^p(t)\, dt\right)^{\frac{1}{p}} \left(\int_0^{\infty} g^q(t)\, dt\right)^{\frac{1}{q}}, \tag{1.2}$$

where the constant $\frac{\pi}{\sin\left(\frac{\pi}{p}\right)}$ is still best possible (see [1]).

The Hardy-Hilbert inequality is important in analysis and its applications (see [2]). In recent years, some new improvements of (1.1) have been given in [3, 4]. By introducing two parameters α, λ $\left(\alpha \in \mathbb{R}, \lambda \in \left(\frac{1}{r}, 1\right] \quad (r = p, q)\right)$, Yang [5] gave a generalisation of (1.2) as:

$$\begin{aligned} &\int_{\alpha}^{\infty} \int_{\alpha}^{\infty} \frac{f(x) g(y)}{(x+y-2\alpha)^{\lambda}} dxdy \\ &\leq \tilde{k}_{\lambda}^{\frac{1}{p}}(p)\, \tilde{k}_{\lambda}^{\frac{1}{q}}(q) \left[\int_{\alpha}^{\infty} (t-\alpha)^{1-\lambda} f^p(t)\, dt\right]^{\frac{1}{p}} \left[\int_{\alpha}^{\infty} (t-\alpha)^{1-\lambda} g^q(t)\, dt\right]^{\frac{1}{q}}, \end{aligned} \tag{1.3}$$

where

$$\tilde{k}_{\lambda}(r) = \int_0^{\infty} \frac{1}{(1+u)^{\lambda}} \left(\frac{1}{u}\right)^{1-\frac{1}{r}} du = B\left(\frac{1}{r}, \lambda - \frac{1}{r}\right) \quad (r = p, q),$$

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($B(u,v)$ is the $\beta-$function), and Kuang [6] gave the same result for $\alpha = 0$ in (1.3). For $T > 0$, and $0 < \lambda \le 1$, Yang [7] gave generalistions of (1.2) when $p = q = 2$ as:

$$\int_0^T \int_0^T \frac{f(x)g(y)}{(x+y)^\lambda} dxdy \tag{1.4}$$
$$\le B\left(\frac{\lambda}{2},\frac{\lambda}{2}\right)\left\{\int_0^T \left[1-\frac{1}{2}\left(\frac{t}{T}\right)^{\frac{\lambda}{2}}\right] t^{1-\lambda} f^2(t)\,dt\right\}^{\frac{1}{2}}$$
$$\times \left\{\int_0^T \left[1-\frac{1}{2}\left(\frac{t}{T}\right)^{\frac{\lambda}{2}}\right] t^{1-\lambda} g^2(t)\,dt\right\}^{\frac{1}{2}} \quad (T<\infty);$$

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+y)^\lambda} dxdy \tag{1.5}$$
$$\le B\left(\frac{\lambda}{2},\frac{\lambda}{2}\right)\left\{\int_0^\infty t^{1-\lambda} f^2(t)\,dt \int_0^\infty t^{1-\lambda} g^2(t)\,dt\right\}^{\frac{1}{2}}.$$

In this paper, following the way of [8, 9] in estimating the weight function, we introduce the $\beta-$function, and build some lemmas. As the main result, a new generalisation with the best constant factor involving the $\beta-$function is given, which is more accurate than (1.3) (see (3.2)). We also consider its more extended form (see (3.1)).

2. Some Lemmas

Lemma 1. *For $a < 1$, $\lambda > 0$, defined $g(a,y)$ as*

$$g(a,y) = y^{a-1}\int_0^y \frac{1}{(1+u)^\lambda}\left(\frac{1}{u}\right)^a du, \quad y \in (0,1].$$

Then we have $g(a,y) > g(a,1)$ $(y \in (0,1))$.

Proof. Integrating by parts, we have

$$\begin{aligned} g'_y(a,y) &= (a-1)y^{a-2}\int_0^y \frac{1}{(1+u)^\lambda}\left(\frac{1}{u}\right)^a du + y^{a-1}\frac{1}{(1+y)^\lambda}\left(\frac{1}{y}\right)^a \\ &= -y^{a-2}\int_0^y \frac{1}{(1+u)^\lambda} du^{1-a} + y^{-1}\frac{1}{(1+y)^\lambda} \\ &= -y^{a-2}\frac{1}{(1+u)^\lambda}y^{1-a} - \lambda y^{a-2}\int_0^y \frac{1}{(1+u)^{\lambda+1}}u^{1-a}du + y^{-1}\frac{1}{(1+y)^\lambda} \\ &= -\lambda y^{a-2}\int_0^y \frac{1}{(1+u)^{\lambda+1}}u^{1-a}du < 0 \quad (y\in(0,1)). \end{aligned}$$

Then $g(a,y)$ is a strictly decreasing function of y. In view of the fact that $g(a,y)$ is left continuous as a function of y at $y = 1$, we have $g(a,y) > g(a,1)$ $(0 < y < 1)$. The lemma is thus proved. ∎

We have a formula of the $\beta-$function as (see [8]):

$$B(p,q) = \int_0^\infty \frac{t^{p-1}}{(1+t)^{p+1}}dt = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} = B(q,p) \quad (p,q>0). \tag{2.1}$$

Lemma 2. *If $r>1$, $\frac{1}{r}+\frac{1}{s}=1$, $\lambda>2-\min\{r,s\}$, and $\alpha<T<\infty$, define the weight function $\tilde{\omega}_\lambda(\alpha,T,r,x)$ as:*

$$\tilde{\omega}_\lambda(\alpha,T,r,x)=\int_\alpha^T \frac{1}{(x+y-2\alpha)^\lambda}\left(\frac{x-\alpha}{y-\alpha}\right)^{\frac{2-\lambda}{r}}dy,\quad x\in(\alpha,T]. \tag{2.2}$$

Setting $\tilde{\omega}_\lambda(\alpha,\infty,r,x)=\lim_{T\to\infty}\tilde{\omega}_\lambda(\alpha,T,r,x)$, and

$$\begin{aligned} k_\lambda(r) &= k_\lambda(s)=B\left(\frac{r+\lambda-2}{r},\frac{s+\lambda-2}{s}\right), \\ \theta_\lambda(s) &= \int_0^1\frac{1}{(1+u)^\lambda}\left(\frac{1}{u}\right)^{\frac{2-\lambda}{s}}du, \end{aligned} \tag{2.3}$$

we have

$$\tilde{\omega}_\lambda(\alpha,\infty,r,x)=k_\lambda(r)(x-\alpha)^{1-\lambda},\quad (x\in(\alpha,\infty)); \tag{2.4}$$

$$\begin{aligned} &\tilde{\omega}_\lambda(\alpha,T,r,x) \\ &> \left[k_\lambda(r)-\theta_\lambda(s)\left(\frac{x-\alpha}{T-\alpha}\right)^{\frac{s+\lambda-2}{s}}\right](x-\alpha)^{1-\lambda},\quad (x\in(\alpha,T)). \end{aligned} \tag{2.5}$$

Proof. Setting $u=\frac{(y-\alpha)}{(x-\alpha)}$, we find

$$\tilde{\omega}_\lambda(\alpha,T,r,x)=(x-\alpha)^{1-\lambda}\int_0^{\frac{T-\alpha}{x-\alpha}}\frac{1}{(1+u)^\lambda}\left(\frac{1}{u}\right)^{\frac{2-\lambda}{r}}du,\ x\in(\alpha,T]. \tag{2.6}$$

Since

$$\lambda=\frac{r+\lambda-2}{r}+\frac{s+\lambda-2}{s}$$

and

$$\frac{2-\lambda}{r}=1-\frac{r+\lambda-2}{r},$$

by (2.1) and (2.6), we have

$$\begin{aligned} \tilde{\omega}_\lambda(\alpha,\infty,r,x) &= (x-\alpha)^{1-\lambda}\int_0^\infty\frac{1}{(1+u)^\lambda}\left(\frac{1}{u}\right)^{\frac{2-\lambda}{r}}du \\ &= (x-\alpha)^{1-\lambda}B\left(\frac{r+\lambda-2}{r},\frac{s+\lambda-2}{r}\right),\quad (x\in(\alpha,\infty)), \end{aligned}$$

and (2.4) is valid. By (2.6) and (2.4), we have

$$\begin{aligned} \tilde{\omega}_\lambda(\alpha,T,r,x) &= (x-\alpha)^{1-\lambda}\left[k_\lambda(r)-\int_{\frac{T-\alpha}{x-\alpha}}^\infty\frac{1}{(1+u)^\lambda}\left(\frac{1}{u}\right)^{\frac{2-\lambda}{r}}du\right] \\ &= (x-\alpha)^{1-\lambda}\left[k_\lambda(r)-\int_0^{\frac{x-\alpha}{T-\alpha}}\frac{1}{(1+v)^\lambda}\left(\frac{1}{v}\right)^{\frac{2-\lambda}{r}}dv\right] \\ &= (x-\alpha)^{1-\lambda}\left\{k_\lambda(r)-\left[\left(\frac{x-\alpha}{T-\alpha}\right)^{\frac{2-\lambda}{s}-1}\right.\right. \\ &\quad \left.\left.\times\int_0^{\frac{x-\alpha}{T-\alpha}}\frac{1}{(1+v)^\lambda}\left(\frac{1}{v}\right)^{\frac{2-\lambda}{s}}dv\right]\left(\frac{x-\alpha}{T-\alpha}\right)^{1+\frac{\lambda-2}{s}}\right\}. \end{aligned} \tag{2.7}$$

For $a=\frac{(2-\lambda)}{s}$ in Lemma 1, we have $a<1$. Since $\min\{r,s\}\leq 2$, it follows that $\lambda>2-\min\{r,s\}\geq 0$. By Lemma 1 and (2.3), we find

$$\begin{aligned}(2.8)\qquad & \left(\frac{x-\alpha}{T-\alpha}\right)^{\frac{2-\lambda}{s}-1}\int_0^{\frac{x-\alpha}{T-\alpha}}\frac{1}{(1+v)^\lambda}\left(\frac{1}{v}\right)^{\frac{2-\lambda}{s}}dv \\ = & \; g\left(\frac{2-\lambda}{s},\frac{x-\alpha}{T-\alpha}\right)>g\left(\frac{2-\lambda}{s},1\right)=\int_0^1\frac{1}{(1+v)^\lambda}\left(\frac{1}{v}\right)^{\frac{2-\lambda}{s}}dv \\ = & \; \theta_\lambda(s)\quad (x\in(\alpha,T)).\end{aligned}$$

Substituting (2.8) in (2.7), we have (2.5). This proves the lemma. ∎

Lemma 3. *If* $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, $\lambda>2-\min\{p,q\}$, $\frac{p+\lambda-2}{p}-\frac{1}{n_0}>0\quad(n_0\in\mathbb{N})$, *and* $0<\varepsilon\leq\frac{q}{n_0}$, *then we have*

$$(2.9)\qquad \int_1^\infty X^{-1-\varepsilon}\left[\int_0^{\frac{1}{X}}\frac{1}{(1+u)^\lambda}\left(\frac{1}{u}\right)^{\frac{2-\lambda}{p}+\frac{\varepsilon}{q}}du\right]dX=O(1)\quad\left(\varepsilon\to 0^+\right).$$

Proof. Since $0<\varepsilon\leq\frac{q}{n_0}$, then we have $\frac{\varepsilon}{q}\leq\frac{1}{n_0}$, and

$$\begin{aligned}0 &< \int_1^\infty X^{-1-\varepsilon}\left[\int_0^{\frac{1}{X}}\frac{1}{(1+u)^\lambda}\left(\frac{1}{u}\right)^{\frac{2-\lambda}{p}+\frac{\varepsilon}{q}}du\right]dX \\ &\leq \int_1^\infty X^{-1}\left[\int_0^{\frac{1}{X}}\left(\frac{1}{u}\right)^{\frac{2-\lambda}{p}+\frac{1}{n_0}}du\right]dX=\frac{1}{\left[\frac{p+\lambda-2}{p}-\frac{1}{n_0}\right]^2}.\end{aligned}$$

Hence (2.9) is valid. The lemma is proved. ∎

Lemma 4. *If* $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, $\lambda>2-\min\{p,q\}$, $\frac{p+\lambda-2}{p}-\frac{1}{n_0}>0\quad(n_0\in\mathbb{N})$, *and* $0<\varepsilon\leq\frac{q}{n_0}$, *then for* $\alpha,T\in\mathbb{R}\ (\alpha<T)$, *we have*

$$\begin{aligned}(2.10)\qquad & \int_\alpha^T\int_\alpha^T\frac{1}{(x+y-2\alpha)^\lambda}\left(\frac{x-\alpha}{T-\alpha}\right)^{\frac{\lambda-2+\delta}{p}}\left(\frac{y-\alpha}{T-\alpha}\right)^{\frac{\lambda-2+\phi}{q}}dxdy \\ & \tilde{}\;\;(T-\alpha)^{2-\lambda}\frac{1}{s}k_\lambda(p)\quad\left(\varepsilon\to 0^+\right).\end{aligned}$$

Proof. Setting $X = \frac{(T-\alpha)}{(x-\alpha)}$, and $Y = \frac{(T-\alpha)}{(y-\alpha)}$, we find

$$\begin{aligned}
(2.11)\quad & \int_{\alpha}^{T}\int_{\alpha}^{T}\frac{1}{(x+y-2\alpha)^{\lambda}}\left(\frac{x-\alpha}{T-\alpha}\right)^{\frac{\lambda-2+\delta}{p}}\left(\frac{y-\alpha}{T-\alpha}\right)^{\frac{\lambda-2+\phi}{q}}dxdy \\
= & (T-\alpha)^{2-\lambda}\int_{1}^{\infty}X^{\frac{\lambda-2}{q}-\frac{\varepsilon}{p}}\left[\int_{1}^{\infty}\frac{1}{(X+Y)^{\lambda}}Y^{\frac{\lambda-2}{q}-\frac{\varepsilon}{p}}dY\right]dX \\
= & (T-\alpha)^{2-\lambda}\int_{1}^{\infty}X^{-1-\varepsilon}\left[\int_{\frac{1}{X}}^{\infty}\frac{1}{(1+u)^{\lambda}}u^{\frac{\lambda-2}{q}-\frac{\varepsilon}{p}}du\right]dX \\
= & (T-\alpha)^{2-\lambda}\left\{\int_{1}^{\infty}X^{-1-\varepsilon}\left[\int_{0}^{\infty}\frac{1}{(1+u)^{\lambda}}u^{\frac{\lambda-2}{q}-\frac{\varepsilon}{p}}du\right]dX\right. \\
& \left.-\int_{1}^{\infty}X^{-1-\varepsilon}\left[\int_{0}^{\frac{1}{X}}\frac{1}{(1+u)^{\lambda}}u^{\frac{\lambda-2}{q}-\frac{\varepsilon}{p}}du\right]dX\right\}.
\end{aligned}$$

Since

$$\lim_{\varepsilon\to 0^{+}}\int_{0}^{\infty}\frac{1}{(1+u)^{\lambda}}u^{\frac{\lambda-2}{q}-\frac{\varepsilon}{p}}du=\int_{0}^{\infty}\frac{1}{(1+u)^{\lambda}}u^{\frac{\lambda-2}{q}}du=k_{\lambda}(p),$$

then we find

$$\int_{0}^{\infty}\frac{1}{(1+u)^{\lambda}}u^{\frac{\lambda-2}{q}-\frac{\varepsilon}{p}}du=k_{\lambda}(p)+o(1)\quad\left(\varepsilon\to 0^{+}\right).$$

By Lemma 3 and (2.11), we have

$$\begin{aligned}
& \int_{\alpha}^{T}\int_{\alpha}^{T}\frac{1}{(x+y-2\alpha)^{\lambda}}\left(\frac{x-\alpha}{T-\alpha}\right)^{\frac{\lambda-2+\delta}{p}}\left(\frac{y-\alpha}{T-\alpha}\right)^{\frac{\lambda-2+\phi}{q}}dxdy \\
= & (T-\alpha)^{2-\lambda}\left[\int_{1}^{\infty}X^{-1-\varepsilon}(k_{\lambda}(p)+o(1))\,dX-O(1)\right] \\
= & (T-\alpha)^{2-\lambda}\left[\frac{1}{\varepsilon}(k_{\lambda}(p)+o(1))-O(1)\right] \\
= & (T-\alpha)^{2-\lambda}\frac{1}{\varepsilon}((k_{\lambda}(p)+o(1))-\varepsilon O(1)) \\
\sim & (T-\alpha)^{2-\lambda}\frac{1}{s}k_{\lambda}(p)\quad\left(\varepsilon\to 0^{+}\right).
\end{aligned}$$

Hence (2.10) is valid, and the lemma is proved. ∎

3. Main Results and Some Corollaries

Theorem 1. *If* $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, $\lambda>2-\min\{p,q\}$, $\alpha<T\leq\infty$, *and* $f(t)$, $g(t)\geq 0$,

$$0<\int_{\alpha}^{T}(t-\alpha)^{1-\lambda}f^{p}(t)\,dt<\infty,\quad 0<\int_{\alpha}^{T}(t-\alpha)^{1-\lambda}g^{q}(t)\,dt<\infty,$$

then,

(i) *for $T<\infty$, we have*

$$\int_{\alpha}^{T}\int_{\alpha}^{T}\frac{f(x)\,g(y)}{(x+y-2\alpha)^{\lambda}}dxdy \tag{3.1}$$

$$< \left\{\int_{\alpha}^{T}\left[k_{\lambda}(p)-\theta_{\lambda}(p)\left(\frac{t-\alpha}{T-\alpha}\right)^{\frac{p+\lambda-2}{p}}\right](t-\alpha)^{1-\lambda}f^{p}(t)\,dt\right\}^{\frac{1}{p}}$$

$$\times\left\{\int_{\alpha}^{T}\left[k_{\lambda}(p)-\theta_{\lambda}(q)\left(\frac{t-\alpha}{T-\alpha}\right)^{\frac{q+\lambda-2}{q}}\right](t-\alpha)^{1-\lambda}g^{q}(t)\,dt\right\}^{\frac{1}{q}},$$

where

$$k_{\lambda}(p) = B\left(\frac{p+\lambda-2}{p},\frac{q+\lambda-2}{q}\right),$$

$$\theta_{\lambda}(r) = \int_{0}^{1}\frac{1}{(1+u)^{\lambda}}u^{\frac{2-\lambda}{r}}du \quad (r=p,q);$$

(ii) *for $T=\infty$, we have*

$$\int_{\alpha}^{\infty}\int_{\alpha}^{\infty}\frac{f(x)\,g(y)}{(x+y-2\alpha)^{\lambda}}dxdy \tag{3.2}$$

$$< k_{\lambda}(p)\left[\int_{\alpha}^{\infty}(t-\alpha)^{1-\lambda}f^{p}(t)\,dt\right]^{\frac{1}{p}}\left[\int_{\alpha}^{\infty}(t-\alpha)^{1-\lambda}g^{q}(t)\,dt\right]^{\frac{1}{q}},$$

where the constant $k_{\lambda}(p)$ in (3.1) and (3.2) is the best possible.

Proof. For $\alpha<T\le\infty$, by Hölder's inequality in $\mathbb{R}^2$ and (2.2), we have

$$\int_{\alpha}^{T}\int_{\alpha}^{T}\frac{f(x)\,g(y)}{(x+y-2\alpha)^{\lambda}}dxdy \tag{3.3}$$

$$= \int_{\alpha}^{T}\int_{\alpha}^{T}\left[\frac{f(x)}{(x+y-2\alpha)^{\frac{\lambda}{p}}}\left(\frac{x-\alpha}{y-\alpha}\right)^{\frac{2-\lambda}{pq}}\right]\left[\frac{g(y)}{(x+y-2\alpha)^{\frac{\lambda}{q}}}\left(\frac{y-\alpha}{x-\alpha}\right)^{\frac{2-\lambda}{pq}}\right]dxdy$$

$$\le \left\{\int_{\alpha}^{T}\int_{\alpha}^{T}\frac{f^{p}(x)}{(x+y-2\alpha)^{\lambda}}\left(\frac{x-\alpha}{y-\alpha}\right)^{\frac{2-\lambda}{q}}dxdy\right\}^{\frac{1}{p}}$$

$$\times\left\{\int_{\alpha}^{T}\int_{\alpha}^{T}\frac{g^{q}(y)}{(x+y-2\alpha)^{\lambda}}\left(\frac{y-\alpha}{x-\alpha}\right)^{\frac{2-\lambda}{p}}dxdy\right\}^{\frac{1}{q}}$$

$$= \left\{\int_{\alpha}^{T}\left[\int_{\alpha}^{T}\frac{1}{(x+y-2\alpha)^{\lambda}}\left(\frac{x-\alpha}{y-\alpha}\right)^{\frac{2-\lambda}{q}}dy\right]f^{p}(x)\,dx\right\}^{\frac{1}{p}}$$

$$\times\left\{\int_{\alpha}^{T}\left[\int_{\alpha}^{T}\frac{1}{(x+y-2\alpha)^{\lambda}}\left(\frac{y-\alpha}{x-\alpha}\right)^{\frac{2-\lambda}{p}}dx\right]g^{q}(y)\,dy\right\}^{\frac{1}{q}}$$

$$= \left\{\int_{\alpha}^{T}\tilde{\omega}_{\lambda}(\alpha,T,q,x)\,f^{p}(x)\,dx\right\}^{\frac{1}{p}}\left\{\int_{\alpha}^{T}\tilde{\omega}_{\lambda}(\alpha,T,p,y)\,g^{q}(y)\,dy\right\}^{\frac{1}{q}}.$$

If (3.3) takes equality, then there exists constants $A, B > 0$ such that (see [10])

$$\begin{aligned} & A\frac{f^p(x)}{(x+y-2\alpha)^\lambda}\left(\frac{x-\alpha}{y-\alpha}\right)^{\frac{2-\lambda}{q}} \\ = \;& B\frac{g^q(y)}{(x+y-2\alpha)^\lambda}\left(\frac{y-\alpha}{x-\alpha}\right)^{\frac{2-\lambda}{p}} \quad \text{a.e. in } [\alpha,T)\times[\alpha,T). \end{aligned}$$

It follows that

$$(x-\alpha)^{2-\lambda} f^p(x) = \frac{B}{A}(y-\alpha)^{2-\lambda} g^q(y) \quad \text{for a.e. in } [\alpha,T)\times[\alpha,T),$$

and

$$(x-\alpha)^{2-\lambda} f^p(x) = \frac{B}{A}(y-\alpha)^{2-\lambda} g^q(y) = c \;\text{ for a.e. in } [\alpha,T)\times[\alpha,T),$$

where c is a constant; which contradicts the fact that

$$0 < \int_\alpha^T (t-\alpha)^{1-\lambda} f^p(t)\,dt < \infty.$$

Hence, by (3.3), we have

$$\begin{aligned} &\int_\alpha^T\int_\alpha^T \frac{f(x)g(y)}{(x+y-2\alpha)^\lambda}dxdy \\ < \;& \left\{\int_\alpha^T \tilde{\omega}_\lambda(\alpha,T,q,t) f^p(t)\,dt\right\}^{\frac{1}{p}}\left\{\int_\alpha^T \tilde{\omega}_\lambda(\alpha,T,p,t) g^q(t)\,dt\right\}^{\frac{1}{q}}. \end{aligned} \tag{3.4}$$

By (2.5) and (2.4) since $k_\lambda(q) = k_\lambda(p)$, we have (3.1) and (3.2).

For $0 < \varepsilon \le \frac{q}{n_0}$, setting $\tilde{f}_\varepsilon(x)$ and $\tilde{g}_\varepsilon(y)$ as

$$\tilde{f}_\varepsilon(x) = \left(\frac{x-\alpha}{T-\alpha}\right)^{\frac{\lambda-2+\varepsilon}{p}}, \quad x\in(\alpha,T],\; \tilde{g}_\varepsilon(y) = \left(\frac{y-\alpha}{T-\alpha}\right)^{\frac{\lambda-2+\varepsilon}{q}}, \quad y\in(\alpha,T],$$

we find

$$\left[\int_\alpha^\infty (t-\alpha)^{1-\lambda}\tilde{f}_\varepsilon^p(t)\,dt\right]^{\frac{1}{p}}\left[\int_\alpha^\infty (t-\alpha)^{1-\lambda}\tilde{g}_\varepsilon^q(t)\,dt\right]^{\frac{1}{q}} = (T-\alpha)^{1-\lambda}\frac{1}{\varepsilon}.$$

By Lemma 4, we have

$$\int_\alpha^T\int_\alpha^T \frac{\tilde{f}_\varepsilon(x)\tilde{g}_\varepsilon(y)}{(x+y-2\alpha)^\lambda}dxdy \;\tilde{}\; (T-\alpha)^{2-\lambda}\frac{1}{\varepsilon}k_\lambda(p) \quad \left(\varepsilon\to 0^+\right).$$

If there exist $\alpha, T\in\mathbb{R}$ $(\alpha < T)$, such that the constant $k_\lambda(p)$ in (3.1) is not the best possible, then there exists K $(0 < K < k_\lambda(p))$, such that (3.1) is valid when we change $k_\lambda(p)$ to K. We find

$$\begin{aligned} & (T-\alpha)^{2-\lambda}\frac{1}{\varepsilon}k_\lambda(p) \;\tilde{}\; \int_\alpha^T\int_\alpha^T \frac{\tilde{f}_\varepsilon(x)\tilde{g}_\varepsilon(y)}{(x+y-2\alpha)^\lambda}dxdy \\ < \;& K\left[\int_\alpha^T (t-\alpha)^{1-\lambda}\tilde{f}_\varepsilon^p(t)\,dt\right]^{\frac{1}{p}}\left[\int_\alpha^T (t-\alpha)^{1-\lambda}\tilde{g}_\varepsilon^q(t)\,dt\right]^{\frac{1}{q}} \\ = \;& K(T-\alpha)^{2-\lambda}\frac{1}{\varepsilon} \quad \left(\varepsilon\to 0^+\right). \end{aligned}$$

Hence we have $k_\lambda(p) \le K$, which contradicts the fact that $K < k_\lambda(p)$. It follows that $k_\lambda(p)$ in (3.1) is the best possible.

If there exists an $\alpha \in \mathbb{R}$, such that the constant $k_\lambda(p)$ in (3.2) is not best possible, then there exists k $(0 < k < k_\lambda(p))$, such that

$$\begin{aligned} (3.5) \qquad & \int_\alpha^\infty \int_\alpha^\infty \frac{f(x)\,g(y)}{(x+y-2\alpha)^\lambda} dxdy \\ < \; & k\left[\int_\alpha^\infty (t-\alpha)^{1-\lambda} f^p(t)\,dt\right]^{\frac{1}{p}} \left[\int_\alpha^\infty (t-\alpha)^{1-\lambda} g^q(t)\,dt\right]^{\frac{1}{q}}. \end{aligned}$$

For any f, g which are suitable to (3.1), setting $f(t) = g(t) = 0$, for $t \in (T, \infty)$, by (3.5), we still have

$$\begin{aligned} & \int_\alpha^T \int_\alpha^T \frac{f(x)\,g(y)}{(x+y-2\alpha)^\lambda} dxdy \\ < \; & k\left[\int_\alpha^T (t-\alpha)^{1-\lambda} f^p(t)\,dt\right]^{\frac{1}{p}} \left[\int_\alpha^T (t-\alpha)^{1-\lambda} g^q(t)\,dt\right]^{\frac{1}{q}}. \end{aligned}$$

This contradicts the fact that $k_\lambda(p)$ is the best possible in (3.1). Hence the constant $k_\lambda(p)$ in (3.2) is the best possible. The theorem is thus proved. ∎

For $\lambda = 1, 2, 3$, we have $\lambda > 2 - \min\{p, q\}$ $\left(p > 1,\ \frac{1}{p} + \frac{1}{q} = 1\right)$. Hence for $r - p, q$, we find

$$\begin{aligned} \theta_1(r) &= \int_0^1 \frac{1}{1+u}\left(\frac{1}{u}\right)^{\frac{1}{r}} du > \int_0^1 \frac{1}{1+u} du = \ln 2, \\ k_1(p) &= B\left(\frac{1}{q}, \frac{1}{p}\right) = \frac{\pi}{\sin\left(\frac{\pi}{p}\right)}; \end{aligned}$$

$$\theta_2(r) = \int_0^1 \frac{1}{(1+u)^2} du = \frac{1}{2},\ k_2(p) = B(1,1) = 1;$$

$$\begin{aligned} \theta_3(r) &= \int_0^1 \frac{1}{(1+u)^3}\left(\frac{1}{u}\right)^{-\frac{1}{r}} du > \int_0^1 \frac{u}{(1+u)^3} du = \frac{1}{8}, \\ k_3(p) &= \frac{1}{2pq} B\left(\frac{1}{q}, \frac{1}{p}\right) = \frac{(p-1)\,\pi}{2p^2 \sin\left(\frac{\pi}{p}\right)}. \end{aligned}$$

By Theorem 1, the following corollaries hold.

Corollary 1. *If* $p > 1,\ \frac{1}{p} + \frac{1}{q} = 1,\ \lambda = 1,\ \alpha < T \le \infty,$ *and*

$$0 < \int_\alpha^T f^p(t)\,dt < \infty, \quad 0 < \int_\alpha^T g^q(t)\,dt < \infty,$$

then we have

$$\int_{\alpha}^{T}\int_{\alpha}^{T}\frac{f(x)\,g(y)}{x+y-2\alpha}dxdy \tag{3.6}$$
$$< \left\{\int_{\alpha}^{T}\left[\frac{\pi}{\sin\left(\frac{\pi}{p}\right)}-\ln 2\left(\frac{t-\alpha}{T-\alpha}\right)^{\frac{1}{q}}\right]f^{p}(t)\,dt\right\}^{\frac{1}{p}}$$
$$\times\left\{\int_{\alpha}^{T}\left[\frac{\pi}{\sin\left(\frac{\pi}{p}\right)}-\ln 2\left(\frac{t-\alpha}{T-\alpha}\right)^{\frac{1}{p}}\right]g^{q}(t)\,dt\right\}^{\frac{1}{q}}\quad (T<\infty);$$

$$\int_{\alpha}^{\infty}\int_{\alpha}^{\infty}\frac{f(x)\,g(y)}{x+y-2\alpha}dxdy < \frac{\pi}{\sin\left(\frac{\pi}{p}\right)}\left(\int_{\alpha}^{\infty}f^{p}(t)\,dt\right)^{\frac{1}{p}}\left(\int_{\alpha}^{\infty}g^{q}(t)\,dt\right)^{\frac{1}{q}}, \tag{3.7}$$

where the constant $\frac{\pi}{sim\left(\frac{\pi}{p}\right)}$ *in (3.6) and (3.7) is the best possible.*

Corollary 2. *If* $p>1,\ \frac{1}{p}+\frac{1}{q}=1,\ \lambda=2,\ \alpha<T\leq\infty,$ *and* $f(t)\,,g(t)\geq 0,$

$$0<\int_{\alpha}^{T}\frac{1}{t-\alpha}f^{p}(t)\,dt<\infty,\ 0<\int_{\alpha}^{T}\frac{1}{t-\alpha}g^{q}(t)\,dt<\infty,$$

then we have

$$\int_{\alpha}^{T}\int_{\alpha}^{T}\frac{f(x)\,g(y)}{(x+y-2\alpha)^{2}}dxdy \tag{3.8}$$
$$< \left\{\int_{\alpha}^{T}\left[1-\frac{1}{2}\left(\frac{t-\alpha}{T-\alpha}\right)\right]\frac{1}{t-\alpha}f^{p}(t)\,dt\right\}^{\frac{1}{p}}$$
$$\times\left\{\int_{\alpha}^{T}\left[1-\frac{1}{2}\left(\frac{t-\alpha}{T-\alpha}\right)\right]\frac{1}{t-\alpha}g^{q}(t)\,dt\right\}^{\frac{1}{q}}\quad (T<\infty),$$

$$\int_{\alpha}^{\infty}\int_{\alpha}^{\infty}\frac{f(x)\,g(y)}{(x+y-2\alpha)^{2}}dxdy \tag{3.9}$$
$$< \left\{\int_{\alpha}^{\infty}\frac{1}{t-\alpha}f^{p}(t)\,dt\right\}^{\frac{1}{p}}\left\{\int_{\alpha}^{\infty}\frac{1}{t-\alpha}g^{q}(t)\,dt\right\}^{\frac{1}{q}},$$

where the constant 1 *in (3.8) and (3.9) is the best possible.*

Corollary 3. *If* $p>1,\ \frac{1}{p}+\frac{1}{q}=1,\ \lambda=3,\ \alpha<T\leq\infty,$ *and* $f(t)\,,g(t)\geq 0,$

$$0<\int_{\alpha}^{T}\frac{1}{(t-\alpha)^{3}}f^{p}(t)\,dt<\infty,\ \ 0<\int_{\alpha}^{T}\frac{1}{(t-\alpha)^{3}}g^{q}(t)\,dt<\infty,$$

then we have

$$\int_{\alpha}^{T}\int_{\alpha}^{T}\frac{f(x)\,g(y)}{(x+y-2\alpha)^{3}}dxdy \tag{3.10}$$
$$< \left\{\int_{\alpha}^{T}\left[\frac{(p-1)\,\pi}{2p^{2}\sin\left(\frac{\pi}{p}\right)}-\frac{1}{8}\left(\frac{t-\alpha}{T-\alpha}\right)^{1+\frac{1}{p}}\right]\frac{1}{(t-\alpha)^{2}}f^{p}(t)\,dt\right\}^{\frac{1}{p}}$$
$$\times\left\{\int_{\alpha}^{T}\left[\frac{(p-1)\,\pi}{2p^{2}\sin\left(\frac{\pi}{p}\right)}-\frac{1}{8}\left(\frac{t-\alpha}{T-\alpha}\right)^{1+\frac{1}{q}}\right]\frac{1}{(t-\alpha)^{2}}g^{q}(t)\,dt\right\}^{\frac{1}{q}}$$

$$\int_{\alpha}^{\infty}\int_{\alpha}^{\infty}\frac{f(x)\,g(y)}{(x+y-2\alpha)^{3}}dxdy \tag{3.11}$$
$$< \frac{(p-1)\,\pi}{2p^{2}\sin\left(\frac{\pi}{p}\right)}\left[\int_{\alpha}^{\infty}\frac{1}{(t-\alpha)^{2}}f^{p}(t)\,dt\right]^{\frac{1}{p}}\left[\int_{\alpha}^{\infty}\frac{1}{(t-\alpha)^{2}}g^{q}(t)\,dt\right]^{\frac{1}{q}},$$

where the constant $\frac{(p-1)\pi}{2p^2\sin\left(\frac{\pi}{p}\right)}$ *in (3.10) and (3.11) is the best possible.*

Since $k_\lambda(2) = B\left(\frac{\lambda}{2},\frac{\lambda}{2}\right)$, $\theta_\lambda(2) = \frac{1}{2}B\left(\frac{\lambda}{2},\frac{\lambda}{2}\right)$, and $\lambda > 2-\min\{2,2\} = 0$, we have:

Corollary 4. *If* $p = q = 2$, $\lambda > 0$, $\alpha < T \le \infty$, *and* $f(t), g(t) \ge 0$,

$$0 < \int_{\alpha}^{T}(t-\alpha)^{1-\lambda}f^{2}(t)\,dt < \infty,\ 0 < \int_{\alpha}^{T}(t-\alpha)^{1-\lambda}g^{2}(t)\,dt < \infty,$$

then we have

$$\int_{\alpha}^{T}\int_{\alpha}^{T}\frac{f(x)\,g(y)}{(x+y-2\alpha)^{\lambda}}dxdy \tag{3.12}$$
$$< B\left(\frac{\lambda}{2},\frac{\lambda}{2}\right)\left\{\int_{\alpha}^{T}\left[1-\frac{1}{2}\left(\frac{t-\alpha}{T-\alpha}\right)^{\frac{1}{2}}\right](t-\alpha)^{1-\lambda}f^{2}(t)\,dt\right\}^{\frac{1}{2}}$$
$$\times\left\{\int_{\alpha}^{T}\left[1-\frac{1}{2}\left(\frac{t-\alpha}{T-\alpha}\right)^{\frac{1}{2}}\right](t-\alpha)^{1-\lambda}g^{2}(t)\,dt\right\}^{\frac{1}{2}},\ (T<\infty);$$

$$\int_{\alpha}^{\infty}\int_{\alpha}^{\infty}\frac{f(x)\,g(y)}{(x+y-2\alpha)^{\lambda}}dxdy \tag{3.13}$$
$$< B\left(\frac{\lambda}{2},\frac{\lambda}{2}\right)\left\{\int_{\alpha}^{\infty}(t-\alpha)^{1-\lambda}f^{2}(t)\,dt\int_{\alpha}^{\infty}(t-\alpha)^{1-\lambda}g^{2}(t)\,dt\right\}^{\frac{1}{2}},$$

where the constant $B\left(\frac{\lambda}{2},\frac{\lambda}{2}\right)$ *in (3.12) and (3.13) is the best possible.*

Remark 1. (1)

(a) *Since the constant* $k_\lambda(p)$ $(\lambda > 2-\min\{p,q\})$ *in (3.2) is the best possible, it follows that (3.2) is more accurate an estimate than (1.3).*

(b) *For* $\alpha = 0$, *inequality (3.7) changes to (1.2), hence inequalities (3.7) and (3.2) are new generalisations of (1.2).*

(c) *where* $T \to \infty$, *(3.1) changes to (3.2), and inequality (3.1) is thus a more extended form of (3.2).*

(d) *Inequalities (3.13) and (3.12) are new generalisations of Hilbert's integral inequality, and are new improvements of (1.5) and (1.4).*

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