

APPROXIMATING THE FINITE HILBERT TRANSFORM VIA OSTROWSKI TYPE INEQUALITIES FOR ABSOLUTELY CONTINUOUS FUNCTIONS

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ABSTRACT. Some inequalities and approximations for the finite Hilbert transform by the use of Ostrowski type inequalities for absolutely continuous functions are given.

1. INTRODUCTION

Cauchy *principal value integrals* of the form

$$(1.1) \quad (Tf)(a, b; t) = PV \int_a^b \frac{f(\tau)}{\tau - t} d\tau := \lim_{\varepsilon \rightarrow 0^+} \left[\int_a^{t-\varepsilon} \frac{f(\tau)}{\tau - t} d\tau + \int_{t+\varepsilon}^b \frac{f(\tau)}{\tau - t} d\tau \right]$$

play an important role in fields like aerodynamics, the theory of elasticity and other areas of the engineering sciences. They are also helpful tools in some methods for finding the solutions of differential equations (cf., e.g. [25]).

For different approaches in approximating the finite Hilbert transform (1.1) including: interpolatory, noninterpolatory, Gaussian, Chebychevian and spline methods, see for example the papers [1] – [12], [16] – [24], [26] – [35] and the references therein.

In contrast with all these methods, we point out here a new method in approximating the finite Hilbert transform by the use of the Ostrowski inequalities for absolutely continuous functions established in [13], [14] and [15].

For a comprehensive list of papers on Ostrowski's inequality, visit the site <http://rgmia.vu.edu.au>.

Estimates for the error bounds and some numerical examples for the obtained approximation are also presented.

2. THE RESULTS

For the sake of completeness, we state and prove the following lemma providing some Ostrowski type inequalities for absolutely continuous functions (see [13], [14] and [15]).

Lemma 1. *Let $u : [a, b] \rightarrow \mathbb{R}$ be an absolutely continuous function on $[a, b]$. Then we have:*

$$(2.1) \quad \left| u(x)(b-a) - \int_a^b u(t) dt \right|$$

Date: May 28, 2001.

1991 Mathematics Subject Classification. Primary 26D15; Secondary 26D20.

Key words and phrases. Hilbert Transform, Cauchy principal value integrals, Ostrowski Inequality.

$$\leq \begin{cases} \left[\frac{1}{4} (b-a)^2 + \left(x - \frac{a+b}{2} \right)^2 \right] \|u'\|_{[a,b],\infty} & \text{if } u' \in L_\infty [a, b]; \\ \frac{1}{(q+1)^{\frac{1}{q}}} \left[(x-a)^{1+\frac{1}{q}} + (b-x)^{1+\frac{1}{q}} \right] \|u'\|_{[a,b],p} & \text{if } u' \in L_p [a, b], \\ & p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \left[\frac{1}{2} (b-a) + \left| x - \frac{a+b}{2} \right| \right] \|u'\|_{[a,b],1} & \text{if } u' \in L [a, b] \end{cases}$$

where $\|\cdot\|_r$ ($r \in [1, \infty]$) are the usual Lebesgue norms, i.e., for $c < d$

$$\|h\|_{[c,d],\infty} := \operatorname{ess\,sup}_{t \in [c,d]} |h(t)|$$

and

$$\|h\|_{[c,d],r} := \left(\int_c^d |h(t)|^r dt \right)^{\frac{1}{r}}, \quad r \geq 1.$$

Proof. Using the integration by parts formula, we have

$$\int_a^x (t-a) u'(t) dt = u(x)(x-a) - \int_a^x u(t) dt$$

and

$$\int_x^b (t-b) u'(t) dt = u(x)(b-x) - \int_x^b u(t) dt.$$

If we add the above two equalities, we get

$$(2.2) \quad u(x)(b-a) - \int_a^b u(t) dt = \int_a^x (t-a) u'(t) dt + \int_x^b (t-b) u'(t) dt$$

for any $x \in [a, b]$.

Taking the modulus, we have

$$\begin{aligned} \left| u(x)(b-a) - \int_a^b u(t) dt \right| &\leq \int_a^x (t-a) |u'(t)| dt + \int_x^b (t-b) |u'(t)| dt \\ &=: M(x). \end{aligned}$$

Now, it is obvious that

$$\begin{aligned} M(x) &\leq \|u'\|_{[a,x],\infty} \int_a^x (t-a) dt + \|u'\|_{[x,b],\infty} \int_x^b (b-t) dt \\ &= \|u'\|_{[a,x],\infty} \cdot \frac{(x-a)^2}{2} + \|u'\|_{[x,b],\infty} \cdot \frac{(b-x)^2}{2} \\ &\leq \|u'\|_{[a,b],\infty} \left[\frac{(x-a)^2 + (b-x)^2}{2} \right] \\ &= \|u'\|_{[a,b],\infty} \left[\frac{1}{4} (b-a)^2 + \left(x - \frac{a+b}{2} \right)^2 \right], \end{aligned}$$

proving the first part of (2.1).

Using Hölder's integral inequality, we may write:

$$\begin{aligned}
M(x) &\leq \|u'\|_{[a,x],p} \left(\int_a^x (t-a)^q dt \right)^{\frac{1}{q}} + \|u'\|_{[x,b],p} \left(\int_x^b (b-t)^q dt \right)^{\frac{1}{q}} \\
&= \|u'\|_{[a,x],p} \cdot \left[\frac{(x-a)^{q+1}}{q+1} \right]^{\frac{1}{q}} + \|u'\|_{[x,b],p} \cdot \left[\frac{(b-x)^{q+1}}{q+1} \right]^{\frac{1}{q}} \\
&\leq \|u'\|_{[a,b],p} \frac{1}{(q+1)^{\frac{1}{q}}} \left[(x-a)^{1+\frac{1}{q}} + (b-x)^{1+\frac{1}{q}} \right],
\end{aligned}$$

proving the second part of (2.1).

Finally, we observe that

$$\begin{aligned}
M(x) &\leq (x-a) \|u'\|_{[a,x],1} + (b-x) \|u'\|_{[x,b],1} \\
&\leq \max \{x-a, b-x\} \left[\|u'\|_{[a,x],1} + \|u'\|_{[x,b],1} \right] \\
&= \left[\frac{1}{2}(b-a) + \left| x - \frac{a+b}{2} \right| \right] \|u'\|_{[a,b],1}
\end{aligned}$$

and the lemma is proved. ■

The best inequalities we can get from (2.1) are embodied in the following corollary.

Corollary 1. *With the assumptions of Lemma 1, we have*

$$(2.3) \quad \left| u \left(\frac{a+b}{2} \right) (b-a) - \int_a^b u(t) dt \right| \leq \begin{cases} \frac{1}{4} (b-a)^2 \|u'\|_{[a,b],\infty} & \text{if } u' \in L_\infty[a,b]; \\ \frac{1}{2(q+1)^{\frac{1}{q}}} (b-a)^{1+\frac{1}{q}} \|u'\|_{[a,b],p} & \text{if } u' \in L_p[a,b], \\ & p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \frac{1}{2} (b-a) \|u'\|_{[a,b],1} & \text{if } u' \in L[a,b] \end{cases}$$

The following theorem providing an estimate for the finite Hilbert transform, holds.

Theorem 1. *Let $f : [a, b] \rightarrow \mathbb{R}$ be a function so that its derivative $f' : [a, b] \rightarrow \mathbb{R}$ is absolutely continuous on $[a, b]$. Then we have the inequalities*

$$(2.4) \quad \left| (Tf)(a, b; t) - \frac{f(t)}{\pi} \ln \left(\frac{b-t}{t-a} \right) - \frac{b-a}{\pi} [f; \lambda t + (1-\lambda)b, \lambda t + (1-\lambda)a] \right|$$

$$\leq \begin{cases} \frac{1}{\pi} \left[\frac{1}{4} + \left(\lambda - \frac{1}{2} \right)^2 \right] \\ \quad \times \left[\frac{1}{4} (b-a)^2 + \left(t - \frac{a+b}{2} \right)^2 \right] \|f''\|_{[a,b],\infty} & \text{if } f'' \in L_\infty[a,b]; \\ \\ \frac{1}{\pi} \frac{q}{(q+1)^{1+\frac{1}{q}}} \left[\lambda^{1+\frac{1}{q}} + (1-\lambda)^{1+\frac{1}{q}} \right] \\ \quad \times \left[(t-a)^{1+\frac{1}{q}} + (b-t)^{1+\frac{1}{q}} \right] \|f''\|_{[a,b],p} & \text{if } f'' \in L_p[a,b], \\ & p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \\ \frac{1}{\pi} \left[\frac{1}{2} + \left| \lambda - \frac{1}{2} \right| \right] \left[\frac{1}{2} (b-a) + \left| t - \frac{a+b}{2} \right| \right] \|f''\|_{[a,b],1}, & \text{if } f'' \in L[a,b] \end{cases}$$

for any $t \in (a, b)$ and $\lambda \in [0, 1)$, where $[f; \alpha, \beta]$ is the divided difference, i.e.,

$$[f; \alpha, \beta] = \frac{f(\alpha) - f(\beta)}{\alpha - \beta}.$$

Proof. Since f' is bounded on $[a, b]$, it follows that f is Lipschitzian on $[a, b]$ and thus the finite Hilbert transform exists everywhere in (a, b) . As for the function $f_0 : (a, b) \rightarrow \mathbb{R}$, $f_0(t) = 1$, $t \in (a, b)$, we have

$$(Tf_0)(a, b; t) = \frac{1}{\pi} \ln \left(\frac{b-t}{t-a} \right), \quad t \in (a, b),$$

then, obviously,

$$\begin{aligned} (Tf)(a, b; t) &= \frac{1}{\pi} PV \int_a^b \frac{f(\tau) - f(t) + f(t)}{\tau - t} d\tau \\ &= \frac{1}{\pi} PV \int_a^b \frac{f(\tau) - f(t)}{\tau - t} d\tau + \frac{f(t)}{\pi} PV \int_a^b \frac{d\tau}{\tau - t}, \end{aligned}$$

from where we get the identity

$$(2.5) \quad (Tf)(a, b; t) - \frac{f(t)}{\pi} \ln \left(\frac{b-t}{t-a} \right) = \frac{1}{\pi} PV \int_a^b \frac{f(\tau) - f(t)}{\tau - t} d\tau.$$

Now, if we choose in (2.1), $u = f'$, $x = \lambda c + (1-\lambda)d$, $\lambda \in [0, 1]$, $c, d \in [a, b]$ then we get

$$\begin{aligned} & |f(d) - f(c) - (d-c)f'(\lambda c + (1-\lambda)d)| \\ & \leq \begin{cases} (d-c)^2 \left[\frac{1}{4} + \left(\lambda - \frac{1}{2} \right)^2 \right] \|f''\|_{[c,d],\infty} & \text{if } f'' \in L_\infty[a,b]; \\ \\ \frac{|d-c|^{1+\frac{1}{q}}}{(q+1)^{\frac{1}{q}}} \left[\lambda^{1+\frac{1}{q}} + (1-\lambda)^{1+\frac{1}{q}} \right] \|f''\|_{[c,d],p} & \text{if } f'' \in L_p[a,b], \\ & p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \\ |d-c| \left[\frac{1}{2} + \left| \lambda - \frac{1}{2} \right| \right] \|f''\|_{[c,d],1}, \end{cases} \end{aligned}$$

which is equivalent to:

$$(2.6) \quad \left| \frac{f(d) - f(c)}{d - c} - f'(\lambda c + (1 - \lambda)d) \right| \leq \begin{cases} |d - c| \left[\frac{1}{4} + \left(\lambda - \frac{1}{2} \right)^2 \right] \|f''\|_{[c,d],\infty} & \text{if } f'' \in L_\infty[a, b]; \\ \frac{|d - c|^{\frac{1}{q}}}{(q + 1)^{\frac{1}{q}}} \left[\lambda^{1 + \frac{1}{q}} + (1 - \lambda)^{1 + \frac{1}{q}} \right] \|f''\|_{[c,d],p} & \text{if } f'' \in L_p[a, b], \\ & p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \left[\frac{1}{2} + \left| \lambda - \frac{1}{2} \right| \right] \|f''\|_{[c,d],1}. \end{cases}$$

Using (2.6), we may write

$$(2.7) \quad \left| \frac{1}{\pi} PV \int_a^b \frac{f(\tau) - f(t)}{\tau - t} d\tau - \frac{1}{\pi} PV \int_a^b f'(\lambda t + (1 - \lambda)\tau) d\tau \right| \leq \begin{cases} \frac{1}{\pi} \left[\frac{1}{4} + \left(\lambda - \frac{1}{2} \right)^2 \right] PV \int_a^b |t - \tau| \|f''\|_{[t,\tau],\infty} d\tau \\ \frac{1}{\pi} \cdot \frac{1}{(q + 1)^{\frac{1}{q}}} \left[\lambda^{1 + \frac{1}{q}} + (1 - \lambda)^{1 + \frac{1}{q}} \right] PV \int_a^b |t - \tau|^{\frac{1}{q}} \|f''\|_{[t,\tau],p} d\tau \\ \frac{1}{\pi} \left[\frac{1}{2} + \left| \lambda - \frac{1}{2} \right| \right] PV \int_a^b \|f''\|_{[t,\tau],1} d\tau \end{cases}$$

$$\leq \begin{cases} \frac{1}{\pi} \left[\frac{1}{4} + \left(\lambda - \frac{1}{2} \right)^2 \right] \|f''\|_{[a,b],\infty} PV \int_a^b |t - \tau| d\tau \\ \frac{1}{\pi} \cdot \frac{1}{(q + 1)^{\frac{1}{q}}} \left[\lambda^{1 + \frac{1}{q}} + (1 - \lambda)^{1 + \frac{1}{q}} \right] \|f''\|_{[a,b],p} PV \int_a^b |t - \tau|^{\frac{1}{q}} d\tau \\ \frac{1}{\pi} \left[\frac{1}{2} + \left| \lambda - \frac{1}{2} \right| \right] PV \left[\int_a^t \|f''\|_{[\tau,t],1} d\tau + \int_t^b \|f''\|_{[t,\tau],1} d\tau \right] \end{cases}$$

$$\leq \begin{cases} \frac{1}{\pi} \left[\frac{1}{4} + \left(\lambda - \frac{1}{2} \right)^2 \right] \|f''\|_{[a,b],\infty} \left[\frac{1}{4} (b - a)^2 + \left(t - \frac{a + b}{2} \right)^2 \right] \\ \frac{1}{\pi} \cdot \frac{1}{(q + 1)^{\frac{1}{q}}} \left[\lambda^{1 + \frac{1}{q}} + (1 - \lambda)^{1 + \frac{1}{q}} \right] \|f''\|_{[a,b],p} \\ \quad \times \frac{q}{(q + 1)} \left[(t - a)^{1 + \frac{1}{q}} + (b - t)^{1 + \frac{1}{q}} \right] \\ \frac{1}{\pi} \left[\frac{1}{2} + \left| \lambda - \frac{1}{2} \right| \right] \left[\frac{1}{2} (b - a) + \left| t - \frac{a + b}{2} \right| \right] \|f''\|_{[a,b],1}. \end{cases}$$

Since (note that $\lambda \neq 1$)

$$\begin{aligned}
& \frac{1}{\pi} PV \int_a^b f'(\lambda t + (1-\lambda)\tau) d\tau \\
&= \frac{1}{\pi} \lim_{\varepsilon \rightarrow 0^+} \left[\int_a^{t-\varepsilon} + \int_{t+\varepsilon}^b \right] (f'(\lambda t + (1-\lambda)\tau) d\tau) \\
&= \frac{1}{\pi} \lim_{\varepsilon \rightarrow 0^+} \left[\frac{1}{1-\lambda} f(\lambda t + (1-\lambda)\tau) \Big|_a^{t-\varepsilon} + \frac{1}{1-\lambda} f(\lambda t + (1-\lambda)\tau) \Big|_{t+\varepsilon}^b \right] \\
&= \frac{1}{\pi} \frac{f(t) - f(\lambda t + (1-\lambda)a) + f(\lambda t + (1-\lambda)a) - f(t)}{1-\lambda} \\
&= \frac{b-a}{\pi} [f; \lambda t + (1-\lambda)b, \lambda t + (1-\lambda)a],
\end{aligned}$$

then by (2.5) and (2.7) we deduce the desired inequality (2.4). ■

The best inequality one may obtain from (2.4) is embodied in the following corollary.

Corollary 2. *With the assumptions of Theorem 1, one has the inequality*

$$(2.8) \left| (Tf)(a, b; t) - \frac{f(t)}{\pi} \ln \left(\frac{b-t}{t-a} \right) - \frac{b-a}{\pi} \left[f; \frac{t+b}{2}, \frac{a+t}{2} \right] \right|$$

$$\leq \begin{cases} \frac{1}{4\pi} \left[\frac{1}{4} (b-a)^2 + \left(t - \frac{a+b}{2} \right)^2 \right] \|f''\|_{[a,b],\infty} & \text{if } f'' \in L_\infty[a, b]; \\ \frac{1}{\pi} \frac{q}{2^{\frac{p}{q}} (q+1)^{1+\frac{1}{q}}} \left[(t-a)^{1+\frac{1}{q}} + (b-t)^{1+\frac{1}{q}} \right] \|f''\|_{[a,b],p} & \text{if } f'' \in L_p[a, b], \\ & p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \frac{1}{2\pi} \left[\frac{1}{2} (b-a) + \left| t - \frac{a+b}{2} \right| \right] \|f''\|_{[a,b],1} & \text{if } f'' \in L[a, b]; \end{cases}$$

for any $t \in (a, b)$.

3. A QUADRATURE FORMULA

The following lemma is of interest in itself.

Lemma 2. *Let $u : [a, b] \rightarrow \mathbb{R}$ be an absolutely continuous function on $[a, b]$. Then for all $n \geq 1$, $\lambda_i \in [0, 1)$ ($i = 0, \dots, n-1$) and $t, \tau \in [a, b]$ with $t \neq \tau$, we have the*

inequality

$$\begin{aligned}
 (3.1) \quad & \left| \frac{1}{\tau-t} \int_t^\tau u(s) ds - \frac{1}{n} \sum_{i=0}^{n-1} u \left[t + (i+1-\lambda_i) \frac{\tau-t}{n} \right] \right| \\
 & \leq \begin{cases} \frac{|t-\tau|}{n} \left[\frac{1}{4} + \frac{1}{n} \sum_{i=0}^{n-1} \left(\lambda_i - \frac{1}{2} \right)^2 \right] \|u'\|_{[t,\tau],\infty}; \\ \frac{|t-\tau|^{\frac{1}{q}}}{(q+1)^{\frac{1}{q}} n} \left[\frac{1}{n} \sum_{i=0}^{n-1} \left(\lambda_i^{1+\frac{1}{q}} + (1-\lambda_i)^{1+\frac{1}{q}} \right)^{q-1} \right]^{\frac{1}{q}} \|u'\|_{[t,\tau],p} \\ \hspace{15em} p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \frac{1}{n} \left[\frac{1}{2} + \max \left| \lambda_i - \frac{1}{2} \right| \right] \|u'\|_{[t,\tau],1}; \end{cases} \\
 & \leq \begin{cases} \frac{|t-\tau|}{2n} \|u'\|_{[t,\tau],\infty}; \\ \frac{|t-\tau|^{\frac{1}{q}}}{(q+1)^{\frac{1}{q}} n} \|u'\|_{[t,\tau],p} \quad p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \frac{1}{n} \|u'\|_{[t,\tau],1}, \end{cases}
 \end{aligned}$$

where

$$\|u'\|_{[t,\tau],p} := \left| \int_t^\tau |u'(s)|^p ds \right|^{\frac{1}{p}}, \quad p \geq 1$$

and

$$\|u'\|_{[t,\tau],\infty} := \operatorname{ess\,sup}_{\substack{s \in [t,\tau] \\ (s \in [\tau,t])}} |u'(s)|.$$

Proof. Consider the equidistant division of $[t, \tau]$ (if $t < \tau$) or $[\tau, t]$ (if $\tau < t$) given by

$$(3.2) \quad E_n : x_i = t + i \cdot \frac{\tau-t}{n}, \quad i = \overline{0, n}.$$

Then the points $\xi_i := \lambda_i \left[t + i \cdot \frac{\tau-t}{n} \right] + (1-\lambda_i) \left[t + (i+1) \cdot \frac{\tau-t}{n} \right]$ ($\lambda_i \in [0, 1)$, $i = \overline{0, n-1}$) are between x_i and x_{i+1} . We observe that we may write for simplicity $\xi_i = t + (i+1-\lambda_i) \frac{\tau-t}{n}$ ($i = \overline{0, n-1}$). We also have

$$\begin{aligned}
 \xi_i - \frac{x_i + x_{i+1}}{2} &= \frac{\tau-t}{n} \left(\frac{1}{2} - \lambda_i \right); \\
 \xi_i - x_i &= (1-\lambda_i) \frac{\tau-t}{n}
 \end{aligned}$$

and

$$x_{i+1} - \xi_i = \lambda_i \cdot \frac{\tau-t}{n}$$

for any $i = \overline{0, n-1}$.

If we apply the inequality (2.1) on the interval $[x_i, x_{i+1}]$ and the intermediate points ξ_i ($i = 0, n-1$), then we may write that

$$(3.3) \quad \left| \frac{\tau-t}{n} u \left[t + (i+1-\lambda_i) \frac{\tau-t}{n} \right] - \int_{x_i}^{x_{i+1}} u(s) ds \right|$$

$$(3.4) \quad \leq \begin{cases} \left[\frac{1}{4} \frac{(t-\tau)^2}{n^2} + \frac{(t-\tau)^2}{4n^2} (1-2\lambda_i)^2 \right] \|u'\|_{[x_i, x_{i+1}], \infty} & \text{if } u' \in L_\infty[a, b], \\ \frac{1}{(q+1)^{\frac{1}{q}}} \left[\frac{|t-\tau|^{1+\frac{1}{q}}}{n^{1+\frac{1}{q}}} \lambda_i^{1+\frac{1}{q}} + \frac{|t-\tau|^{1+\frac{1}{q}}}{n^{1+\frac{1}{q}}} (1-\lambda_i)^{1+\frac{1}{q}} \right] \|u'\|_{[x_i, x_{i+1}], p} & \text{if } u' \in L_p[a, b], \\ \left[\frac{1}{2} \frac{|\tau-t|}{n} + \frac{|\tau-t|}{n} \left| \frac{1}{2} - \lambda_i \right| \right] \|u'\|_{[t, \tau], 1} & p > 1, \frac{1}{p} + \frac{1}{q} = 1; \end{cases}$$

Summing (2.3), we get:

$$(3.5) \quad \left| \int_t^\tau u(s) ds - \frac{\tau-t}{n} \sum_{i=0}^{n-1} u \left[t + (i+1-\lambda_i) \frac{\tau-t}{n} \right] \right| \\ \leq \begin{cases} \frac{(t-\tau)^2}{n^2} \sum_{i=0}^{n-1} \left[\frac{1}{4} + \left(\lambda_i - \frac{1}{2} \right)^2 \right] \|u'\|_{[x_i, x_{i+1}], \infty}; \\ \frac{|t-\tau|^{1+\frac{1}{q}}}{(q+1)^{\frac{1}{q}} n^{1+\frac{1}{q}}} \sum_{i=0}^{n-1} \left[\lambda_i^{1+\frac{1}{q}} + (1-\lambda_i)^{1+\frac{1}{q}} \right] \|u'\|_{[x_i, x_{i+1}], p} & p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \frac{|t-\tau|}{n} \sum_{i=0}^{n-1} \left[\frac{1}{2} + \left| \lambda_i - \frac{1}{2} \right| \right] \|u'\|_{[x_i, x_{i+1}], 1}. \end{cases}$$

However,

$$(3.6) \quad \sum_{i=0}^{n-1} \left[\frac{1}{4} + \left(\lambda_i - \frac{1}{2} \right)^2 \right] \|u'\|_{[x_i, x_{i+1}], \infty} \\ = \|u'\|_{[t, \tau], \infty} \left[\frac{1}{4} n + \sum_{i=0}^{n-1} \left(\lambda_i - \frac{1}{2} \right)^2 \right],$$

$$(3.7) \quad \sum_{i=0}^{n-1} \left[\lambda_i^{1+\frac{1}{q}} + (1-\lambda_i)^{1+\frac{1}{q}} \right] \|u'\|_{[x_i, x_{i+1}], p} \\ \leq \left(\sum_{i=0}^{n-1} \left[\lambda_i^{1+\frac{1}{q}} + (1-\lambda_i)^{1+\frac{1}{q}} \right]^q \right)^{\frac{1}{q}} \left(\sum_{i=0}^{n-1} \|u'\|_{[x_i, x_{i+1}], p}^p \right)^{\frac{1}{p}} \\ = \|u'\|_{[t, \tau], p} \left[\sum_{i=0}^{n-1} \left(\lambda_i^{1+\frac{1}{q}} + (1-\lambda_i)^{1+\frac{1}{q}} \right)^q \right]^{\frac{1}{q}}$$

and

$$(3.8) \quad \sum_{i=0}^{n-1} \left[\frac{1}{2} + \left| \lambda_i - \frac{1}{2} \right| \right] \|u'\|_{[x_i, x_{i+1}], 1} \\ \leq \left[\frac{1}{2} + \max \left| \lambda_i - \frac{1}{2} \right| \right] \sum_{i=0}^{n-1} \|u'\|_{[x_i, x_{i+1}], 1} = \left[\frac{1}{2} + \max \left| \lambda_i - \frac{1}{2} \right| \right] \|u'\|_{[t, \tau], 1}.$$

Now, using (3.5) – (3.8), we deduce the first part of (3.1).

The second part is obvious. ■

We may now state the following theorem in approximating the finite Hilbert transform of a differentiable function whose derivative is absolutely continuous.

Theorem 2. *Let $f : [a, b] \rightarrow \mathbb{R}$ be a differentiable function so that its derivative f' is absolutely continuous on $[a, b]$. If $\lambda = (\lambda_i)_{i=0, n-1}$, $\lambda_i \in [0, 1]$ ($i = 0, n-1$) and*

$$(3.9) \quad S_n(f; \lambda, t) := \frac{b-a}{n\pi} \sum_{i=0}^{n-1} \left[f; t + (i+1-\lambda_i) \frac{b-t}{n}, t - (i+1-\lambda_i) \frac{t-a}{n} \right]$$

then we have

$$(3.10) \quad (Tf)(a, b; t) = \frac{f(t)}{\pi} \ln \left(\frac{b-t}{t-a} \right) + S_n(f; \lambda, t) + R_n(f; \lambda, t)$$

and the remainder $R_n(f; \lambda, t)$ satisfies the estimate:

$$(3.11) \quad |R_n(f; \lambda, t)| \\ \leq \frac{1}{\pi} \times \begin{cases} \frac{1}{n} \left[\frac{1}{4} + \frac{1}{n} \sum_{i=0}^{n-1} \left(\lambda_i - \frac{1}{2} \right)^2 \right] \\ \quad \times \left[\frac{1}{4} (b-a)^2 + \left(t - \frac{a+b}{2} \right)^2 \right] \|f''\|_{[a, b], \infty} & \text{if } f'' \in L_\infty[a, b]; \\ \frac{1}{n} \cdot \frac{q}{(q+1)^{\frac{1}{q}+1}} \left[\frac{1}{n} \sum_{i=0}^{n-1} \left(\lambda_i^{1+\frac{1}{q}} + (1-\lambda_i)^{1+\frac{1}{q}} \right)^q \right]^{\frac{1}{q}} \\ \quad \times \left[(t-a)^{1+\frac{1}{q}} + (b-t)^{1+\frac{1}{q}} \right] \|f''\|_{[a, b], p} & \text{if } f'' \in L_p[a, b], \\ & p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \frac{1}{n} \left[\frac{1}{2} + \max \left| \lambda_i - \frac{1}{2} \right| \right] \left[\frac{1}{2} (b-a) + \left| t - \frac{a+b}{2} \right| \right] \|f''\|_{[a, b], 1}, \end{cases}$$

$$(3.12) \quad \leq \frac{1}{\pi} \times \begin{cases} \frac{1}{4n} \|f''\|_{[a, b], \infty} (b-a)^2 & \text{if } f'' \in L_\infty[a, b]; \\ \frac{q}{(q+1)^{\frac{1}{q}+1} n} (b-a)^{1+\frac{1}{q}} \|f''\|_{[a, b], p} & \text{if } f'' \in L_p[a, b], p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \frac{1}{n} (b-a) \|f''\|_{[a, b], 1}, \end{cases}$$

Proof. Applying Lemma 2 for the function f' , we may write that

$$(3.13) \quad \left| \frac{f(t) - f(\tau)}{\tau - t} - \frac{1}{n} \sum_{i=0}^{n-1} f' \left[t + (i+1 - \lambda_i) \frac{\tau - t}{n} \right] \right|$$

$$\leq \begin{cases} \frac{|t - \tau|}{n} \left[\frac{1}{4} + \frac{1}{n} \sum_{i=0}^{n-1} \left(\lambda_i - \frac{1}{2} \right)^2 \right] \|f''\|_{[t, \tau], \infty}; \\ \frac{|t - \tau|^{\frac{1}{q}}}{(q+1)^{\frac{1}{q}} n} \left[\frac{1}{n} \sum_{i=0}^{n-1} \left(\lambda_i^{1+\frac{1}{q}} + (1 - \lambda_i)^{1+\frac{1}{q}} \right)^q \right]^{\frac{1}{q}} \|f''\|_{[t, \tau], p}; \\ \frac{1}{n} \left[\frac{1}{2} + \max \left| \lambda_i - \frac{1}{2} \right| \right] \|f''\|_{[t, \tau], 1} \end{cases}$$

for any $t, \tau \in [a, b]$, $t \neq \tau$.

Taking the PV, we may write

$$(3.14) \quad \left| \frac{1}{\pi} PV \int_a^b \frac{f(t) - f(\tau)}{\tau - t} d\tau - \frac{1}{n\pi} \sum_{i=0}^{n-1} PV \int_a^b f' \left[t + (i+1 - \lambda_i) \frac{\tau - t}{n} \right] d\tau \right|$$

$$(3.15) \quad \leq \frac{1}{\pi} \times \begin{cases} \frac{1}{n} \left[\frac{1}{4} + \frac{1}{n} \sum_{i=0}^{n-1} \left(\lambda_i - \frac{1}{2} \right)^2 \right] PV \int_a^b |t - \tau| \|f''\|_{[t, \tau], \infty} d\tau; \\ \frac{1}{n(q+1)^{\frac{1}{q}}} \left[\frac{1}{n} \sum_{i=0}^{n-1} \left(\lambda_i^{1+\frac{1}{q}} + (1 - \lambda_i)^{1+\frac{1}{q}} \right)^q \right]^{\frac{1}{q}} PV \int_a^b |t - \tau|^{\frac{1}{q}} \|f''\|_{[t, \tau], p} d\tau; \\ \frac{1}{n} \left[\frac{1}{2} + \max \left| \lambda_i - \frac{1}{2} \right| \right] PV \int_a^b \|f''\|_{[t, \tau], 1} d\tau. \end{cases}$$

However,

$$PV \int_a^b |t - \tau| \|f''\|_{[t, \tau], \infty} d\tau \leq \|f''\|_{[a, b], \infty} PV \int_a^b |t - \tau| d\tau$$

$$= \|f''\|_{[a, b], \infty} \left[\frac{(t-a)^2 + (b-t)^2}{2} \right],$$

$$PV \int_a^b |t - \tau|^{\frac{1}{q}} \|f''\|_{[t, \tau], p} d\tau \leq \|f''\|_{[a, b], p} PV \int_a^b |t - \tau|^{\frac{1}{q}} d\tau$$

$$= \|f''\|_{[a, b], p} \left[\frac{(t-a)^{\frac{1}{q}+1} + (b-t)^{\frac{1}{q}+1}}{\frac{1}{q} + 1} \right]$$

$$= \frac{q}{(q+1)} \left[(t-a)^{\frac{1}{q}+1} + (b-t)^{\frac{1}{q}+1} \right] \|f''\|_{[a, b], p},$$

$$\begin{aligned}
PV \int_a^b \|f''\|_{[t,\tau],1} d\tau &= PV \left[\int_a^t \|f''\|_{[\tau,t],1} d\tau + \int_t^b \|f''\|_{[t,\tau],1} d\tau \right] \\
&\leq \max\{t-a, b-t\} \|f''\|_{[a,b],1} \\
&= \left[\frac{1}{2}(b-a) + \left| t - \frac{a+b}{2} \right| \right] \|f''\|_{[a,b],1}
\end{aligned}$$

and using the inequality (3.14) we obtain the desired estimate (3.11). ■

The following particular case which may be easily numerically implemented holds.

Corollary 3. *let f be as in Theorem 2. Define*

$$S_{M,n}(f; t) := \frac{b-a}{n\pi} \sum_{i=0}^{n-1} \left[f; t + \left(i + \frac{1}{2}\right) \frac{b-t}{n}, t - \left(i + \frac{1}{2}\right) \frac{t-a}{n} \right]$$

and the remainder $R_{M,n}(f; t)$ satisfies the estimate

$$(Tf)(a, b; t) = \frac{f(t)}{\pi} \ln \left(\frac{b-t}{t-a} \right) + S_{M,n}(f; t) + R_{M,n}(f; t)$$

Then we have the representation:

$$\begin{aligned}
& |R_{M,n}(f; t)| \\
& \leq \frac{1}{\pi} \times \begin{cases} \frac{1}{4n\pi} \left[\frac{1}{4}(b-a)^2 + \left(t - \frac{a+b}{2}\right)^2 \right] \|f''\|_{[a,b],\infty} & \text{if } f'' \in L_\infty[a, b]; \\ \frac{1}{n} \cdot \frac{1}{2^{\frac{1}{q}} \pi n (q+1)^{\frac{1}{q}+1}} \left[(t-a)^{1+\frac{1}{q}} + (b-t)^{1+\frac{1}{q}} \right] \|f''\|_{[a,b],p} & \text{if } f'' \in L_p[a, b], \\ & p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \frac{1}{2\pi n} \left[\frac{1}{2}(b-a) + \left| t - \frac{a+b}{2} \right| \right] \|f''\|_{[a,b],1}, & \end{cases}
\end{aligned}$$

for any $t \in (a, b)$.

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