SHARP ERROR BOUNDS OF A QUADRATURE RULE WITH ONE MULTIPLE NODE FOR THE FINITE HILBERT TRANSFORM IN SOME CLASSES OF CONTINUOUS DIFFERENTIABLE FUNCTIONS

S.S. DRAGOMIR

Abstract. Some inequalities and approximations for the finite Hilbert transform by the use of Taylor's formula with the integral remainder are given.

1. Introduction

Cauchy principal value integrals of the form

$$(Tf)(a,b;t) = PV\int_a^b \frac{f(\tau)}{\tau - t}d\tau := \lim_{\varepsilon\to 0+}\left[\int_a^{t-\varepsilon}\frac{f(\tau)}{\tau - t}d\tau + \int_{t+\varepsilon}^b \frac{f(\tau)}{\tau - t}d\tau\right] \tag{1.1}$$

play an important role in fields like aerodynamics, the theory of elasticity and other areas of the engineering sciences. They are also helpful tools in some methods for the solution of differential equations (cf., e.g. [22]).

For different approaches in approximating the finite Hilbert transform (1.1) including: interpolatory, noninterpolatory, Gaussian, Chebychevian and spline methods, see for example the papers [1] – [12], [13] – [21], [23] – [32] and the references therein.

In contrast with all these methods, we point out here a new method in approximating the finite Hilbert transform by the use of the Taylor formula with integral remainder for functions whose n-th derivatives are absolutely continuous.

For a comprehensive list of papers on generalized Taylor formulae and Ostrowski's inequality, visit the site `http://rgmia.vu.edu.au`.

Estimates for the error bounds and some numerical examples for the obtained approximation are also presented.

2. Inequalities on the Whole Interval $[a,b]$

The following result holds.

Theorem 1. *Let $f:[a,b]\to\mathbb{R}$ be such that $f^{(n-1)}$ $(n\geq 1)$ is absolutely continuous on $[a,b]$. Then we have the bounds:*

(2.1)

$$\left|(Tf)(a,b;t) - f(t)\ln\left(\frac{b-t}{t-a}\right) - \sum_{k=1}^{n-1}\frac{f^{(k)}(t)}{k!}\cdot\left[\frac{(b-t)^k + (-1)^{k+1}(t-a)^k}{k}\right]\right|$$

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$$\leq \begin{cases} \dfrac{\left\|f^{(n)}\right\|_{[a,b],\infty}}{n\cdot n!}\left[(b-t)^{n}+(t-a)^{n}\right], & \textit{if } \ f^{(n)}\in L_{\infty}[a,b]; \\ \dfrac{q\left\|f^{(n)}\right\|_{[a,b],p}\left[(b-t)^{n-1+\frac{1}{q}}+(t-a)^{n-1+\frac{1}{q}}\right]}{(n-1)!\left[(n-1)q+1\right]^{1+\frac{1}{q}}}, & \textit{if } \ f^{(n)}\in L_{p}[a,b], \\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \dfrac{\left\|f^{(n)}\right\|_{[a,b],1}}{(n-1)\cdot(n-1)!}\left[(b-t)^{n-1}+(t-a)^{n-1}\right], & \end{cases}$$

$$\leq \begin{cases} \dfrac{(b-a)^{n}}{n\cdot n!}\left\|f^{(n)}\right\|_{[a,b],\infty}, & \textit{if } \ f^{(n)}\in L_{\infty}[a,b]; \\ \dfrac{q(b-a)^{n-1+\frac{1}{q}}}{(n-1)!\left[(n-1)q+1\right]^{1+\frac{1}{q}}}\left\|f^{(n)}\right\|_{[a,b],p}, & \textit{if } \ f^{(n)}\in L_{p}[a,b], \\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \dfrac{1}{(n-1)\cdot(n-1)!}(b-a)^{n-1}\left\|f^{(n)}\right\|_{[a,b],1}, & \end{cases}$$

for any $t\in(a,b)$.

Proof. Start with Taylor's formula for a function $g:I\to\mathbb{R}$ (I is a compact interval) with the property that $g^{(n-1)}$ $(n\geq 1)$ is absolutely continuous on I, then we have

$$g(x)=\sum_{k=0}^{n-1}\frac{(x-a)^{k}}{k!}g^{(k)}(a)+\frac{1}{(n-1)!}\int_{a}^{x}(x-t)^{n-1}g^{(n)}(t)\,dt,$$

where $a,x\in\mathring{I}$ ($\mathring{I}$is the interior of I). This implies that

$$\begin{aligned}\left|g(x)-\sum_{k=0}^{n-1}\frac{(x-a)^{k}}{k!}g^{(k)}(a)\right| &\leq \frac{1}{(n-1)!}\left|\int_{a}^{x}|x-t|^{n-1}\left|g^{(n)}(t)\right|dt\right| \\ &=: \frac{1}{(n-1)!}\cdot M(x)\end{aligned}$$

for any $a,x\in\mathring{I}$.

Before we estimate $M(x)$, let us introduce the following notations

$$\|h\|_{[a,x],p}:=\left|\int_{a}^{x}|h(t)|^{p}\,dt\right|^{\frac{1}{p}}\quad\text{if}\ \ p\geq 1$$

and

$$\|h\|_{[a,x],\infty}:=ess\sup_{\substack{t\in[a,x]\\ t\in([x,a])}}|h(t)|,$$

where $a,x\in\mathring{I}$.

It is obvious now that

$$M(x)\leq\sup_{\substack{t\in[a,x]\\ (t\in[x,a])}}\left|g^{(n)}(t)\right|\left|\int_{a}^{x}|x-t|^{n-1}\,dt\right|=\left\|g^{(n)}\right\|_{[a,x],\infty}\frac{|x-a|^{n}}{n!}$$

for any $a,x\in\mathring{I}$.

Using Hölder's integral inequality, we may state that

$$\begin{aligned} M(x) &\leq \left|\int_a^x \left|g^{(n)}(t)\right|^p dt\right|^{\frac{1}{p}} \left|\int_a^x |x-t|^{(n-1)q}\, dt\right|^{\frac{1}{q}} \\ &= \left\|g^{(n)}\right\|_{[a,x],p} \frac{|x-a|^{n-1+\frac{1}{q}}}{[(n-1)q+1]^{\frac{1}{q}}} \end{aligned}$$

for any $a, x \in \mathring{I}$.

Also, we observe that

$$M(x) \leq |x-a|^{n-1} \left|\int_a^x \left|g^{(n)}(t)\right| dt\right| = \left\|g^{(n)}\right\|_{[a,x],1} |x-a|^{n-1}$$

for all $a, x \in \mathring{I}$.

In conclusion, we may state the following inequality which will be used in the sequel

$$\left|g(x) - \sum_{k=0}^{n-1} \frac{(x-a)^k}{k!} g^{(k)}(a)\right| \leq \begin{cases} \dfrac{|x-a|^n}{n!} \left\|g^{(n)}\right\|_{[a,x],\infty} & \text{if } \ g^{(n)} \in L_\infty(\mathring{\mathrm{I}}); \\ \dfrac{|x-a|^{n-1+\frac{1}{q}}}{(n-1)!\,[(n-1)q+1]^{\frac{1}{q}}} \left\|g^{(n)}\right\|_{[a,x],p} & \text{if } \ g^{(n)} \in L_p(\mathring{\mathrm{I}}), \\ & \quad p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \dfrac{|x-a|^{n-1}}{(n-1)!} \left\|g^{(n)}\right\|_{[a,x],1} & \end{cases} \tag{2.2}$$

for any $a, x \in \mathring{\mathrm{I}}$.

Now, let us note for the function $f_0 : [a,b] \to \mathbb{R}$, $f_0(t) = 1$, we have that

$$(Tf_0)(a,b;t) = \ln\left(\frac{b-t}{t-a}\right),\ t \in (a,b),$$

and then

$$\begin{aligned} (Tf)(a,b;t) &= PV \int_a^b \frac{f(\tau) - f(t) + f(t)}{\tau - t} d\tau \\ &= PV \int_a^b \frac{f(\tau) - f(t)}{\tau - t} d\tau + f(t) \ln\left(\frac{b-t}{t-a}\right), \end{aligned}$$

giving the equality

$$(Tf)(a,b;t) - f(t) \ln\left(\frac{b-t}{t-a}\right) = PV \int_a^b \frac{f(\tau) - f(t)}{\tau - t} d\tau. \tag{2.3}$$

Writing (2.2) for $g=f$, $x=\tau$, $a=t$, we get

$$\left|\frac{f(\tau)-f(t)}{\tau-t}-\sum_{k=1}^{n-1}\frac{(\tau-t)^{k-1}}{k!}f^{(k)}(t)\right| \tag{2.4}$$
$$\leq\begin{cases}\dfrac{|\tau-t|^{n-1}}{n!}\left\|f^{(n)}\right\|_{[t,\tau],\infty} & \text{if } f^{(n)}\in L_{\infty}[a,b];\\[2ex] \dfrac{|\tau-t|^{n-2+\frac{1}{q}}}{(n-1)!\left[(n-1)q+1\right]^{\frac{1}{q}}}\left\|f^{(n)}\right\|_{[t,\tau],p} & \text{if } f^{(n)}\in L_p[a,b],\\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1;\\[2ex] \dfrac{|\tau-t|^{n-2}}{(n-1)!}\left\|f^{(n)}\right\|_{[t,\tau],1}\end{cases}$$

for any $t,\tau\in(a,b)$, $t\neq\tau$.

If we take the PV in (2.4), then we may write

$$\left|PV\int_a^b\frac{f(\tau)-f(t)}{\tau-t}d\tau-\sum_{k=1}^{n-1}\frac{f^{(k)}(t)}{k!}PV\int_a^b(\tau-t)^{k-1}d\tau\right| \tag{2.5}$$
$$\leq PV\int_a^b\left|\frac{f(\tau)-f(t)}{\tau-t}-\sum_{k=1}^{n-1}\frac{(\tau-t)^{k-1}}{k!}f^{(k)}(t)\right|d\tau$$
$$\leq\begin{cases}\dfrac{1}{n!}PV\int_a^b|\tau-t|^{n-1}\left\|f^{(n)}\right\|_{[t,\tau],\infty}d\tau\\[2ex] \dfrac{1}{(n-1)!\left[(n-1)q+1\right]^{\frac{1}{q}}}PV\int_a^b|\tau-t|^{n-2+\frac{1}{q}}\left\|f^{(n)}\right\|_{[t,\tau],p}d\tau\\[2ex] \dfrac{1}{(n-1)!}PV\int_a^b|\tau-t|^{n-2}\left\|f^{(n)}\right\|_{[t,\tau],1}d\tau\end{cases}$$
$$\leq\begin{cases}\dfrac{1}{n!}\left\|f^{(n)}\right\|_{[a,b],\infty}PV\int_a^b|\tau-t|^{n-1}d\tau\\[2ex] \dfrac{1}{(n-1)!\left[(n-1)q+1\right]^{\frac{1}{q}}}\left\|f^{(n)}\right\|_{[a,b],p}PV\int_a^b|\tau-t|^{n-2+\frac{1}{q}}d\tau\\[2ex] \dfrac{1}{(n-1)!}\left\|f^{(n)}\right\|_{[a,b],1}PV\int_a^b|\tau-t|^{n-2}d\tau.\end{cases}$$

However,

$$PV\int_a^b|\tau-t|^{n-1}d\tau=\frac{1}{n}\left[(b-t)^n+(t-a)^n\right],$$

$$PV\int_a^b|\tau-t|^{n-2+\frac{1}{q}}d\tau=\frac{q}{\left[(n-1)q+1\right]}\left[(b-t)^{n-1+\frac{1}{q}}+(t-a)^{n-1+\frac{1}{q}}\right],$$

$$PV\int_a^b|\tau-t|^{n-2}d\tau=\frac{1}{n-1}\left[(b-t)^{n-1}+(t-a)^{n-1}\right]$$

and

$$PV\int_a^b (\tau - t)^{k-1}\, d\tau = \frac{1}{k}\left[(b-t)^k + (-1)^{k+1}(t-a)^k\right]$$

and then by (2.5) we deduce the desired result (2.1). ∎

It is obvious that the best inequality one would deduce from (2.1) is the one for $t = \frac{a+b}{2}$, getting the following corollary.

Corollary 1. *With the assumptions of Theorem 1, we have*

$$\left|(Tf)\left(a,b;\frac{a+b}{2}\right) - \sum_{k=1}^{n-1}\frac{(b-a)^k}{2^k\cdot k\cdot k!}\left[1+(-1)^{k+1}\right]f^{(k)}\left(\frac{a+b}{2}\right)\right| \tag{2.6}$$

$$\leq \begin{cases} \dfrac{(b-a)^n}{2^{n-1}\cdot n\cdot n!}\left\|f^{(n)}\right\|_{[a,b],\infty}, & \text{if } f^{(n)}\in L_\infty[a,b]; \\ \dfrac{q(b-a)^{n-1+\frac{1}{q}}}{2^{n-2+\frac{1}{q}}(n-1)!\left[(n-1)q+1\right]^{1+\frac{1}{q}}}\left\|f^{(n)}\right\|_{[a,b],p}, & \text{if } f^{(n)}\in L_p[a,b], \\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \dfrac{(b-a)^{n-1}}{2^{n-2}\cdot(n-1)\cdot(n-1)!}\left\|f^{(n)}\right\|_{[a,b],1}. \end{cases}$$

It is important to note that for small intervals, we basically have the following representation:

Corollary 2. *Assume that $f\in C^\infty[a,b]$ and $0<b-a\leq 1$. Then*

$$(Tf)(a,b;t) = f(t)\ln\left(\frac{b-t}{t-a}\right) + \sum_{k=1}^{\infty}\frac{f^{(k)}(t)}{k!}\cdot\left[\frac{(b-t)^k+(-1)^{k+1}(t-a)^k}{k}\right]$$

and the convergence is uniform by rapport of $t\in[a,b]$.

3. The Composite Case

The following lemma holds.

Lemma 1. *Let $g:[a,b]\to\mathbb{R}$ be such that $g^{(n-1)}$ $(n\geq 1)$ is absolutely continuous on $[a,b]$. Then for any $m\in\mathbb{N}$, $m\geq 1$, we have the inequality:*

$$\left|\frac{1}{b-a}\int_a^b g(u)\,du - \sum_{i=0}^{m-1}\sum_{k=1}^{n}\frac{(b-a)^{k-1}}{m^k k!}\cdot g^{(k-1)}\left(a+i\cdot\frac{b-a}{m}\right)\right| \tag{3.1}$$

$$\leq \begin{cases} \dfrac{(b-a)^n}{m^n(n+1)!}\left\|g^{(n)}\right\|_{[a,b],\infty}, & \text{if } g^{(n)}\in L_\infty[a,b]; \\ \dfrac{(b-a)^{n-1+\frac{1}{q}}}{m^n n!(nq+1)^{\frac{1}{q}}}\left\|g^{(n)}\right\|_{[a,b],p}, & \text{if } g^{(n)}\in L_p[a,b], \\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \dfrac{(b-a)^{n-1}}{m^n n!}\left\|g^{(n)}\right\|_{[a,b],1}. \end{cases}$$

Proof. Write Taylor's formula with the integral remainder for $\varphi(x)=\int_\alpha^x g(u)\,du$ and then choose $x=\beta$, to get:

$$\left|\int_\alpha^\beta g(u)\,du-\sum_{k=1}^n \frac{(\beta-\alpha)^k}{k!}g^{(k-1)}(\alpha)\right| \tag{3.2}$$
$$\leq \begin{cases} \dfrac{|\beta-\alpha|^{n+1}}{(n+1)!}\left\|g^{(n)}\right\|_{[\alpha,\beta],\infty}, & \text{if } g^{(n)}\in L_\infty[a,b]; \\[2ex] \dfrac{|\beta-\alpha|^{n+\frac{1}{q}}}{n!\,(nq+1)^{\frac{1}{q}}}\left\|g^{(n)}\right\|_{[\alpha,\beta],p}, & \text{if } g^{(n)}\in L_p[a,b], \\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \dfrac{|\beta-\alpha|^{n}}{n!}\left\|g^{(n)}\right\|_{[\alpha,\beta],1} & \end{cases}$$

for any $\alpha,\beta\in[a,b]$.

Now, if we consider the division

$$I_n: x_i=a+i\cdot\frac{b-a}{m},\quad i=\overline{0,m},$$

and apply (3.2) on the intervals $[x_{i,}x_{i+1}]$ $\left(i=\overline{0,m-1}\right)$, we can write:

$$\left|\int_{x_{i,}}^{x_{i+1}} g(u)\,du-\sum_{k=1}^n \frac{(b-a)^k}{m^k k!}\cdot g^{(k-1)}\left(a+i\cdot\frac{b-a}{m}\right)\right|$$

$$\leq \begin{cases} \dfrac{(b-a)^{n+1}}{m^{n+1}(n+1)!}\left\|g^{(n)}\right\|_{[x_i,x_{i+1}],\infty}, \\[2ex] \dfrac{(b-a)^{n+\frac{1}{q}}}{m^{n+\frac{1}{q}}n!\,(nq+1)^{\frac{1}{q}}}\left\|g^{(n)}\right\|_{[x_i,x_{i+1}],p}, \\[2ex] \dfrac{(b-a)^{n}}{m^n n!}\left\|g^{(n)}\right\|_{[x_i,x_{i+1}],1}. \end{cases}$$

Summing over i from 0 to $m-1$ and using the generalized triangle inequality, we deduce (3.2). ∎

The following main result holds.

Theorem 2. *Let $f:[a,b]\to\mathbb{R}$ be such that $f^{(n)}$ $(n\geq 0)$ is absolutely continuous on $[a,b]$. Then for any $m\in\mathbb{N}$, $m\geq 1$, we have:*

$$(Tf)(a,b;t)=f(t)\ln\left(\frac{b-t}{t-a}\right)+A_{n,m}(f,t)+R_{n,m}(f,t),$$

where

$$\begin{aligned}(3.3)\qquad & A_{n,m}(f,t) \\ = & \sum_{k=1}^{n}\frac{f^{(k)}(t)}{m^k k!}\cdot\left[\frac{(b-t)^k+(-1)^{k+1}(t-a)^k}{k}\right]+(b-a)\sum_{i=1}^{m-1}\sum_{k=1}^{n}\frac{1}{m^k k!} \\ & \times\left\{\sum_{\nu=1}^{k-1}(-1)^{\nu-1}(k-1)\cdots(k-\nu)\left(\frac{m}{i}\right)^{\nu-1}\right. \\ & \times\left[f^{(k-\nu)};t+\frac{i}{m}(b-t),t-\frac{i}{m}(t-a)\right] \\ & \left.+(-1)^{k-1}\left(\frac{m}{i}\right)^{k-1}(k-1)!\left[f;t+\frac{i}{m}(b-t),t-\frac{i}{m}(t-a)\right]\right\}\end{aligned}$$

and the remainder $R_{n,m}(f,t)$ *satisfies the estimate*

$$(3.4)\qquad |R_{n,m}(f,t)|$$

$$\leq\begin{cases}\dfrac{\left\|f^{(n+1)}\right\|_{[a,b],\infty}}{m^n(n+1)!\cdot(n+1)}\left[(b-t)^{n+1}+(t-a)^{n+1}\right], & \text{if } f^{(n+1)}\in L_\infty[a,b]; \\ \dfrac{q\left[(b-t)^{n+\frac{1}{q}}+(t-a)^{n+\frac{1}{q}}\right]}{m^n n!(nq+1)^{1+\frac{1}{q}}}\left\|f^{(n+1)}\right\|_{[a,b],p}, & \text{if } f^{(n+1)}\in L_p[a,b], \\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \dfrac{\left\|f^{(n+1)}\right\|_{[a,b],1}}{m^n n!\cdot n}\left[(b-t)^n+(t-a)^n\right]; & \end{cases}$$

$$\leq\begin{cases}\dfrac{(b-a)^{n+1}}{m^n(n+1)\cdot(n+1)!},\left\|f^{(n+1)}\right\|_{[a,b],\infty} & \text{if } f^{(n+1)}\in L_\infty[a,b]; \\ \dfrac{q(b-a)^{n+\frac{1}{q}}}{m^n n!(nq+1)^{1+\frac{1}{q}}}\left\|f^{(n+1)}\right\|_{[a,b],p}, & \text{if } f^{(n+1)}\in L_p[a,b], \\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \dfrac{(b-a)^n}{m^n n!\cdot n}\left\|f^{(n+1)}\right\|_{[a,b],1}; & \end{cases}$$

Proof. We have (see (2.3)) that:

$$(Tf)(a,b;t)-f(t)\ln\left(\frac{b-t}{t-a}\right)=PV\int_a^b\frac{f(\tau)-f(t)}{\tau-t}d\tau.$$

If we write the inequality (3.1) for $g=f'$, we get

$$(3.5)\qquad \left|\frac{f(t)-f(\tau)}{\tau-t}-\sum_{i=0}^{m-1}\sum_{k=1}^{n}\frac{(\tau-t)^{k-1}}{m^k k!}\cdot f^{(k)}\left(t+i\cdot\frac{\tau-t}{m}\right)\right|$$

$$(3.6)\qquad \leq \begin{cases} \dfrac{|\tau-t|^{n}}{m^{n}(n+1)!}\left\|f^{(n+1)}\right\|_{[t,\tau],\infty}, & \text{if } f^{(n+1)}\in L_{\infty}[a,b];\\[2ex] \dfrac{|\tau-t|^{n-1+\frac{1}{q}}}{m^{n}n!(nq+1)^{\frac{1}{q}}}\left\|f^{(n+1)}\right\|_{[t,\tau],p}, & \text{if } f^{(n+1)}\in L_{p}[a,b],\\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1;\\[1ex] \dfrac{|\tau-t|^{n-1}}{m^{n}n!}\left\|f^{(n+1)}\right\|_{[t,\tau],1}. \end{cases}$$

If we apply PV to (3.5), we may write that:

$$(3.7)\qquad \left| PV\int_{a}^{b}\frac{f(t)-f(\tau)}{\tau-t}d\tau - \sum_{i=0}^{m-1}\sum_{k=1}^{n}PV\int_{a}^{b}\frac{(\tau-t)^{k-1}}{m^{k}k!}\cdot f^{(k)}\left(t+i\cdot\frac{\tau-t}{m}\right)d\tau\right|$$

$$\leq \begin{cases} \dfrac{1}{m^{n}(n+1)!}PV\int_{a}^{b}|\tau-t|^{n}\left\|f^{(n+1)}\right\|_{[t,\tau],\infty}d\tau,\\[2ex] \dfrac{1}{m^{n}n!(nq+1)^{\frac{1}{q}}}PV\int_{a}^{b}|\tau-t|^{n-1+\frac{1}{q}}\left\|f^{(n+1)}\right\|_{[t,\tau],p}d\tau,\\[2ex] \dfrac{1}{m^{n}n!}PV\int_{a}^{b}|\tau-t|^{n-1}\left\|f^{(n+1)}\right\|_{[t,\tau],1}d\tau, \end{cases}$$

$$\leq \begin{cases} \dfrac{1}{m^{n}(n+1)!}\left\|f^{(n+1)}\right\|_{[a,b],\infty}PV\int_{a}^{b}|\tau-t|^{n}d\tau,\\[2ex] \dfrac{1}{m^{n}n!(nq+1)^{\frac{1}{q}}}\left\|f^{(n+1)}\right\|_{[a,b],p}PV\int_{a}^{b}|\tau-t|^{n-1+\frac{1}{q}}d\tau,\\[2ex] \dfrac{1}{m^{n}n!}\left\|f^{(n+1)}\right\|_{[a,b],1}PV\int_{a}^{b}|\tau-t|^{n-1}d\tau, \end{cases}$$

$$\leq \begin{cases} \dfrac{1}{m^{n}(n+1)!}\left\|f^{(n+1)}\right\|_{[a,b],\infty}\left[\dfrac{(b-t)^{n+1}+(t-a)^{n+1}}{n+1}\right],\\[2ex] \dfrac{1}{m^{n}n!(nq+1)^{\frac{1}{q}}}\left\|f^{(n+1)}\right\|_{[a,b],p}\left[\dfrac{(b-t)^{n+\frac{1}{q}}+(t-a)^{n+\frac{1}{q}}}{n+\frac{1}{q}}\right],\\[2ex] \dfrac{1}{m^{n}n!}\left\|f^{(n+1)}\right\|_{[a,b],1}\left[\dfrac{(b-t)^{n}+(t-a)^{n}}{n}\right]. \end{cases}$$

Now, let us denote

$$I_{i,k}:=PV\int_{a}^{b}(\tau-t)^{k-1}f^{(k)}\left(t+\frac{i}{m}(\tau-t)\right)d\tau,$$

where $i=0,\dots,m-1$, $k=1,\dots,n$.

For $i=0$, we have

$$I_{0,k}:=PV\int_a^b(\tau-t)^{k-1}f^{(k)}(t)\,d\tau=f^{(k)}(t)\cdot\frac{(b-t)^k+(-1)^{k+1}(t-a)^k}{k}$$

for any $k=1,\dots,n$.

For $k=1,\dots,n$ and $i=1,\dots,m-1$, we have

$$\begin{aligned}
I_{i,k} &= \lim_{\varepsilon\to0+}\left[\int_a^{t-\varepsilon}(\tau-t)^{k-1}f^{(k)}\left[t+\frac{i}{m}(\tau-t)\right]d\tau\right. \\
&\quad\left.+\int_{t+\varepsilon}^b(\tau-t)^{k-1}f^{(k)}\left[t+\frac{i}{m}(\tau-t)\right]d\tau\right] \\
&= \lim_{\varepsilon\to0+}\left[\frac{m}{i}f^{(k-1)}\left[t+\frac{i}{m}(\tau-t)\right](\tau-t)^{k-1}\Big|_a^{t-\varepsilon}\right. \\
&\quad-\frac{m}{i}\int_a^{t-\varepsilon}(k-1)(\tau-t)^{k-2}f^{(k-1)}\left[t+\frac{i}{m}(\tau-t)\right]d\tau \\
&\quad+\frac{m}{i}f^{(k-1)}\left[t+\frac{i}{m}(\tau-t)\right](\tau-t)^{k-1}\Big|_{t+\varepsilon}^{b} \\
&\quad\left.-\frac{m}{i}\int_{t+\varepsilon}^b(k-1)(\tau-t)^{k-2}f^{(k-1)}\left[t+\frac{i}{m}(\tau-t)\right]d\tau\right] \\
&= \frac{m}{i}\left[f^{(k-1)}\left[t+\frac{i}{m}(b-t)\right]-f^{(k-1)}\left[t-\frac{i}{m}(t-a)\right]\right] \\
&\quad-\frac{m}{i}(k-1)\,PV\int_a^b(\tau-t)^{k-2}f^{(k-1)}\left(t+\frac{i}{m}(\tau-t)\right)d\tau \\
&= (b-a)\left[f^{(k-1)};t+\frac{i}{m}(b-t),t-\frac{i}{m}(t-a)\right]-\frac{m}{i}(k-1)\,I_{i,k-1}
\end{aligned}\tag{3.8}$$

for any $k=2,\dots,n$.

For $k=1$, we have

$$\begin{aligned}
I_{i,1} &= PV\int_a^b f^{(1)}\left(t+\frac{i}{m}(\tau-t)\right)d\tau \\
&= \frac{m}{i}\left[f\left[t+\frac{i}{m}(b-t)\right]-f\left[t-\frac{i}{m}(t-a)\right]\right] \\
&= (b-a)\left[f;t+\frac{i}{m}(b-t),t-\frac{i}{m}(t-a)\right].
\end{aligned}$$

Using the recursive relation (3.8), we may write

$$\begin{aligned}
&I_{i,k} \\
&= (b-a)\left[f^{(k-1)};t+\frac{i}{m}(b-t),t-\frac{i}{m}(t-a)\right]-\left(\frac{m}{i}\right)(k-1)\,I_{i,k-1} \\
&= (b-a)\left[f^{(k-1)};t+\frac{i}{m}(b-t),t-\frac{i}{m}(t-a)\right]-\left(\frac{m}{i}\right)(k-1) \\
&\times\left[(b-a)\left[f^{(k-2)};t+\frac{i}{m}(b-t),t-\frac{i}{m}(t-a)\right]-\left(\frac{m}{i}\right)(k-2)\,I_{i,k-2}\right]
\end{aligned}\tag{3.9}$$

$$
\begin{aligned}
&= (b-a)\left[f^{(k-1)};t+\frac{i}{m}(b-t),t-\frac{i}{m}(t-a)\right] \\
&\quad -(b-a)\left(\frac{m}{i}\right)(k-1)\left[f^{(k-2)};t+\frac{i}{m}(b-t),t-\frac{i}{m}(t-a)\right] \\
&\quad +\left(\frac{m}{i}\right)^{2}(k-1)(k-2)I_{i,k-2} \\
&= \cdots\cdots\cdots\cdots = \\
&= (b-a)\sum_{\nu=1}^{k-1}(-1)^{\nu-1}(k-1)\cdots(k-\nu)\left(\frac{m}{i}\right)^{\nu-1} \\
&\quad \times\left[f^{(k-\nu)};t+\frac{i}{m}(b-t),t-\frac{i}{m}(t-a)\right]+(-1)^{k-1}\left(\frac{m}{i}\right)^{k-1}(k-1)!I_{i,1} \\
&= (b-a)\left[\sum_{\nu=1}^{k-1}(-1)^{\nu-1}(k-1)\cdots(k-\nu)\left(\frac{m}{i}\right)^{\nu-1}\right. \\
&\quad \times\left[f^{(k-\nu)};t+\frac{i}{m}(b-t),t-\frac{i}{m}(t-a)\right] \\
&\quad \left.+(-1)^{k-1}\left(\frac{m}{i}\right)^{k-1}(k-1)!\left[f;t+\frac{i}{m}(b-t),t-\frac{i}{m}(t-a)\right]\right].
\end{aligned}
$$

Replacing $I_{i,k}$ in (3.7), we deduce the estimate (3.4) with $A_{m,n}$ as defined by (3.3). ∎

4. A Numerical Example

Consider the function $f:[1,2]\to R, f(t)=\sin t$. The exact finite Hilbert transform of f provided by Maple is

$$
(Tf)(1,2;t)=-\operatorname{Si}(t-2)\cos t+\operatorname{Ci}(2-t)\sin t+\operatorname{Si}(t-1)\cos t-\operatorname{Ci}(t-1) \tag{4.1}
$$

where

$$
\operatorname{Si}(x):=\int_0^x\frac{\sin u}{u}du,\operatorname{Ci}(x):=\gamma+\ln x+\int_0^x\frac{\cos u-1}{u}du.
$$

The plot of the finite Hilbert transform is incorporated in Figure 1.

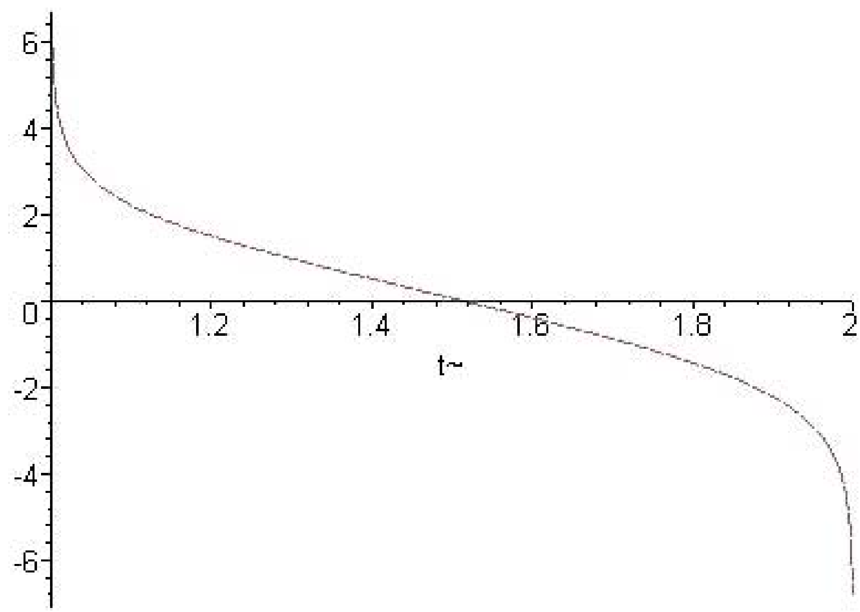

Figure 1.

Define by $A_n(1,2;t)$ the approximation for the finite Hilbert transform provided by the formula (2.1), i.e.,

$$A_n(1,2;t) = \sin t \cdot \ln\left(\frac{2-t}{t-1}\right) + \sum_{k=1}^{n-1} \frac{[\sin t]^{(k)}}{k!} \cdot \left[\frac{(b-t)^k + (-1)^{k+1}(t-a)^k}{k}\right].$$

Figure 2 contains the plot of the error

$$Er_5(1,2;t) := (Tf)(1,2;t) - A_5(1,2;t), t \in [1,2].$$

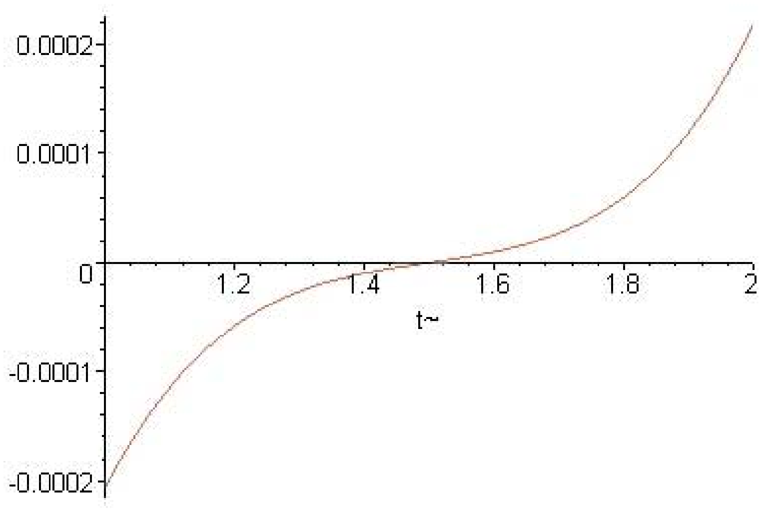

Figure 2.

Figure 3 contains the plot of the error

$$Er_{10}\left(1,2;t\right) := \left(Tf\right)\left(1,2;t\right) - A_{10}\left(1,2;t\right), t \in [1,2].$$

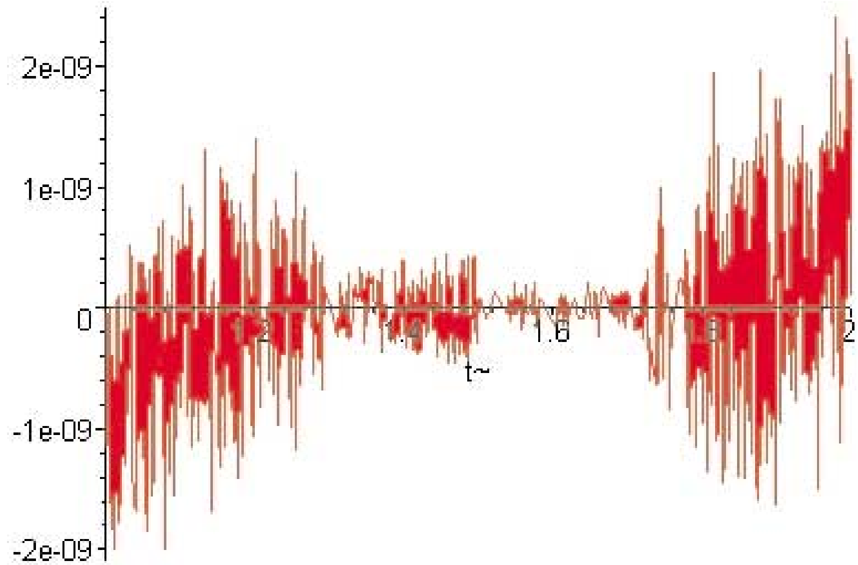

Figure 3.

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References

[1] G. CRISCUOLO and G. MASTROIANNI, Formule gaussiane per il calcolo di integrali a valor principale secondo e Cauchy loro convergenza, *Calcolo,* **22** (1985), 391-411.

[2] G. CRISCUOLO and G. MASTROIANNI, Sulla convergenza di alcune formule di quadratura per il calcolo di integrali a valor principale secondo Cauchy, *Anal. Numér. Théor. Approx.,* **14** (1985), 109-116.

[3] G. CRISCUOLO and G. MASTROIANNI, On the convergence of an interpolatory product rule for evaluating Cauchy principal value integrals, *Math. Comput.,* **48** (1987), 725-735.

[4] G. CRISCUOLO and G. MASTROIANNI, Mean and uniform convergence of quadrature rules for evaluating the finite Hilbert transform, in: A. Pinkus and P. Nevai, Eds., *Progress in Approximation Theory* (Academic Press, Orlando, 1991), 141-175.

[5] C. DAGNINO, V. DEMICHELIS and E. SANTI, Numerical integration based on quasi-interpolating splines, *Computing*, **50** (1993), 149-163.

[6] C. DAGNINO, V. DEMICHELIS and E. SANTI, An algorithm for numerical integration based on quasi-interpolating splines, *Numer. Algorithms*, **5** (1993), 443-452.

[7] C. DAGNINO, V. DEMICHELIS and E. SANTI, Local spline approximation methods for singular product integration, *Approx. Theory Appl.,* **12** (1996), 1-14.

[8] C. DAGNINO and P. RABINOWITZ, Product integration of singular integrands based on quasi-interpolating splines, *Comput. Math.,* to appear, special issue dedicated to 100th birthday of Cornelius Lanczos.

[9] P.J. DAVIS and P. RABINOWITZ, *Methods of Numerical Integration,* (Academic Press, Orlando, 2nd Ed., 1984).

[10] K. DIETHELM, Non-optimality of certain quadrature rules for Cauchy principal value integrals, *Z. Angew. Math. Mech.,* **74**(6) (1994), T689-T690.

[11] K. DIETHELM, Peano kernels and bounds for the error constants of Gaussian and related quadrature rules for Cauchy principal value integrals, *Numer. Math.,* to appear.

[12] K. DIETHELM, Uniform convergence to optimal order quadrature rules for Cauchy principal value integrals, *J. Comput. Appl. Math.,* **56**(3) (1994), 321-329.

[13] D. ELLIOTT, On the convergence of Hunter's quadrature rule for Cauchy principal value integrals, *BIT,* **19** (1979), 457-462.

[14] D. ELLIOTT and D.F. PAGET, On the convergence of a quadrature rule for evaluating certain Cauchy principal value integrals, *Numer. Math.,* **23** (1975), 311-319.

[15] D. ELLIOTT and D.F. PAGET, On the convergence of a quadrature rule for evaluating certain Cauchy principal value integrals: An addendum, *Numer. Math.,* **25** (1976), 287-289.

[16] D. ELLIOTT and D.F. PAGET, Gauss type quadrature rules for Cauchy principal value integrals, *Math. Comput.,* **33** (1979), 301-309.

[17] T. HASEGAWA and T. TORII, An automatic quadrature for Cauchy principal value integrals, *Math. Comput.,* **56** (1991), 741-754.

[18] T. HASEGAWA and T. TORII, Hilbert and Hadamard transforms by generalised Chebychev expansion, *J. Comput. Appl. Math.,* **51** (1994), 71-83.

[19] D.B. HUNTER, Some Gauss type formulae for the evaluation of Cauchy principal value integrals, *Numer. Math.,* **19** (1972), 419-424.

[20] N.I. IOAKIMIDIS, Further convergence results for two quadrature rules for Cauchy type principal value integrals, *Apl. Mat.,* **27** (1982), 457-466.

[21] N.I. IOAKIMIDIS, On the uniform convergence of Gaussian quadrature rules for Cauchy principal value integrals and their derivatives, *Math. Comput.,* **44** (1985), 191-198.

[22] A.I. KALANDIYA, *Mathematical Methods of Two-Dimensional Elasticity,* (Mir, Moscow, 1st English ed., 1978).

[23] G. MONEGATO, The numerical evaluation of one-dimensional Cauchy principal value integrals, *Computing,* **29** (1982), 337-354.

[24] G. MONEGATO, On the weights of certain quadratures for the numerical evaluation of Cauchy principal value integrals and their derivatives, *Numer. Math.,* **50** (1987), 273-281.

[25] D.F. PAGET and D. ELLIOTT, An algorithm for the numerical evaluation of certain Cauchy principal value integrals, *Numer. Math.,* **19** (1972), 373-385.

[26] P. RABINOWITZ, On an interpolatory product rule for evaluating Cauchy principal value integrals, *BIT,* **29** (1989), 347-355.

[27] P. RABINOWITZ, Numerical integration based on approximating splines, *J. Comput. Appl. Math.,* **33** (1990), 73-83.

[28] P. RABINOWITZ, Application of approximating splines for the solution of Cauchy singular integral equations, *Appl. Numer. Math.,* **15** (1994), 285-297.

[29] P. RABINOWITZ, Numerical evaluation of Cauchy principal value integrals with singular integrands, *Math. Comput.,* **55** (1990), 265-276.

[30] P. RABINOWITZ and E. SANTI, On the uniform convergence of Cauchy principle value of quasi-interpolating splines, *BIT,* **35** (1995), 277-290.

[31] M.S. SHESHKO, On the convergence of quadrature processes for a singular integral, *Izv., Vyssh. Uohebn. Zaved Mat.,* **20** (1976), 108-118.

[32] G. TSAMASPHYROS and P.S. THEOCARIS, On the convergence of some quadrature rules for Cauchy principal-value and finite-part integrals, *Computing*, **31** (1983), 105-114.

School of Communications and Informatics, Victoria University of Technology, PO Box 14428, Melbourne City MC 8001, Victoria, Australia

E-mail address: `sever@matilda.vu.edu.au`

URL: `http://rgmia.vu.edu.au/SSDragomirWeb.html`