SCHUR-CONVEXITY OF THE EXTENDED MEAN VALUES

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Abstract. In this article, the Schur-convexity of the extended mean values are proved. Consequently, an inequality between the logarithmic mean values and the identity (exponential) mean values is deduced.

1. Introduction

It is well-known that, in 1975, the extended mean values $E(r,s;x,y)$ were defined in [19] by K. B. Stolarsky as follows

$$E(r,s;x,y) = \left[\frac{r}{s}\cdot\frac{y^s - x^s}{y^r - x^r}\right]^{1/(s-r)}, \qquad rs(r-s)(x-y) \neq 0; \tag{1}$$

$$E(r,0;x,y) = \left[\frac{1}{r}\cdot\frac{y^r - x^r}{\ln y - \ln x}\right]^{1/r}, \qquad r(x-y) \neq 0; \tag{2}$$

$$E(r,r;x,y) = \frac{1}{e^{1/r}}\left(\frac{x^{x^r}}{y^{y^r}}\right)^{1/(x^r - y^r)}, \qquad r(x-y) \neq 0; \tag{3}$$

$$E(0,0;x,y) = \sqrt{xy}, \qquad x \neq y; \tag{4}$$

$$E(r,s;x,x) = x, \qquad x = y;$$

where $x, y > 0$ and $r, s \in \mathbb{R}$.

For $x, y > 0$ and $t \in \mathbb{R}$, let us define a function g by

$$g(t) = g(t;x,y) = \begin{cases} \dfrac{(y^t - x^t)}{t}, & t \neq 0; \\ \ln y - \ln x, & t = 0. \end{cases} \tag{5}$$

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It is easy to see that g can be expressed in integral form as

$$g(t;x,y)=\int_x^y u^{t-1}\,\mathrm{d}u, \tag{6}$$

and

$$g^{(n)}(t)=\int_x^y (\ln u)^n u^{t-1}\,\mathrm{d}u. \tag{7}$$

Therefore, in [2, 7, 10, 17], the extended mean values $E(r,s;x,y)$ were represented in terms of g by

$$E(r,s;x,y)=\begin{cases}\left(\dfrac{g(s;x,y)}{g(r;x,y)}\right)^{1/(s-r)}, & (r-s)(x-y)\neq 0;\\ \exp\left(\dfrac{\partial g(r;x,y)/\partial r}{g(r;x,y)}\right), & r=s,\ x-y\neq 0\end{cases} \tag{8}$$

and

$$\ln E(r,s;x,y)=\begin{cases}\dfrac{1}{s-r}\displaystyle\int_r^s \dfrac{\partial g(t;x,y)/\partial t}{g(t;x,y)}\,\mathrm{d}t, & (r-s)(x-y)\neq 0;\\ \dfrac{\partial g(r;x,y)/\partial r}{g(r;x,y)}, & r=s,\ x-y\neq 0.\end{cases} \tag{9}$$

In 1978, Leach and Sholander [3] showed that $E(r,s;x,y)$ are increasing with both r and s, or with both x and y. Later, the monotonicities of E have also been researched by the author and others in [2], [12]–[15] and [17, 18] using different ideas and simpler approaches.

In 1983 and 1988, Leach and Sholander [4] and Páles [5] respectively solved the problem of comparison of E; that is, they found necessary and sufficient conditions for the parameters r,s and u,v in order that $E(r,s;x,y)\le E(u,v;x,y)$ be satisfied for all positive x and y.

The concepts of mean values have been generalized or extended by the author in [7]–[9] and [11, 12].

Recently, the author verified the logarithmic convexity of $E(r,s;x,y)$ with two parameters r and s as follows

Theorem A ([10]). *For all fixed $x,y>0$ and $s\in[0,+\infty)$ (or $r\in[0,+\infty)$, respectively), the extended mean values $E(r,s;x,y)$ are logarithmically concave in r (or in s, respectively) on $[0,+\infty)$; For all fixed $x,y>0$ and $s\in(-\infty,0]$ (or $r\in(-\infty,0]$, respectively), the extended mean values $E(r,s;x,y)$ are logarithmically convex in r (or in s, respectively) on $(-\infty,0]$.*

Definition 1 ([6, p. 75–76])**.** A function f with n arguments defined on I^n is Schur-convex on I^n if $f(x) \le f(y)$ for each two n-tuples $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ in I^n such that $x \prec y$ holds, where I is an interval with nonempty interior.

The relationship majorization $x \prec y$ means that

$$\sum_{i=1}^{k} x_{[i]} \le \sum_{i=1}^{k} y_{[i]}, \qquad \sum_{i=1}^{n} x_{[i]} \le \sum_{i=1}^{n} y_{[i]}, \tag{10}$$

where $1 \le k \le n-1$, $x_{[i]}$ denotes the ith largest component in x.

A function f is Schur-concave if and only if $-f$ is Schur-convex.

In this article, our main purpose is to prove the Schur-convexity of the extended mean values $E(r,s;x,y)$ with (r,s), and then we obtain the following

Theorem 1. *For fixed $x > 0$ and $y > 0$, the extended mean values $E(r,s;x,y)$ are Schur-concave on $\mathbb{R}^2_+$ and Schur-convex on $\mathbb{R}^2_-$ with (r,s), where $\mathbb{R}^2_+$ and $\mathbb{R}^2_-$ denote $[0,+\infty) \times [0,+\infty)$ and $(-\infty,0] \times (-\infty,0]$, the first and third quadrants, respectively.*

Considering $(r_1, s_1) = (0, 2r)$ and $(r_2, s_2) = (r, r)$ for $r \ne 0$, as a direct consequence of Theorem 1, we obtain an inequality between the logarithmic mean values (2) and the identity (exponential) mean values (3) as follows

Corollary 1. *Let $x, y > 0$ and $x \ne y$. Then, for $r > 0$, we have*

$$\left[\frac{1}{2r} \cdot \frac{y^{2r} - x^{2r}}{\ln y - \ln x}\right]^{1/(2r)} \le \frac{1}{e^{1/r}} \left(\frac{x^{x^r}}{y^{y^r}}\right)^{1/(x^r - y^r)}. \tag{11}$$

For $r < 0$, inequality (11) reverses.

2. Lemmae

In order to prove Theorem 1, we need the following lemmae.

Lemma 1 ([1])**.** *Let f be a continuous function on I. Then the arithmetic mean of function f (or the integral arithmetic mean),*

$$\phi(u,v) = \begin{cases} \dfrac{1}{v-u} \displaystyle\int_u^v f(t)\,\mathrm{d}t, & u \ne v, \quad u, v \in I; \\ f(r), & u = v, \end{cases} \tag{12}$$

is Schur-convex (Schur-concave) on I^2 if and only if f is convex (concave) on I.

By formula (9) and Lemma 1, it is easy to see that, to prove the Schur-convexity of the extended mean values $E(r,s;x,y)$ with (r,s), it suffices to verify the convexity of the function

$$\frac{g'(t)}{g(t)} \triangleq \frac{g'_t(t;x,y)}{g(t;x,y)} \triangleq \frac{\partial g(t;x,y)}{\partial t} \cdot \frac{1}{g(t;x,y)} \tag{13}$$

with respect to t, where $g(t) = g(t;x,y)$ is defined by (5) or (6).

Straightforward computation results in

$$\left(\frac{g'(t)}{g(t)}\right)' = \frac{g''(t)g(t) - [g'(t)]^2}{g^2(t)}, \tag{14}$$

$$\left(\frac{g'(t)}{g(t)}\right)'' = \frac{g^2(t)g'''(t) - 3g(t)g'(t)g''(t) + 2[g'(t)]^3}{g^3(t)}. \tag{15}$$

Lemma 2 ([10])**.** *If $y > x = 1$, then, for $t \geq 0$,*

$$g^2(t;1,y)g'''_t(t;1,y) - 3g(t;1,y)g'_t(t;1,y)g''_t(t;1,y) + 2[g'_t(t;1,y)]^3 \leq 0. \tag{16}$$

Lemma 3. *If $y > x = 1$, then, for $t \geq 0$, the function $\frac{g'(t)}{g(t)}$ is concave.*

Proof. This follows from using a combination of formulae (13), (14) and (15) with Lemma 2 easily. □

3. Proof of Theorem 1

It is evident that $E(r,s;x,y)$ is symmetric with (r,s) since we have $E(r,s;x,y) = E(s,r;x,y)$.

Combining Lemma 2 with equality (15) shows that the function $\frac{g'_t(t;1,y)}{g(t;1,y)}$ is concave on $[0,+\infty)$ with t for $y > x = 1$. Therefore, from Lemma 1, it follows that the extended mean values $E(r,s;1,y)$ are Schur-concave with (r,s) on $[0,+\infty)\times[0,+\infty)$ for $y > x = 1$.

By standard arguments, we obtain

$$E(r,s;x,y) = xE(r,s;1,\frac{y}{x}), \tag{17}$$

$$E(-r,-s;x,y) = \frac{xy}{E(r,s;x,y)}. \tag{18}$$

Hence, for fixed x and y, the extended mean values $E(r,s;x,y)$ are Schur-concave with (r,s) on $[0,+\infty)\times[0,+\infty)$ and Schur-convex with (r,s) on $(-\infty,0]\times(-\infty,0]$. The proof of Theorem 1 is complete.

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