INEQUALITIES INVOLVING THE SEQUENCE $\sqrt[3]{a+\sqrt[3]{a+\cdots+\sqrt[3]{a}}}$

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Abstract. In this article, the convergence of the sequence

$$\underbrace{\sqrt[3]{a+\sqrt[3]{a+\cdots+\sqrt[3]{a}}}}_{n}$$

is proved, and some inequalities involving this sequence are established for $a > 0$. As by-product, two identities involving irrational numbers are obtained. Two open problems are proposed.

1. Introduction

Let $a > 0$ and $\mathbb{N}$ be the set of natural numbers. Denote

$$S_n(a) = \underbrace{\sqrt{a+\sqrt{a+\cdots+\sqrt{a}}}}_{n}, \tag{1}$$

$$f_n(a) = \frac{a - S_{n+1}(a)}{a - S_n(a)}. \tag{2}$$

In 1993, J.-Ch. Kuang sought the lower and upper bounds of $f_n(a)$, and conjectured that

$$f_n(a) > \frac{1}{a^2} \tag{3}$$

for all $n \in \mathbb{N}$. See [2, pp. 505–506 and p. 778].

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In 1999, as a reading note in [2], the second author raised the issue of the convergence of $S_{n,t}(a)$ and the bounds of $f_{n,t}(a)$, where, for $a>0$ and $t\neq 0$,

$$S_{n,t}(a)=\underbrace{\sqrt[t]{a+\sqrt[t]{a+\cdots+\sqrt[t]{a}}}}_{n}, \tag{4}$$

$$f_{n,t}(a)=\frac{a-S_{n+1,t}(a)}{a-S_{n,t}(a)}. \tag{5}$$

Recently, the conjecture made by by J.-Ch. Kuang was considered in [3], and the following result obtained.

Theorem A. *Let $a>0$ and $n\in\mathbb{N}$.*

(1) *For $a\geq 2$, we have*

$$\frac{1}{a^2}<\frac{2\left(a+\sqrt{a}-a^2\right)}{\left(\sqrt{a}-a\right)\left(\sqrt{1+4a}+2a+1\right)}<f_n(a)<1; \tag{6}$$

(2) *For $1\leq a<2$, there is a number $n_0\in\mathbb{N}$ such that*

$$f_n(a)>1\geq\frac{1}{a^2} \tag{7}$$

holds for $n>n_0$;

(3) *For $0<a<1$, we have*

$$1<f_n(a)\leq\frac{\sqrt{a+\sqrt{a}}-a}{\sqrt{a}-a}. \tag{8}$$

In this article, motivated by the reading note in [2] and the paper [3], we give an explicit solution to the problem involving the convergence of $S_{n,t}(a)$ and the bounds of $f_{n,t}(a)$ defined by (4) and (5) in the case of $t=3$.

2. Convergence and Inequalities for $S_{n,t}(a)$

In this section, we first discuss the convergence of the sequence $S_{n,t}(a)$, and then obtain several inequalities for it.

Theorem 1. *Let $a>0$ and $n\in\mathbb{N}$. The sequence $\{S_{n,3}(a)\}_{n=1}^{\infty}$ is strictly increasing.*

(1) *If $0<a\leq\frac{2}{3\sqrt{3}}$, we have*

$$\lim_{n\to\infty}S_{n,3}(a)=\frac{2}{\sqrt{3}}\cos\left(\frac{1}{3}\arccos\frac{3a\sqrt{3}}{2}\right); \tag{9}$$

(2) *If* $a > \frac{2}{3\sqrt{3}}$, *we have*

$$\lim_{n\to\infty} S_{n,3}(a) = \sqrt[3]{\frac{a}{2}+\sqrt{\frac{a^2}{4}-\frac{1}{27}}}+\sqrt[3]{\frac{a}{2}-\sqrt{\frac{a^2}{4}-\frac{1}{27}}}. \tag{10}$$

Proof. By induction, it is easy to prove that the sequence $\{S_{n,3}(a)\}_{n=1}^{\infty}$ is strictly increasing for $a > 0$ and $\sqrt[3]{a} \le S_{n,3}(a) < \sqrt[3]{a}+1$, therefore, the sequence $\{S_{n,3}(a)\}_{n=1}^{\infty}$ converges.

Suppose $\lim_{n\to\infty} S_{n,3}(a) = x$, then, from $S_{n,3}^3(a) = a + S_{n-1,3}(a)$, it can be deduced that $x^3 - x - a = 0$.

From Cardano's formula [1] for the solution of a cubic equation of a single variable, the proof of Theorem 1 follows. □

Using monotonicity of the sequence $\{S_{n,3}(a)\}_{n=1}^{\infty}$ and Theorem 1, the following inequalities are obtained.

Theorem 2. *Let* $a > 0$ *and* $n \in \mathbb{N}$.

(1) *If* $0 < a \le \frac{2}{3\sqrt{3}}$, *then*

$$a < \sqrt[3]{a} \le S_{n,3}(a) \le \frac{2}{\sqrt{3}}\cos\left(\frac{1}{3}\arccos\frac{3a\sqrt{3}}{2}\right); \tag{11}$$

(2) *If* $\frac{2}{3\sqrt{3}} < a < 1$, *we have*

$$a < \sqrt[3]{a} \le S_{n,3}(a) < \sqrt[3]{\frac{a}{2}+\sqrt{\frac{a^2}{4}-\frac{1}{27}}}+\sqrt[3]{\frac{a}{2}-\sqrt{\frac{a^2}{4}-\frac{1}{27}}}; \tag{12}$$

(3) *If* $1 \le a < \sqrt{2}$, *there exists a number* $n_0 \in \mathbb{N}$ *such that*

$$\sqrt[3]{a} \le S_{n_0,3}(a) \le a < S_{n,3}(a) < \sqrt[3]{\frac{a}{2}+\sqrt{\frac{a^2}{4}-\frac{1}{27}}}+\sqrt[3]{\frac{a}{2}-\sqrt{\frac{a^2}{4}-\frac{1}{27}}} \tag{13}$$

holds for $n > n_0$;

(4) *If* $a \ge \sqrt{2}$, *then*

$$\sqrt[3]{a} \le S_{n,3}(a) < \sqrt[3]{\frac{a}{2}+\sqrt{\frac{a^2}{4}-\frac{1}{27}}}+\sqrt[3]{\frac{a}{2}-\sqrt{\frac{a^2}{4}-\frac{1}{27}}} \le a. \tag{14}$$

Proof. We verify the inequalities (13) and (14), the rest follow similarly.

For $x \ge \frac{2}{3\sqrt{3}}$, we introduce a function $\psi(x)$ defined by

$$\psi(x) \triangleq g(x) - x \triangleq \sqrt[3]{\frac{x}{2}+\sqrt{\frac{x^2}{4}-\frac{1}{27}}}+\sqrt[3]{\frac{x}{2}-\sqrt{\frac{x^2}{4}-\frac{1}{27}}} - x. \tag{15}$$

We also claim that $\psi(x) \le 0$ if and only if $x \ge \sqrt{2}$.

Direct calculation reveals that

$$g^3(x) = g(x) + x. \tag{16}$$

We have, then

$$g'(x) = \frac{1}{3g^2(x) - 1}, \quad g''(x) = -\frac{6g(x)}{[3g^2(x) - 1]^3}. \tag{17}$$

It is clear that both the terms of $g(x)$ are positive for $x \ge \frac{2}{3\sqrt{3}}$. Using the arithmetic-geometric mean inequality yields that $g(x) > \frac{2\sqrt{3}}{3}$ for $x \ge \frac{2}{3\sqrt{3}}$. This leads to $3g^2(x) - 1 > 3$ for $x \ge \frac{2}{3\sqrt{3}}$. Therefore, the first derivative of $g(x)$ satisfies $g'(x) > 0$ and the second derivative $g''(x) < 0$ for $x \ge \frac{2}{3\sqrt{3}}$. This means that the function $g(x)$ is increasing and concave on $\left[\frac{2}{3\sqrt{3}}, \infty\right)$.

Straightforward computation yields

$$\psi\left(\frac{2}{3\sqrt{3}}\right) = \frac{4}{3\sqrt{3}}, \quad \lim_{x\to\infty} \psi(x) = -\infty. \tag{18}$$

This implies that the curve $y = g(x)$ and the straight line $y = x$ intersect at a unique point on $\left[\frac{2}{3\sqrt{3}}, \infty\right)$. Thus, there exists a unique point $x_0 \in \left(\frac{2}{3\sqrt{3}}, \infty\right)$ such that $\psi(x) > 0$ for $x \in \left(\frac{2}{3\sqrt{3}}, x_0\right)$ and $\psi(x) < 0$ for (x_0, ∞).

Since $\psi\left(\sqrt{2}\right) = 0$, consequently $x_0 = \sqrt{2}$. The proof is complete. $\square$

Remark 1. Now we provide another proof for the claim that $\psi(x) \le 0$ if and only if $x \ge \sqrt{2}$.

Firstly, we prove that $g(x) = x$ holds if and only if $x = \sqrt{2}$. Letting $x = \sqrt{2}$ in (16), we have $g^3\left(\sqrt{2}\right) - g\left(\sqrt{2}\right) - \sqrt{2} = 0$, which is equivalent to $\left[g\left(\sqrt{2}\right) - \sqrt{2}\right]\left[g^2\left(\sqrt{2}\right) + \sqrt{2}\, g\left(\sqrt{2}\right) + 1\right] = 0$, thus $g\left(\sqrt{2}\right) = \sqrt{2}$. Conversely, letting $g(x) = x \ge \frac{2}{3\sqrt{3}}$, then equation (16) reduces to $x^3 - 2x = 0$, and so $x = \sqrt{2}$.

Secondly, we verify that $g(x) < x$ is valid if and only if $x > \sqrt{2}$. If $g(x) < x$, then equation (16) can be rewritten as $x - g(x) = g^3(x) - 2g(x) = g(x)[g^2(x) - 2] > 0$, then $x > g(x) > \sqrt{2}$. Conversely, if $x > \sqrt{2}$, then $g^3(x) - g(x) - \sqrt{2} > g^3(x) - g(x) - x = 0$, which is equivalent to $[g(x) - \sqrt{2}][g^2(x) + \sqrt{2}\, g(x) + 1] > 0$, and so $g(x) > \sqrt{2}$. Therefore, $g(x) - x = 2g(x) - g^3(x) = g(x)[2 - g^2(x)] < 0$, which means that $g(x) < x$.

The proof is complete.

Corollary 1. *The irrational number $\sqrt{2}$ can be expressed as*

$$\sqrt{2} = \sqrt[3]{\frac{1}{\sqrt{2}} - \frac{5}{3\sqrt{6}}} + \sqrt[3]{\frac{1}{\sqrt{2}} + \frac{5}{3\sqrt{6}}}, \tag{19}$$

which is equivalent to

$$\sqrt[3]{3\sqrt{3}-5}+\sqrt[3]{3\sqrt{3}+5}=\sqrt{3}\cdot\sqrt[3]{4}\,. \tag{20}$$

Proof. Identity (20) follows from simplifying (19) directly.

Raising both sides of $A=\sqrt[3]{3\sqrt{3}-5}+\sqrt[3]{3\sqrt{3}+5}$ to the power of 3 shows that A satisfies the cubic equation $x^3-3\sqrt[3]{2}\,x-6\sqrt{3}=0$. By Cardano's formula in [1], it follows that $A=\sqrt{3}\cdot\sqrt[3]{4}$. The proof is complete. □

3. Inequalities for the Sequence $f_{n,3}(a)$

From the monotonicity and inequalities for the sequence $\{S_{n,3}(a)\}_{n=1}^{\infty}$, we will derive some inequalities for the sequence $\{f_{n,3}(a)\}_{n=1}^{\infty}$.

Theorem 3. *Let $a>0$ and $n\in\mathbb{N}$.*

(1) *When $0<a<1$, we have*

$$1<f_{n,3}(a)\le\frac{\sqrt[3]{a+\sqrt[3]{a}}-a}{\sqrt[3]{a}-a}; \tag{21}$$

(2) *When $1\le a<\sqrt{2}$, there exists a number $n_0\in\mathbb{N}$ such that*

$$f_{n,3}(a)>1>\frac{1}{a}>\frac{1}{a^2} \tag{22}$$

holds for all $n>n_0$;

(3) *When $a\ge\sqrt{2}$, we have*

$$1>f_{n,3}(a)>\frac{1}{a^2+a\alpha+\alpha^2}\left(1+\frac{a^3-2a}{a-\sqrt[3]{a}}\right), \tag{23}$$

where

$$\alpha=\sqrt[3]{\frac{a}{2}+\sqrt{\frac{a^2}{4}-\frac{1}{27}}}+\sqrt[3]{\frac{a}{2}-\sqrt{\frac{a^2}{4}-\frac{1}{27}}}\,. \tag{24}$$

Proof. For $0<a<1$, since the sequence $\{S_{n,3}(a)\}_{n=1}^{\infty}$ is strictly increasing with $S_{n,3}(a)>a$, then $a-S_{n+1,3}(a)<a-S_{n,3}(a)<0$, and $f_{n,3}(a)=\frac{a-S_{n+1,3}(a)}{a-S_{n,3}(a)}>1$.

By standard argument, it follows that

$$\begin{aligned}
f_{n+1,3}(a) &= \frac{a - S_{n+2,3}(a)}{a - S_{n+1,3}(a)} \\
&= \frac{1}{a^2 + aS_{n+2,3}(a) + S_{n+2,3}^2(a)} \frac{a^3 - S_{n+2,3}^3(a)}{a - S_{n+1,3}(a)} \\
&= \frac{1}{a^2 + aS_{n+2,3}(a) + S_{n+2,3}^2(a)} \left[1 + \frac{2a - a^3}{S_{n+1,3}(a) - a}\right] \\
&< \frac{1}{a^2 + aS_{n+1,3}(a) + S_{n+1,3}^2(a)} \left[1 + \frac{2a - a^3}{S_{n,3}(a) - a}\right] \\
&= \frac{a - S_{n+1,3}(a)}{a - S_{n,3}(a)} \\
&= f_{n,3}(a).
\end{aligned} \tag{25}$$

This implies that the sequence $\{f_{n,3}(a)\}_{n=1}^{\infty}$ is strictly decreasing, therefore

$$f_{n,3}(a) \le f_{1,3}(a) = \frac{\sqrt[3]{a + \sqrt[3]{a}} - a}{\sqrt[3]{a} - a}. \tag{26}$$

For $1 \le a < \sqrt{2}$, and (13), there exists a number $n_0 \in \mathbb{N}$ such that $a - S_{n+1,3}(a) < a - S_{n,3}(a) < 0$ holds for $n > n_0$. Hence

$$f_{n,3}(a) = \frac{a - S_{n+1,3}(a)}{a - S_{n,3}(a)} > 1 > \frac{1}{a} > \frac{1}{a^2}, \quad n > n_0.$$

For $n > n_0$, the formula (25) is also valid. Thus, the sequence $\{f_{n,3}(a)\}_{n=n_0+1}^{\infty}$ is strictly decreasing, and

$$\frac{1}{a^2} < \frac{1}{a} < 1 < f_{n,3}(a) < \frac{a - S_{n_0+2,3}(a)}{a - S_{n_0+1,3}(a)}, \quad n > n_0. \tag{27}$$

For $a \ge \sqrt{2}$, and (14), we have $0 < a - S_{n+1,3}(a) < a - S_{n,3}(a)$ for $n \in \mathbb{N}$. Then $f_{n,3}(a) = \frac{a - S_{n+1,3}(a)}{a - S_{n,3}(a)} < 1$. From a combination of the following formula (28),

$$f_{n,3}(a) = \frac{a - S_{n+1,3}(a)}{a - S_{n,3}(a)} = \frac{1}{a^2 + aS_{n+1,3}(a) + S_{n+1,3}^2(a)} \left[1 + \frac{2a - a^3}{S_{n,3}(a) - a}\right], \tag{28}$$

and the inequalities in (14), then (23) follows.

The proof is complete. □

4. Open Problems

It is natural to pose the following questions.

(1) Can we prove or disprove the convergence of the sequence $\{S_{n,t}(a)\}_{n=1}^{\infty}$ for a positive real number a and nonzero real number $t \ne 0$?

(2) Can we establish sharp lower and upper sharp bounds for the sequence $\{f_{n,t}(a)\}_{n=1}^{\infty}$ for a positive real number a and nonzero real number $t \neq 0$?

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