LOWER AND UPPER BOUNDS OF $\zeta(3)$

QIU-MING LUO, ZONG-LI WEI, AND FENG QI

Abstract. In this short note, using refinements of Jordan's inequality and an integral expression of $\zeta(3)$, the lower and upper bounds of $\zeta(3)$ are obtained, and some related results are improved.

1. Introduction

The Riemann zeta function can be defined by the integral

$$\zeta(x) = \frac{1}{\Gamma(x)} \int_0^\infty \frac{u^{x-1}}{e^u - 1} \, \mathrm{d}u, \tag{1}$$

where $x > 1$. If x is an integer n, we obtain the most common form of the function $\zeta(n)$, which is given by

$$\zeta(n) = \sum_{k=1}^{\infty} \frac{1}{k^n}. \tag{2}$$

For $n = 1$, the zeta function reduces to the harmonic series, which is divergent.

The Riemann zeta function can also be defined in terms of multiple integrals by

$$\zeta(n) = \underbrace{\int_0^1 \cdots \int_0^1}_{n} \frac{\prod_{i=1}^n \mathrm{d}x_i}{1 - \prod_{i=1}^n x_i}. \tag{3}$$

An additional identity is

$$\lim_{s \to 1} \zeta(s) - \frac{1}{s-1} = \gamma, \tag{4}$$

where γ is the Euler-Mascheroni constant.

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The Euler product formula can also be written as

$$\zeta(x) = \left[\prod_{i=2}^{\infty}(1 - i^{-x})\right]^{-1}. \tag{5}$$

The function $\zeta(n)$ was proved to be transcendental for all even n. For $n = 2k$ we have:

$$\zeta(n) = \frac{2^{n-1}\,|B_n|\,\pi^n}{n!}, \tag{6}$$

where B_n is the Bernoulli number. Another intimate connection with the Bernoulli numbers is provided by

$$B_n = (-1)^{n+1} n\zeta(1 - n). \tag{7}$$

The Riemann zeta function is related to the gamma function and it has important applications in mathematics, especially in the Analytic Number Theory.

The Riemann zeta function may be computed analytically for even n using either Contour integration or Parseval's theorem with the appropriate Fourier series.

No analytic form for $\zeta(n)$ is known for odd $n = 2k + 1$, but $\zeta(2k + 1)$ can be expressed as the sum limit

$$\zeta(2k+1) = \left(\frac{\pi}{2}\right)^{2k+1} \lim_{t\to\infty} \frac{1}{t^{2k+1}} \sum_{i=1}^{\infty}\left[\cot\left(\frac{i}{2t+1}\right)\right]^{2k+1}. \tag{8}$$

The values for the first few integral arguments are

$$\zeta(0) = -\frac{1}{2}, \qquad \zeta(1) = \infty, \qquad \zeta(2) = \frac{\pi^2}{6},$$
$$\zeta(3) = 1.2020569032\cdots, \qquad \zeta(4) = \frac{\pi^4}{90}, \qquad \zeta(5) = 1.0369277551\cdots.$$

In [9, p. 81], using the Jordan inequality $1 < \frac{x}{\sin x} \le \frac{\pi}{2}$ for $x \in (0, \frac{\pi}{2}]$ and two integral expressions

$$\sum_{i=0}^{\infty}\frac{1}{(2i+1)^3} = \frac{1}{4}\int_0^{\pi/2}\frac{x(\pi - x)}{\sin x}\,\mathrm{d}x, \tag{9}$$

$$\sum_{i=1}^{\infty}\frac{1}{i^3} = \frac{1}{6\pi}\int_0^{\pi}\left(\frac{x(\pi - x)}{\sin x}\right)^2\mathrm{d}x, \tag{10}$$

the following estimates were given

$$\frac{3\pi^2}{32} \le \sum_{i=0}^{\infty}\frac{1}{(2i+1)^3} \le \frac{3\pi^3}{64}, \qquad \sum_{i=1}^{\infty}\frac{1}{i^3} < \frac{29}{24}. \tag{11}$$

There is much literature devoted to evaluations and proofs of $\zeta(2) = \frac{\pi^2}{6}$. Please refer to the related references in this paper.

In this short note, using an integral expression of $\zeta(3)$ and refinements of the Jordan inequality, we obtain the following

Theorem 1. *The value of $\zeta(3)$ can be evaluated by the following*

$$\begin{aligned}
1.201\cdots = {} & \frac{1}{14\sqrt{5}}\Biggl\{\pi\sqrt{30\bigl(\sqrt{30}-5\bigr)}\ln\Bigl[\pi^2+4\pi\sqrt{5+\sqrt{30}}+8\sqrt{30}\Bigr] \\
& -\pi\sqrt{30\bigl(\sqrt{30}-5\bigr)}\ln\Bigl[\pi^2-4\pi\sqrt{5+\sqrt{30}}+8\sqrt{30}\Bigr] \\
& +2\pi\sqrt{30\bigl(5+\sqrt{30}\bigr)}\arctan\left[\frac{\pi+2\sqrt{\sqrt{30}+5}}{2\sqrt{\sqrt{30}-5}}\right] \\
& +2\pi\sqrt{30\bigl(5+\sqrt{30}\bigr)}\arctan\left[\frac{\pi-2\sqrt{\sqrt{30}+5}}{2\sqrt{\sqrt{30}-5}}\right] \\
& +120\arctan\left[\frac{\pi+2\sqrt{\sqrt{30}+5}}{2\sqrt{\sqrt{30}-5}}\right] \\
& -120\arctan\left[\frac{\pi-2\sqrt{\sqrt{30}+5}}{2\sqrt{\sqrt{30}-5}}\right]-240\arctan\sqrt{\frac{\sqrt{30}+5}{\sqrt{30}-5}}\Biggr\} \\
& <\zeta(3)=\sum_{i=0}^{\infty}\frac{1}{i^3}=\frac{8}{7}\sum_{i=0}^{\infty}\frac{1}{(2i+1)^3} \\
& <\frac{2}{7}\biggl\{3\ln(24-\pi^2)-9\ln 2-3\ln 3+\pi\sqrt{6}\operatorname{arctanh}\frac{\pi\sqrt{6}}{12}\biggr\} \\
& =7\biggl\{6\ln\frac{24-\pi^2}{4}-6\ln 6-\pi\sqrt{6}\ln\frac{12-\pi\sqrt{6}}{2}+\pi\sqrt{6}\ln\frac{12+\pi\sqrt{6}}{2}\biggr\} \\
& =1.217\cdots.
\end{aligned} \tag{12}$$

It is obvious that inequality (12) improves inequality (11).

2. Lower and upper bounds of $\zeta(3)$

It is well-known that, for $x \in [0, \frac{\pi}{2}]$, we have

$$x-\frac{1}{6}x^3 \le \sin x \le x-\frac{1}{6}x^3+\frac{1}{120}x^5. \tag{13}$$

The inequalities in (13) can be found in [7, 10, 11] and [9, p. 309]. The inequalities in (13) are a refinement of the well-known Jordan inequality $\frac{2}{\pi}x \le \sin x \le x$ for $x \in [0, \frac{\pi}{2}]$.

It is easy to see that

$$\sum_{i=0}^{\infty}\frac{1}{(2i+1)^3}=\frac{7}{8}\zeta(3). \tag{14}$$

From formula (9) and inequality (13), it follows by direct calculation that

$$
\begin{aligned}
1.051\cdots = \frac{1}{16\sqrt{5}}\Bigg\{&\pi\sqrt{30\big(\sqrt{30}-5\big)}\ln\Big[\pi^2+4\pi\sqrt{5+\sqrt{30}}+8\sqrt{30}\Big] \\
&-\pi\sqrt{30\big(\sqrt{30}-5\big)}\ln\Big[\pi^2-4\pi\sqrt{5+\sqrt{30}}+8\sqrt{30}\Big] \\
&+2\pi\sqrt{30\big(5+\sqrt{30}\big)}\arctan\left[\frac{\pi+2\sqrt{\sqrt{30}+5}}{2\sqrt{\sqrt{30}-5}}\right] \\
&+2\pi\sqrt{30\big(5+\sqrt{30}\big)}\arctan\left[\frac{\pi-2\sqrt{\sqrt{30}+5}}{2\sqrt{\sqrt{30}-5}}\right] \\
&+120\arctan\left[\frac{\pi+2\sqrt{\sqrt{30}+5}}{2\sqrt{\sqrt{30}-5}}\right] \\
&-120\arctan\left[\frac{\pi-2\sqrt{\sqrt{30}+5}}{2\sqrt{\sqrt{30}-5}}\right]-240\arctan\sqrt{\frac{\sqrt{30}+5}{\sqrt{30}-5}}\Bigg\} \qquad (15)\\
=\ &\frac{1}{4}\int_0^{\pi/2}\frac{\pi-x}{1-\frac{1}{6}x^2+\frac{1}{120}x^4}\,\mathrm{d}x \\
<\ &\sum_{i=0}^{\infty}\frac{1}{(2i+1)^3} \\
<\ &\frac{1}{4}\int_0^{\pi/2}\frac{\pi-x}{1-\frac{1}{6}x^2}\,\mathrm{d}x \\
=\ &\frac{3\ln(24-\pi^2)-9\ln2-3\ln3+\pi\sqrt{6}\operatorname{arctanh}\frac{\pi\sqrt{6}}{12}}{4} \\
=\ &\frac{6\ln\frac{24-\pi^2}{4}-6\ln6-\pi\sqrt{6}\ln\frac{12-\pi\sqrt{6}}{2}+\pi\sqrt{6}\ln\frac{12+\pi\sqrt{6}}{2}}{8} \\
=\ &1.0654\cdots.
\end{aligned}
$$

The proof of Theorem 1 follows from a combination of (14) with (15).

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(Luo) Department of Broadcast-Television-Teaching, Jiaozuo University, Jiaozuo City, Henan 454002, China

(Wei) Department of Mathematics, Luoyang Normal College, Luoyang City, Henan 471022, China

E-mail address: `weizl@lync.edu.cn`

(Qi) Department of Mathematics, Jiaozuo Institute of Technology, Jiaozuo City, Henan 454000, China

E-mail address: `qifeng@jzit.edu.cn`

URL: `http://rgmia.vu.edu.au/qi.html`