

# THE STRENGTHENED HARDY INEQUALITIES AND ITS NEW GENERALIZATIONS

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ABSTRACT. In this article, using the properties of power mean, new generalizations of the strengthened Hardy Inequalities are proved.

## 1. INTRODUCTION

It is well known that the following Hardy's Inequality (see [4, Theorem 326]):  
if  $p > 1$  and  $a_n \geq 0$ , then

$$(1.1) \quad \sum \left( \frac{a_1 + a_2 + \dots + a_n}{n} \right)^p < \left( \frac{p}{p-1} \right)^p \sum a_n^p,$$

unless all the  $a$  are zero. The constant is the best possible.

This theorem was discovered in the course of attempts to simplify the proofs then known of Hilbert's double series theorems (see [4, Theorem 315]). Hilbert's double series theorem was completed by the above inequality. This inequality was first proved by Hardy [3], except that Hardy was unable to fit the constant in inequality (1.1). If in inequality (1.1) we write  $a_n$  for  $a_n^p$ , we obtain

$$(1.2) \quad \sum \left( \frac{a_1^{1/p} + a_2^{1/p} + \dots + a_n^{1/p}}{n} \right)^p < \left( \frac{p}{p-1} \right)^p \sum a_n.$$

If we make  $p \rightarrow \infty$ , and use the elementary mean values

$$\lim_{p \rightarrow 0} \left( \sum_{i=1}^n \frac{1}{n} a_i^p \right)^{1/p} = \left( \prod_{i=1}^n a_i \right)^{1/n},$$

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we obtain

$$\sum_{n=1}^{\infty} (a_1 a_2 \cdots a_n)^{1/n} < e \sum_{n=1}^{\infty} a_n,$$

and this suggests the more complete theorem which follow;

$$(1.3) \quad \sum_{n=1}^{\infty} (a_1 a_2 \cdots a_n)^{1/n} < e \sum_{n=1}^{\infty} a_n,$$

unless  $(a_n)$  is null. The constant is the best possible.

The inequality given in (1.3) which later went by the name of *Carleman's inequality*, led to a great many papers dealing with alternative proofs, various generalizations, and numerous variants and applications in analysis. It is natural to attempt to prove the complete inequality by means of following

$$(1.4) \quad \left( \prod_{i=1}^n a_i \right)^{1/n} < \sum_{i=1}^n \frac{1}{n} a_i,$$

unless all the  $a_i$  are equal. But a direct application of inequality (1.4) to the left-hand side of the inequality (1.2) is insufficient. To remedy this, we apply inequality (1.4) not to  $a_1, a_2, \dots, a_n$  but to  $c_1 a_1, c_2 a_2, \dots, c_n a_n$ , and choose the  $c$  so that when  $\sum a_n$  is near the boundary of convergence, these numbers shall be 'roughly equal'. This requires that  $c_n$  shall be roughly of the order of  $n$ .

By Hardy (see, [4, Theorem 349]), the Carleman's inequality was generalized as follows:

If  $a_n \geq 0, \lambda_n > 0, \Lambda_n = \sum_{m=1}^n \lambda_m (n \in N)$  and  $0 < \sum_{n=1}^{\infty} \lambda_n a_n < \infty$ , then

$$(1.5) \quad \sum_{n=1}^{\infty} \lambda_n (a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_n^{\lambda_n})^{1/\Lambda_n} < e \sum_{n=1}^{\infty} \lambda_n a_n.$$

Recently, Z. Xie and Y. Zhong [7] gave an improvement of the inequality (1.5) as follows: If  $a_n \geq 0, 0 < \lambda_{n+1} \leq \lambda_n, \Lambda_n = \sum_{m=1}^n \lambda_m (n \in N)$  and  $0 < \sum_{n=1}^{\infty} \lambda_n a_n < \infty$ , then

$$(1.6) \quad \sum_{n=1}^{\infty} \lambda_{n+1} (a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_n^{\lambda_n})^{1/\Lambda_n} < e \sum_{n=1}^{\infty} \left( 1 - \frac{6\lambda_n}{12\Lambda_n + 11\lambda_n} \right) \lambda_n a_n.$$

Most recently, Z. Yang [11] obtained the strengthened Carleman's inequality as follows: *If  $a_n \geq 0$ ,  $n = 1, 2, \dots$ , and  $0 < \sum_{n=1}^{\infty} a_n < \infty$ . Then*

$$(1.7) \quad \sum_{n=1}^{\infty} (a_1 a_2 \cdots a_n)^{1/n} < e \sum_{n=1}^{\infty} \left( 1 - \frac{1}{2(1+n)} - \frac{1}{24(1+n)^2} - \frac{1}{48(1+n)^3} \right) a_n.$$

It is immediate from the proof of inequality (1.6) and the inequality (1.7) that we can deduce the following new strengthened Hardy's inequality:

$$(1.8) \quad \sum_{n=1}^{\infty} \lambda_{n+1} (a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_n^{\lambda_n})^{1/\Lambda_n} < e \sum_{n=1}^{\infty} \left( 1 - \frac{\lambda_n}{2(\Lambda_n + \lambda_n)} - \frac{\lambda_n^2}{24(\Lambda_n + \lambda_n)^2} - \frac{\lambda_n^3}{48(\Lambda_n + \lambda_n)^3} \right) \lambda_n a_n.$$

But we know that the inequality (1.8) is a better improvement of the inequality (1.6), as a result of following

$$\left( 1 - \frac{\lambda_n}{2(\Lambda_n + \lambda_n)} - \frac{\lambda_n^2}{24(\Lambda_n + \lambda_n)^2} - \frac{\lambda_n^3}{48(\Lambda_n + \lambda_n)^3} \right) < \left( 1 - \frac{6\lambda_n}{12\Lambda_n + 11\lambda_n} \right)$$

for  $\Lambda_n/\lambda_n \geq 1$ .

The purpose of this paper is to prove new extension of the strengthened Hardy's inequality in the spirit of the strict monotonicity of the power mean of  $n$  distinct positive numbers.

For any positive values  $a_1, a_2, \dots, a_n$  and positive weights  $\alpha_1, \alpha_2, \dots, \alpha_n$ ,  $\sum_{i=1}^n \alpha_i = 1$ , and any real  $p \neq 0$ , we defined the power mean, or the mean of order  $p$  of the value  $a$  with weights  $\alpha$  by

$$M_p(a; \alpha) = M_p(a_1, a_2, \dots, a_n; \alpha_1, \alpha_2, \dots, \alpha_n) = \left( \sum_{i=1}^n \alpha_i a_i^p \right)^{1/p}.$$

An easy application of L'Hospital's rule shows that

$$\lim_{p \rightarrow 0} M_p(a; \alpha) = \prod_{i=1}^n a_i^{\alpha_i},$$

the geometric mean. Accordingly, we define  $M_0(a; \alpha) = \prod_{i=1}^n a_i^{\alpha_i}$ . It is well known that  $M_p(a; \alpha)$  is a nondecreasing function of  $p$  for  $-\infty \leq p \leq \infty$ , and is strictly increasing unless all the  $a_i$  are equal (cf. [1]).

## 2. STRENGTHENED HARDY'S INEQUALITIES

The main results of this paper are presented as follows:

**Lemma 2.1** [7]. *Let  $x \geq 1$ , then we have the following inequality:*

$$(2.1) \quad \frac{12x + 11}{12x + 5} \left(1 + \frac{1}{x}\right)^x < e < \frac{14x + 12}{14x + 5} \left(1 + \frac{1}{x}\right)^x.$$

We can deduce the following improvement results of the inequality (1.6):

**Theorem 2.2.** *Let  $0 < \lambda_{n+1} \leq \lambda_n$ ,  $\Lambda_n = \sum_{m=1}^n \lambda_m$  ( $\Lambda_n \geq 1$ ),  $a_n \geq 0$  ( $n \in N$ ),  $0 < p \leq 1$  and  $0 < \sum_{n=1}^{\infty} \lambda_n a_n < \infty$ . Then*

$$(2.2) \quad \sum_{n=1}^{\infty} \lambda_{n+1} (a_1^{\lambda_1} a_2^{\lambda_2} \dots a_n^{\lambda_n})^{1/\Lambda_n} < \frac{e^p}{p} \sum_{n=1}^{\infty} \left(1 - \frac{6\lambda_n}{12\Lambda_n + 11\lambda_n}\right)^p \lambda_n (a_n)^p \Lambda_n^{p-1} \left(\sum_{k=1}^n \lambda_k (c_k a_k)^p\right)^{(1-p)/p}.$$

where  $c_k^{\lambda_n} = (\Lambda_{n+1})^{\Lambda_n} / (\Lambda_n)^{\Lambda_{n-1}}$ .

*Proof.* By the power mean inequality, we have

$$\alpha_1^{q_1} \alpha_2^{q_2} \dots \alpha_n^{q_n} \leq \left(\sum_{m=1}^n q_m (\alpha_m)^p\right)^{1/p},$$

for  $\alpha_m \geq 0$ ,  $p \geq 0$  and  $q_m > 0$  ( $m = 1, 2, \dots, n$ ) with  $\sum_{m=1}^n q_m = 1$ . Setting  $c_m > 0$ ,  $\alpha_m = c_m a_m$  and  $q_m = \lambda_m / \Lambda_n$ , we obtain

$$(c_1 a_1)^{\lambda_1 / \Lambda_n} (c_2 a_2)^{\lambda_2 / \Lambda_n} \dots (c_n a_n)^{\lambda_n / \Lambda_n} \leq \left(\frac{1}{\Lambda_n} \sum_{m=1}^n \lambda_m (c_m a_m)^p\right)^{1/p}.$$

Using the above inequality, we have

$$\begin{aligned}
 (2,3) \quad & \sum_{n=1}^{\infty} \lambda_{n+1} (a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_n^{\lambda_n})^{1/\Lambda_n} \\
 &= \sum_{n=1}^{\infty} \lambda_{n+1} \frac{(c_1 a_1)^{\lambda_1/\Lambda_n} (c_2 a_2)^{\lambda_2/\Lambda_n} \cdots (c_n a_n)^{\lambda_n/\Lambda_n}}{(c_1^{\lambda_1} c_2^{\lambda_2} \cdots c_n^{\lambda_n})^{1/\Lambda_n}} \\
 &\leq \sum_{n=1}^{\infty} \left[ \frac{\lambda_{n+1}}{(c_1^{\lambda_1} c_2^{\lambda_2} \cdots c_n^{\lambda_n})^{1/\Lambda_n}} \right] \left( \frac{1}{\Lambda_n} \sum_{m=1}^n \lambda_m (c_m a_m)^p \right)^{1/p}.
 \end{aligned}$$

By using the following inequality (see [2], [6]),

$$\left( \sum_{m=1}^n z_m \right)^t \leq t \sum_{m=1}^n z_m \left( \sum_{k=1}^m z_k \right)^{t-1},$$

where  $t \geq 1$  is constant and  $z_m \geq 0 (m = 1, 2, \dots)$ , it is easy to observe that

$$\begin{aligned}
 (2.4) \quad & \left( \frac{1}{\Lambda_n} \sum_{m=1}^n \lambda_m (c_m a_m)^p \right)^{1/p} \\
 &\leq \frac{1}{\Lambda_n} \left( \sum_{m=1}^n \lambda_m (c_m a_m)^p \right)^{1/p} \\
 &\leq \frac{1}{p \Lambda_n} \sum_{m=1}^n \lambda_m (c_m a_m)^p \left( \sum_{k=1}^m \lambda_k (c_k a_k)^p \right)^{(1-p)/p}
 \end{aligned}$$

for  $\Lambda_n \geq 1$  and  $0 < p \leq 1$ . Then, by (2.3) and (2.4), we obtain

$$\begin{aligned}
 & \sum_{n=1}^{\infty} \lambda_{n+1} (a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_n^{\lambda_n})^{1/\Lambda_n} \\
 &\leq \frac{1}{p} \sum_{m=1}^n \lambda_m (c_m a_m)^p \sum_{n=m}^{\infty} \left( \frac{\lambda_{n+1}}{\Lambda_n (c_1^{\lambda_1} c_2^{\lambda_2} \cdots c_n^{\lambda_n})^{1/\Lambda_n}} \right) \left( \sum_{k=1}^m \lambda_k (c_k a_k)^p \right)^{(1-p)/p}
 \end{aligned}$$

Choosing  $c_1^{\lambda_1} c_2^{\lambda_2} \cdots c_n^{\lambda_n} = (\Lambda_{n+1})^{\Lambda_n}$  ( $n \in N$ ) and setting  $\Lambda_0 = 0$ , from  $\lambda_{n+1} \leq \lambda_n$ , it follows that

$$\begin{aligned}
 c_n &= \left[ \frac{(\Lambda_{n+1})^{\Lambda_n}}{(\Lambda_n)^{\Lambda_{n-1}}} \right]^{1/\lambda_n} = \left( 1 + \frac{\lambda_{n+1}}{\Lambda_n} \right)^{\Lambda_n/\lambda_n} \cdot \Lambda_n \\
 &\leq \left( 1 + \frac{\lambda_n}{\Lambda_n} \right)^{\Lambda_n/\lambda_n} \cdot \Lambda_n.
 \end{aligned}$$

This implies that

$$\begin{aligned}
& \sum_{n=1}^{\infty} \lambda_{n+1} (a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_n^{\lambda_n})^{1/\Lambda_n} \\
& \leq \frac{1}{p} \sum_{m=1}^n \lambda_m (c_m a_m)^p \sum_{n=m}^{\infty} \frac{\lambda_{n+1}}{\Lambda_n \Lambda_{n+1}} \left( \sum_{k=1}^m \lambda_k (c_k a_k)^p \right)^{(1-p)/p} \\
& = \frac{1}{p} \sum_{m=1}^n \lambda_m (c_m a_m)^p \sum_{n=m}^{\infty} \left( \frac{1}{\Lambda_n} - \frac{1}{\Lambda_{n+1}} \right) \left( \sum_{k=1}^m \lambda_k (c_k a_k)^p \right)^{(1-p)/p} \\
& = \frac{1}{p} \sum_{m=1}^n \lambda_m (c_m a_m)^p \frac{1}{\Lambda_m} \left( \sum_{k=1}^m \lambda_k (c_k a_k)^p \right)^{(1-p)/p} \\
& \leq \frac{1}{p} \sum_{m=1}^{\infty} \left( 1 + \frac{1}{\Lambda_m/\lambda_m} \right)^{p\Lambda_m/\lambda_m} \lambda_m (a_m)^p \Lambda_m^{p-1} \left( \sum_{k=1}^m \lambda_k (c_k a_k)^p \right)^{(1-p)/p}.
\end{aligned}$$

Hence, by the above inequality and Lemma 2.1, we have

$$\begin{aligned}
& \sum_{n=1}^{\infty} \lambda_{n+1} (a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_n^{\lambda_n})^{1/\Lambda_n} \\
& < \frac{e^p}{p} \sum_{n=1}^{\infty} \left( 1 - \frac{6\lambda_n}{12\Lambda_n + 11\lambda_n} \right)^p \lambda_n (a_n)^p \Lambda_n^{p-1} \left( \sum_{k=1}^n \lambda_k (c_k a_k)^p \right)^{(1-p)/p}.
\end{aligned}$$

Thus Theorem 2.2 is proved.

Setting  $p \equiv 1$  in Theorem 2.2, then, from inequality (2.2) we have the inequality (1.6). Also assuming that  $\lambda_n = 1$  in the Theorem, we have an extension of the strengthened Carleman's inequality as following:

**Corollary 2.3.** *Let  $a_n \geq 0 (n \in N)$ ,  $0 < p \leq 1$  and  $0 < \sum_{n=1}^{\infty} a_n < \infty$ . Then*

$$\begin{aligned}
& \sum_{n=1}^{\infty} (a_1 a_2 \cdots a_n)^{1/n} \\
& < \frac{e^p}{p} \sum_{n=1}^{\infty} \left( 1 - \frac{6}{12n + 11} \right)^p (a_n)^p n^{p-1} \left( \sum_{k=1}^n (c_k a_k)^p \right)^{(1-p)/p}.
\end{aligned}$$

where  $c_k = (1 + 1/k)^k \cdot k$ .

Similarly to Theorem 2.2, we can consider a generalization version of the inequality (1.8) as following theorem:

**Theorem 2.4.** *Let  $0 < \lambda_{n+1} \leq \lambda_n$ ,  $\Lambda_n = \sum_{m=1}^n \lambda_m$ ,  $a_n \geq 0 (n \in N)$ ,  $0 < p \leq 1$  and  $0 < \sum_{n=1}^{\infty} \lambda_n a_n < \infty$ . Then*

$$(2.5) \quad \begin{aligned} & \sum_{n=1}^{\infty} \lambda_{n+1} (a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_n^{\lambda_n})^{1/\Lambda_n} \\ & < \frac{e}{p} \sum_{n=1}^{\infty} \left( 1 - \frac{\lambda_n}{2(\Lambda_n + \lambda_n)} - \frac{\lambda_n^2}{24(\Lambda_n + \lambda_n)^2} - \frac{\lambda_n^3}{48(\Lambda_n + \lambda_n)^3} \right)^p \\ & \quad \times \lambda_n (a_n)^p \Lambda_n^{p-1} \left( \sum_{k=1}^n \lambda_k (c_k a_k)^p \right)^{(1-p)/p}. \end{aligned}$$

The proof is almost the same as in proving Theorem 2.2. We here only need to note that

$$\left( 1 + \frac{1}{x} \right)^x < e \left( 1 - \frac{1}{2(1+x)} - \frac{1}{2(1+x)^2} - \frac{1}{2(1+x)^3} \right)$$

for  $x > 0$ , which proved in [11, Lemma 1].

**Corollary 2.5.** *Let  $a_n \geq 0 (n \in N)$ ,  $0 < p \leq 1$  and  $0 < \sum_{n=1}^{\infty} a_n < \infty$ . Then*

$$\begin{aligned} & \sum_{n=1}^{\infty} (a_1 a_2 \cdots a_n)^{1/n} \\ & < \frac{e^p}{p} \sum_{n=1}^{\infty} \left( 1 - \frac{1}{2(1+n)} - \frac{1}{24(1+n)^2} - \frac{1}{48(1+n)^3} \right)^p \\ & \quad \times (a_n)^p n^{p-1} \left( \sum_{k=1}^n (c_k a_k)^p \right)^{(1-p)/p}. \end{aligned}$$

where  $c_k = (1 + 1/k)^k \cdot k$ .

**Lemma 2.6.** *If  $a_1, a_2, \dots, a_n > 0$  and  $\alpha_1, \alpha_2, \dots, \alpha_n > 0$  with  $\sum_{i=1}^n \alpha_i = 1$ , then we have the following inequality:*

$$\left( \prod_{i=1}^n a_i^{\alpha_i} \right)^k \leq \left( \sum_{i=1}^n \alpha_i (a_i)^p \right)^{k/p}$$

for  $0 < k, p$  with the equality holding if and only if all  $a_i$  are same.

Note that Lemma 2.6 is easily deduced from the fact that  $M_p(a; \alpha)$  is a continuous strictly increasing function of  $p$ .

Now, we are ready to introduce the following new general strengthened Hardy's inequality.

**Theorem 2.7.** *Let  $0 < \lambda_{n+1} \leq \lambda_n$ ,  $\Lambda_n = \sum_{m=1}^n \lambda_m$  ( $\Lambda_n \geq 1$ ),  $a_n \geq 0$  ( $n \in N$ ) and  $0 < \sum_{n=1}^{\infty} \lambda_n (a_n)^t < \infty$  for  $0 < p \leq t < \infty$ . Then*

$$(2.6) \quad \sum_{n=1}^{\infty} \lambda_{n+1} (a_1^{\lambda_1} a_2^{\lambda_2} \dots a_n^{\lambda_n})^{t/\Lambda_n} < \frac{te^{p/t}}{p} \sum_{n=1}^{\infty} \left(1 - \frac{6\lambda_n}{12\Lambda_n + 11\lambda_n}\right)^{p/t} \\ \times \lambda_n (a_n)^p \Lambda_n^{(p-t)/t} \left(\sum_{k=1}^n \lambda_k c_k a_k\right)^{(t-p)/p}.$$

*Proof.* The proof is almost the same as in Theorem 2.2. By Lemma 2.6, we have

$$(\alpha_1^{q_1} \alpha_2^{q_2} \dots \alpha_n^{q_n})^t \leq \left(\sum_{m=1}^n q_m (\alpha_m)^p\right)^{t/p}, \quad p, t \geq 0,$$

where  $\alpha_m \geq 0$  and  $q_m > 0$  ( $m = 1, 2, \dots, n$ ) with  $\sum_{m=1}^n q_m = 1$ . Setting  $c_m > 0$ ,  $\alpha_m = c_m a_m$  and  $q_m = \lambda_m / \Lambda_n$ , we obtain

$$((c_1 a_1)^{\lambda_1/\Lambda_n} (c_2 a_2)^{\lambda_2/\Lambda_n} \dots (c_n a_n)^{\lambda_n/\Lambda_n})^t \leq \left(\frac{1}{\Lambda_n} \sum_{m=1}^n \lambda_m (c_m a_m)^p\right)^{t/p}.$$

Using the above inequality, we have

$$(2.7) \quad \sum_{n=1}^{\infty} \lambda_{n+1} (a_1^{\lambda_1} a_2^{\lambda_2} \dots a_n^{\lambda_n})^{t/\Lambda_n} \\ \leq \sum_{n=1}^{\infty} \left[ \frac{\lambda_{n+1}}{(c_1^{\lambda_1} c_2^{\lambda_2} \dots c_n^{\lambda_n})^{t/\Lambda_n}} \right] \frac{1}{\Lambda_n} \left(\sum_{m=1}^n \lambda_m (c_m a_m)^p\right)^{t/p}$$

for  $\Lambda_n \geq 1$  and  $t \geq p$ . By using the following inequality (see [2], [6]),

$$\left(\sum_{m=1}^n z_m\right)^t \leq t \sum_{m=1}^n z_m \left(\sum_{k=1}^m z_k\right)^{t-1},$$

where  $t \geq 1$  is constant and  $z_m \geq 0$  ( $m = 1, 2, \dots$ ), it is easy to observe that

$$(2.8) \quad \left(\sum_{m=1}^n \lambda_m (c_m a_m)^p\right)^{t/p} \leq \frac{t}{p} \sum_{m=1}^n \lambda_m (c_m a_m)^p \left(\sum_{k=1}^m \lambda_k (c_k a_k)^p\right)^{(t-p)/p}.$$



for  $\Lambda_n \geq 1$  and  $t \geq p$ . Then, by (2.7) and (2.8), we obtain

$$(2.9) \quad \sum_{n=1}^{\infty} \lambda_{n+1} (a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_n^{\lambda_n})^{t/\Lambda_n} \leq \sum_{n=1}^{\infty} \left[ \frac{\lambda_{n+1}}{(c_1^{\lambda_1} c_2^{\lambda_2} \cdots c_n^{\lambda_n})^{t/\Lambda_n}} \right] \frac{1}{\Lambda_n} \frac{t}{p} \\ \times \sum_{m=1}^n \lambda_m (c_m a_m)^p \left( \sum_{k=1}^m \lambda_k (c_k a_k)^p \right)^{(t-p)/p}.$$

Choosing  $c_1^{\lambda_1} c_2^{\lambda_2} \cdots c_n^{\lambda_n} = (\Lambda_{n+1})^{\Lambda_n/t}$  ( $n \in N$ ) and setting  $\Lambda_0 = 0$ , from  $\lambda_{n+1} \leq \lambda_n$ , we have

$$c_n = \left[ \frac{(\Lambda_{n+1})^{\Lambda_n}}{(\Lambda_n)^{\Lambda_{n-1}}} \right]^{1/t\lambda_n} = \left( 1 + \frac{\lambda_{n+1}}{\Lambda_n} \right)^{\Lambda_n/t\lambda_n} \cdot \Lambda_n^{1/t} \\ \leq \left( 1 + \frac{\lambda_n}{\Lambda_n} \right)^{\Lambda_n/t\lambda_n} \cdot \Lambda_n^{1/t}.$$

This implies that

$$\sum_{n=1}^{\infty} \lambda_{n+1} (a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_n^{\lambda_n})^{t/\Lambda_n} \\ \leq \frac{t}{p} \sum_{m=1}^{\infty} \left[ \left( 1 + \frac{1}{\Lambda_m/\lambda_m} \right)^{\Lambda_m/\lambda_m} \right]^{p/t} \lambda_m (a_m)^p \Lambda^{(p-t)/t} \left( \sum_{k=1}^m \lambda_k (c_k a_k)^p \right)^{(t-p)/p}.$$

Hence, by the above inequality and Lemma 2.1, we have

$$\sum_{n=1}^{\infty} \lambda_{n+1} (a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_n^{\lambda_n})^{t/\Lambda_n} < \frac{te^{p/t}}{p} \sum_{m=1}^{\infty} \left( 1 - \frac{6\lambda_m}{12\Lambda_m + 11\lambda_m} \right)^{p/t} \\ \times \lambda_m (a_m)^p \Lambda_m^{(p-t)/t} \left( \sum_{k=1}^m \lambda_k (c_k a_k)^p \right)^{(t-p)/p}.$$

Thus the inequality (2.6) is proved.

**Remark.** Setting  $t \equiv 1$  in Theorem 2.7, then from (2.6), we obtain the inequality (2.2) in Theorem 2.2. Hence the inequality (2.6) is a new generalization of Hardy's inequality.

Moreover, we can consider a generalization version of the inequality (2.5) as following theorem:

**Theorem 2.8.** *Let  $0 < \lambda_{n+1} \leq \lambda_n$ ,  $\Lambda_n = \sum_{m=1}^n \lambda_m$  ( $\Lambda_n \geq 1$ ),  $a_n \geq 0$  ( $n \in N$ ) and  $0 < \sum_{n=1}^{\infty} \lambda_n (a_n)^t < \infty$  for  $0 < p \leq t < \infty$ . Then*

$$\begin{aligned} & \sum_{n=1}^{\infty} \lambda_{n+1} (a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_n^{\lambda_n})^{t/\Lambda_n} \\ & < \frac{te^{p/t}}{p} \sum_{n=1}^{\infty} \left( 1 - \frac{\lambda_n}{2(\Lambda_n + \lambda_n)} - \frac{\lambda_n^2}{24(\Lambda_n + \lambda_n)^2} - \frac{\lambda_n^3}{48(\Lambda_n + \lambda_n)^3} \right)^{p/t} \\ & \quad \times \lambda_n (a_n)^p \Lambda_n^{(p-t)/t} \left( \sum_{k=1}^n \lambda_k c_k a_k \right)^{(t-p)/p}. \end{aligned}$$

*Proof.* The proof is similar to the proof of theorem 2.7.

Also assuming that  $\lambda_n = 1$  in the Theorem 2.7 and Theorem 2.8, we have further extension of the strengthened Carleman's inequality as following:

**Corollary 2.9.** *Let  $a_n \geq 0$  ( $n \in N$ ),  $0 < p \leq 1$  and  $0 < \sum_{n=1}^{\infty} a_n < \infty$ . Then*

$$\begin{aligned} & \sum_{n=1}^{\infty} (a_1 a_2 \cdots a_n)^{t/n} \\ & < \frac{te^{p/t}}{p} \sum_{n=1}^{\infty} \left( 1 - \frac{6}{12n + 11} \right)^{p/t} (a_n)^p n^{(p-t)/t} \left( \sum_{k=1}^n (c_k a_k)^p \right)^{(t-p)/p}. \end{aligned}$$

where  $c_k = (1 + 1/k)^k \cdot k$ .

**Corollary 2.10.** *Let  $a_n \geq 0$  ( $n \in N$ ),  $0 < p \leq 1$  and  $0 < \sum_{n=1}^{\infty} a_n < \infty$ . Then*

$$\begin{aligned} & \sum_{n=1}^{\infty} (a_1 a_2 \cdots a_n)^{t/n} \\ & < \frac{te^{p/t}}{p} \sum_{n=1}^{\infty} \left( 1 - \frac{1}{2(1+n)} - \frac{1}{24(1+n)^2} - \frac{1}{48(1+n)^3} \right)^{p/t} \\ & \quad \times (a_n)^p n^{(p-t)/t} \left( \sum_{k=1}^n (c_k a_k)^p \right)^{(t-p)/p}. \end{aligned}$$

where  $c_k = (1 + 1/k)^k \cdot k$ .

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