AN APPROXIMATION OF THE FOURIER SINE TRANSFORM VIA GRÜSS TYPE INEQUALITIES AND APPLICATIONS FOR ELECTRICAL CIRCUITS

S.S. DRAGOMIR AND A. KALAM

Abstract. An approximation of the Fourier Sine Transform via Grüss, Chebychev and Lupaş integral inequalities and application for an electrical curcuit containing an inductance L, a condenser of capacity C and a source of electromotive force $E_0 P(t)$, where $P(t)$ is an L_2-integrable function, are given.

1. Introduction

Consider the electrical oscillation in a circuit containing a resistance R, an inductance L, a condenser of capacity C, and a source of electromotive force $E_0 P(t)$, where E_0 is a constant and $P(t)$ is a known function of the time t.

If the charge on the plates of the condenser is q, then the potential difference across the plates is $\frac{q}{c}$. Similarly, if i is the current flowing through the resistance and the inductance, the differences of potential between their ends are Ri and $L\left(\frac{di}{dt}\right)$, respectively. By the equation of continuity

$$i=\frac{dq}{dt} \tag{1.1}$$

so that these potential differences may be written as $R\frac{dq}{dt}$ and $L\frac{d^2q}{dt^2}$ respectively. Thus we obtain the ordinary differential equation [6, p. 93]

$$L\frac{d^2q}{dt^2}+R\frac{dq}{dt}+\frac{q}{c}=E_0P(t) \tag{1.2}$$

for the determination of the charge q which accumulates on the plates of the condenser.

If we assume that initially this charge is Q and that a current I is flowing in the circuit, then we obtain the initial conditions

$$\begin{cases} q(0)=Q, \\ \dfrac{dq(0)}{dt}=I. \end{cases} \tag{1.3}$$

It is well known that if the resistance of the circuit is zero, i.e., $R=0$, then the solution of (1.2) with the initial conditions (1.3) is given by (see for example [6, p.

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$$q(t) = Q\cos(\omega t) + \frac{I}{\omega}\sin(\omega t) + \frac{E_0}{\omega L}\int_0^t P(s)\sin[\omega(t-s)]\,ds, \tag{1.4}$$

where $\omega^2 = \frac{1}{LC}$.

Consequently, there is a practical need in computing the following Fourier Sine Transform:

$$A(0,t;P,\omega,t) := \int_0^t P(s)\sin[\omega(t-s)]\,ds,$$

which may be a difficult task if P is complicated enough. In this case, a numerical approach would satisfy the user if the accuracy is good.

The main aim of the present paper is to point out some estimates for the Fourier Sine Transform by the use of a Grüss type inequality.

2. Inequalities for the Fourier Sine Transform

Let us consider the Fourier Sine transform

$$A(a,b;P,\omega,t) := \int_a^b P(s)\sin[\omega(t-s)]\,ds, \tag{2.1}$$

where t may be an arbitrary real number.

The following result holds.

Theorem 1. *If $P \in L_2[a,b]$, then we have the inequality:*

$$\left|A(a,b;P,\omega,t) + COS(\omega(t-b),\omega(t-a))\int_a^b P(s)\,ds\right| \tag{2.2}$$
$$\leq (b-a)\left[\frac{1}{b-a}\int_a^b P^2(s)\,ds - \left(\frac{1}{b-a}\int_a^b P(s)\,ds\right)^2\right]^{\frac{1}{2}}$$
$$\times\left[\frac{1}{2}\left(1 - SIN(2\omega(t-b),2\omega(t-a)) - COS^2[\omega(t-b),\omega(t-a)]\right)\right],$$

where

$$SIN(z,\omega) := \frac{\sin z - \sin\omega}{z-\omega}, \quad z \neq w,$$

and

$$COS(z,\omega) := \frac{\cos z - \cos\omega}{z-\omega}, \quad z \neq w$$

are the trigonometric means.

Proof. Using Korkine-Andréief's integral identity [5, p. 242, 243], i.e., we recall it

$$\frac{1}{b-a}\int_a^b f(t)g(t)\,dt - \frac{1}{b-a}\int_a^b f(t)\,dt\cdot\frac{1}{b-a}\int_a^b g(t)\,dt \tag{2.3}$$
$$= \frac{1}{2(b-a)^2}\int_a^b\int_a^b (f(t)-f(s))(g(t)-g(s))\,dtds,$$

we may write:

$$\begin{aligned}
& \frac{1}{b-a}\int_a^b P(s)\sin[\omega(t-s)]\,ds - \frac{1}{b-a}\int_a^b P(s)\,ds \cdot \frac{1}{b-a}\int_a^b \sin[\omega(t-s)]\,ds \\
= & \frac{1}{2(b-a)^2}\int_a^b\int_a^b (P(u)-P(s))(\sin[\omega(t-u)]-\sin[\omega(t-s)])\,duds,
\end{aligned}$$

from where we get, via the Cauchy-Schwartz inequality, that

$$\begin{aligned}
& \left|\frac{1}{b-a}\int_a^b P(s)\sin[\omega(t-s)]\,ds\right. \\
& \left. - \frac{1}{b-a}\int_a^b P(s)\,ds \cdot \frac{1}{b-a}\int_a^b \sin[\omega(t-s)]\,ds\right| \\
\leq & \frac{1}{2(b-a)^2}\int_a^b\int_a^b |P(u)-P(s)|\,|\sin[\omega(t-u)]-\sin[\omega(t-s)]|\,duds \\
\leq & \frac{1}{2(b-a)^2}\left[\int_a^b\int_a^b [P(u)-P(s)]^2\,duds\right]^{\frac{1}{2}} \\
& \times\left[\int_a^b\int_a^b (\sin[\omega(t-u)]-\sin[\omega(t-s)])^2\,duds\right]^{\frac{1}{2}} \\
= & \frac{1}{2(b-a)^2}\left[2\left((b-a)\int_a^b P^2(s)\,ds - \left(\int_a^b P(s)\,ds\right)^2\right)\right]^{\frac{1}{2}} \\
& \times\left[2\left((b-a)\int_a^b \left(\sin^2[\omega(t-s)]\right)ds - \left(\int_a^b (\sin[\omega(t-s)])\,ds\right)^2\right)\right]^{\frac{1}{2}} \\
= & :K.
\end{aligned} \tag{2.4}$$

However,

$$\begin{aligned}
\int_a^b \sin[\omega(t-s)]\,ds &= \frac{\cos[\omega(t-b)]-\cos[\omega(t-a)]}{\omega} \\
&= -(b-a)\,COS(\omega(t-b),\omega(t-a)),
\end{aligned}$$

$$\begin{aligned}
\int_a^b \sin^2[\omega(t-s)]\,ds &= \int_a^b \frac{1-\cos[2\omega(t-s)]}{2}ds \\
&= \frac{1}{2}(b-a) - \frac{1}{2}\int_a^b \cos[2\omega(t-s)]\,ds \\
&= \frac{1}{2}(b-a) + \frac{1}{2}\cdot\frac{\sin 2\omega(t-b)-\sin 2\omega(t-a)}{2\omega} \\
&= \frac{1}{2}(b-a) - \frac{1}{2}(b-a)\,SIN(2\omega(t-b),2\omega(t-a)) \\
&= \frac{1}{2}(b-a)\left[1-SIN(2\omega(t-b),2\omega(t-a))\right]
\end{aligned}$$

and then

$$\begin{aligned} K &= \frac{1}{(b-a)^2}\left[\left((b-a)\int_a^b P^2(s)\,ds-\left(\int_a^b P(s)\,ds\right)^2\right)\right]^{\frac{1}{2}} \\ &\quad\times\left[\frac{1}{2}(b-a)^2\left[1-SIN\left(2\omega(t-b),2\omega(t-a)\right)\right]\right. \\ &\quad\left.-(b-a)^2 COS^2\left[\omega(t-b),\omega(t-a)\right]\right]^{\frac{1}{2}} \\ &= \left[\frac{1}{b-a}\int_a^b P^2(s)\,ds-\left(\frac{1}{b-a}\int_a^b P(s)\,ds\right)^2\right]^{\frac{1}{2}} \\ &\quad\times\left[\frac{1}{2}\left[1-SIN\left(2\omega(t-b),2\omega(t-a)\right)\right]-COS^2\left[\omega(t-b),\omega(t-a)\right]\right]^{\frac{1}{2}}. \end{aligned}$$

Using (2.4), we deduce the desired result (2.2). ∎

Corollary 1. *If $m \leq P \leq M$ a.e. on $[a,b]$, where m and M are real numbers, then, for any $t \in \mathbb{R}$,*

$$\begin{aligned} &\left|A(a,b;P,\omega,t)+COS\left(\omega(t-b),\omega(t-a)\right)\int_a^b P(s)\,ds\right| \\ &\leq \frac{1}{2}(b-a)(M-m)\,B(a,b,\omega,t), \end{aligned} \tag{2.5}$$

where

$$\begin{aligned} &B(a,b,\omega,t) \\ &:= \left[\frac{1}{2}\left[1-SIN\left(2\omega(t-b),2\omega(t-a)\right)\right]-COS^2\left[\omega(t-b),\omega(t-a)\right]\right]^{\frac{1}{2}}. \end{aligned}$$

Proof. Using Grüss' integral inequality [5, p. 296] for the function P, we may write:

$$0 \leq \frac{1}{b-a}\int_a^b P^2(s)\,ds-\left(\frac{1}{b-a}\int_a^b P(s)\,ds\right)^2 \leq \frac{1}{4}(M-m)^2, \tag{2.6}$$

and then, by (2.2), we deduce (2.5). ∎

Corollary 2. *If the function $P:[a,b]\to\mathbb{R}$ is absolutely continuous on $[a,b]$ and $P' \in L_\infty[a,b]$, then, for any $t\in\mathbb{R}$, we have*

$$\begin{aligned} &\left|A(a,b;P,\omega,t)+COS\left(\omega(t-b),\omega(t-a)\right)\int_a^b P(s)\,ds\right| \\ &\leq \frac{1}{2\sqrt{3}}\left\|P'\right\|_\infty (b-a)^2\,B(a,b,\omega,t), \end{aligned} \tag{2.7}$$

where $\|P'\|_\infty := ess\sup_{t\in[a,b]}|P'(t)|$.

Proof. Using Chebychev's integral inequality [5, p. 297] for P, we may write that

$$0 \leq \frac{1}{b-a}\int_a^b P^2(s)\,ds - \left(\frac{1}{b-a}\int_a^b P(s)\,ds\right)^2 \leq \frac{1}{12}\|P'\|_\infty^2 (b-a)^2 \tag{2.8}$$

and then, by (2.2), we obtain (2.7). ∎

Corollary 3. *If the function $P:[a,b]\to\mathbb{R}$ is absolutely continuous on $[a,b]$ and $P'\in L_2[a,b]$, then, for any $t\in\mathbb{R}$, we have*

$$\left|A(a,b;P,\omega,t) + COS(\omega(t-b),\omega(t-a))\int_a^b P(s)\,ds\right| \tag{2.9}$$

$$\leq \frac{(b-a)^{1+\frac{1}{q}}\|P'\|_2}{\pi} B(a,b,\omega,t).$$

Proof. Using Lupaş' inequality [5, p. 301], we may write that

$$0 \leq \frac{1}{b-a}\int_a^b P^2(s)\,ds - \left(\frac{1}{b-a}\int_a^b P(s)\,ds\right)^2 \leq \frac{b-a}{\pi^2}\|P'\|_2^2, \tag{2.10}$$

and then, by (2.2) we obtain (2.9). ∎

For other Grüss type inequalities which may be applied in a similar fashion, see the recent papers [1] – [4].

We may now state the following result in estimating the solutions of (1.2) with the initial conditions (1.3).

Theorem 2. *Assume that P is absolutely continuous on any compact interval $[0,t]$, $t\in\mathbb{R}^+$. If $q(\cdot)$ is the solution of equation (1.2) with $R=0$ and the initial conditions (1.3), then we have the estimate:*

$$\left|q(t) - Q\cos\omega t - \frac{I}{\omega}\sin\omega t + \frac{\cos\omega t - 1}{\omega t}\cdot\frac{E_0}{\omega L}\int_0^t P(s)\,ds\right| \tag{2.11}$$

$$\leq \frac{tE_0}{2\omega L}U(\omega,t)\cdot\left[\frac{1}{t}\int_0^t P^2(s)\,ds - \left(\frac{1}{t}\int_0^t P(s)\,ds\right)^2\right]^{\frac{1}{2}}$$

$$\leq \begin{cases} \dfrac{E_0}{2\omega L}t(M_t-m_t)U(\omega,t) & \text{if } m_t\leq P(s)\leq M_t,\ s\in[0,t]; \\[2ex] \dfrac{1}{2\sqrt{3}}\cdot\dfrac{E_0}{\omega L}t^2\|P'\|_{\infty,[0,t]}U(\omega,t) & \text{if } P'\in L_\infty[0,t]; \\[2ex] \dfrac{1}{\pi}\cdot\dfrac{E_0}{\omega L}t^{\frac{3}{2}}\|P'\|_{2,[0,t]}U(\omega,t) & \text{if } P'\in L_2[0,t], \end{cases}$$

where

$$U(\omega,t) := \left[\frac{1}{2}\left(1-\frac{\sin 2\omega t}{2\omega t}\right) - \left(\frac{\cos\omega t - 1}{\omega t}\right)^2\right]^{\frac{1}{2}},\quad t>0.$$

Proof. If in (2.2), we choose $a=0$, $b=t$, we get

$$\left|A(0,t;P,\omega,t)+\int_0^t P(s)\,ds\cdot\frac{\cos\omega t-1}{\omega t}\right| \tag{2.12}$$
$$\le t\left[\frac{1}{t}\int_0^t P^2(s)\,ds-\left(\frac{1}{t}\int_0^t P(s)\,ds\right)^2\right]^{\frac{1}{2}}$$
$$\times\left[\frac{1}{2}\left(1-\frac{\sin 2\omega t}{2\omega t}\right)-\left(\frac{\cos\omega t-1}{\omega t}\right)^2\right]^{\frac{1}{2}}.$$

By (1.4), we get

$$A(0,t;P,\omega,t)=\frac{\omega L}{E_0}\left[q(t)-Q\cos\omega t-\frac{I}{\omega}\sin\omega t\right]. \tag{2.13}$$

Inserting (2.13) in (2.12), we easily deduce the first part of (2.11).

The second part follows by Corollaries $1-3$, and we omit the details. ∎

Remark 1. *We observe that* $U(\omega,t)$ *is actually*

$$\left[\frac{1}{t}\int_0^t \sin^2[\omega(t-s)]\,ds-\left(\frac{1}{t}\int_0^t\sin[\omega(t-s)]\,ds\right)^2\right]^{\frac{1}{2}}.$$

If we consider the function $f(s)=\sin[\omega(t-s)]$, then we may state that;

$$\begin{aligned}-1 &\le f(s)\le 1 \text{ for any } s\in[0,t],\\ f'(s) &= -\omega\cos[\omega(t-s)],\ \|f'\|_{[0,t],\infty}\le\omega\end{aligned}$$

and

$$\begin{aligned}\|f'\|_{[0,t],2} &= \left(\int_0^t[f'(s)]^2\,ds\right)^{\frac{1}{2}}=\omega\left(\int_0^t\cos^2[\omega(t-s)]\,ds\right)^{\frac{1}{2}}\\ &= \omega\left(\int_0^t\frac{1+\cos[2\omega(t-s)]}{2}ds\right)^{\frac{1}{2}}\\ &= \omega\left[\frac{1}{2}t+\frac{1}{2\omega}\sin(2\omega t)\right]^{\frac{1}{2}}.\end{aligned}$$

Consequently, using Grüss', Chebychev's and Lupaş's inequalities for $f(s)=\sin[\omega(t-s)]$, we may state that

$$U(\omega,t)\le\begin{cases}1,\\ \dfrac{1}{2\sqrt{3}}\omega t,\\ \dfrac{1}{\pi}\cdot\dfrac{\sqrt{t}}{\sqrt{2}}\omega\left[t+\dfrac{1}{\omega}\sin(2\omega t)\right]^{\frac{1}{2}},\end{cases}$$

implying that

$$U(\omega,t)\le\min\left\{1,\frac{1}{2\sqrt{3}}\omega t,\frac{1}{\sqrt{2}\pi}\sqrt{t}\omega\left[t+\frac{1}{\omega}\sin(2\omega t)\right]^{\frac{1}{2}}\right\}$$

for $t\ge 0$.

3. Some Quadrature Formulae

Consider the division of the interval $[a,b]$ given by

$$I_n : a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b,$$

with

$$h_i := x_{i+1} - x_i, \quad i = \overline{0, n-1} \text{ and } \nu(h) = \max_{i=\overline{0,n-1}} h_i,$$

and define the quadrature formula:

$$A_n(I_n; P, \omega, t) := -\sum_{i=0}^{n-1} COS(\omega(t - x_{i+1}), \omega(t - x_i)) \int_{x_i}^{x_{i+1}} P(s)\, ds. \tag{3.1}$$

Then we may state the following result in approximating the Fourier-Sine Operator $A(a,b;P,\omega,t)$ in terms of the sums defined above in (3.1).

Theorem 3. *Assume that $P : [a,b] \to \mathbb{R}$ is integrable on $[a,b]$. Then we have*

$$A(a,b;P,\omega,t) = A_n(I_n, P, \omega, t) + E_n(I_n, P, \omega, t), \tag{3.2}$$

where $A_n(I_n, P, \omega, t)$ are as defined in (3.1) and the error $E_n(I_n, P, \omega, t)$ satisfies the estimates:

$$|E_n(I_n, P, \omega, t)| \tag{3.3}$$

$$\leq \begin{cases} \frac{1}{2}\sum_{i=0}^{n-1} h_i (M_i - m_i) B(x_i, x_{i+1}, \omega, t) & \text{if } m_i \leq P(s) \leq M_i, \\ & s \in [x_i, x_{i+1}], \ i = \overline{0, n-1} \\ \frac{1}{2\sqrt{3}} \cdot \|P'\|_{[a,b],\infty} \sum_{i=0}^{n-1} h_i^2 B(x_i, x_{i+1}, \omega, t) & \text{if } P' \in L_\infty[a,b]; \\ \frac{1}{\pi} \cdot \|P'\|_{[a,b],2} \sum_{i=0}^{n-1} h_i^{\frac{3}{2}} B(x_i, x_{i+1}, \omega, t) & \text{if } P' \in L_2[a,b], \end{cases}$$

where

$$B(\alpha, \beta, \omega, t)$$
$$: = \left[\frac{1}{\beta - \alpha}\int_\alpha^\beta \sin^2[\omega(t-s)]\, ds - \left(\frac{1}{\beta - \alpha}\int_\alpha^\beta \sin[\omega(t-s)]\, ds\right)^2\right]^{\frac{1}{2}}$$
$$= \left[\frac{1}{2}\left(1 - SIN(2\omega(t-\beta), 2\omega(t-\alpha)) - COS^2[\omega(t-\beta), \omega(t-\alpha)]\right)\right]^{\frac{1}{2}}$$

and $\alpha, \beta \in [a,b]$, $t \in \mathbb{R}$, ω are as given above.

Proof. If we apply Corollary 1 on the intervals $[x_i, x_{i+1}]$ $\left(i = \overline{0, n-1}\right)$, we may write that

$$\left|\int_{x_i}^{x_{i+1}} P(s) \sin[\omega(t-s)]\, ds + COS(\omega(t - x_{i+1}), \omega(t - x_i)) \int_{x_i}^{x_{i+1}} P(s)\, ds\right|$$
$$\leq \frac{1}{2} h_i (M_i - m_i) B(x_i, x_{i+1}, \omega, t),$$

where $m_i = \inf_{s\in[x_i,x_{i+1}]} P(s)$ and $M_i = \sup_{s\in[x_i,x_{i+1}]} P(s)$.

Summing over i from 0 to $n-1$ and using the triangle inequality, we may write:

$$\begin{aligned}
& |A(a,b;P,\omega,t) - A_n(I_n,P,\omega,t)| \\
\leq & \sum_{i=0}^{n-1} \left| \int_{x_i}^{x_{i+1}} P(s)\sin[\omega(t-s)]\,ds \right. \\
& \left. + COS(\omega(t-x_{i+1}),\omega(t-x_i)) \int_{x_i}^{x_{i+1}} P(s)\,ds \right| \\
\leq & \frac{1}{2}\sum_{i=0}^{n-1} h_i(M_i - m_i) B(x_i,x_{i+1},\omega,t),
\end{aligned}$$

and the first part of (3.3) is proved.

If we use Corollary 2, on the intervals $[x_i,x_{i+1}]$, we may write that:

$$\begin{aligned}
& \left| \int_{x_i}^{x_{i+1}} P(s)\sin[\omega(t-s)]\,ds + COS(\omega(t-x_{i+1}),\omega(t-x_i)) \int_{x_i}^{x_{i+1}} P(s)\,ds \right| \\
\leq & \frac{1}{2\sqrt{3}} \|P'\|_\infty h_i^2 B(x_i,x_{i+1},\omega,t).
\end{aligned}$$

Doing in a similar way as above, we get the second part of (3.3).

Finally, if we apply Corollary 3 on the intervals $[x_i,x_{i+1}]$, we may write that

$$\begin{aligned}
& \left| \int_{x_i}^{x_{i+1}} P(s)\sin[\omega(t-s)]\,ds + COS(\omega(t-x_{i+1}),\omega(t-x_i)) \int_{x_i}^{x_{i+1}} P(s)\,ds \right| \\
\leq & \frac{1}{\pi} \|P'\|_{[a,b],2} \sum_{i=0}^{n-1} h_i^{\frac{3}{2}} B(x_i,x_{i+1},\omega,t), \quad i=\overline{0,n-1},
\end{aligned}$$

from where we get the last part of (3.3). ∎

Remark 2. *Since the quantities $B(x_i,x_{i+1},\omega,t)$ $(i=0,\dots,n-1)$ play an important role in evaluating the error estimate in the quadrature formula (3.2), and in practice may be difficult to compute, we point out here a way of upper bounding them by the use of Chebychev's inequality (2.8).*

Applying (2.8) on the intervals $[x_i,x_{i+1}]$, $(i=0,\dots,n-1)$, we may write that:

$$\begin{aligned}
& B(x_i,x_{i+1},\omega,t) \\
= & \left[\frac{1}{h_i}\int_{x_i}^{x_{i+1}} \sin^2[\omega(t-s)]\,ds - \left(\frac{1}{h_i}\int_{x_i}^{x_{i+1}} \sin[\omega(t-s)]\,ds \right)^2 \right]^{\frac{1}{2}} \\
\leq & \frac{1}{2\sqrt{3}} h_i \left\| \frac{d}{ds}\sin[\omega(t-s)] \right\|_{\infty,[x_i,x_{i+1}]} \leq \frac{1}{2\sqrt{3}} h_i\omega,
\end{aligned} \tag{3.4}$$

for any $i=\overline{0,n-1}$.

Now, using the evaluation (3.4), we may point out the following corollary which will contain simpler bounds for the error estimate than Theorem 2.

Corollary 4. *With the assumption in Theorem 2, we have (3.2) and the error* $E_n(I_n, P, \omega, f)$ *satisfies the estimate*

$$|E_n(I_n, P, \omega, t)| \tag{3.5}$$

$$\leq \begin{cases} \frac{\omega}{4\sqrt{3}} \sum_{i=0}^{n-1} (M_i - m_i) h_i^2 & \text{if} \quad m_i \leq P(s) \leq M_i, \\ & \quad s \in [x_i, x_{i+1}], \\ \frac{\omega}{12} \cdot \|P'\|_{[a,b],\infty} \sum_{i=0}^{n-1} h_i^3 & \text{if} \quad P' \in L_\infty[a,b]; \\ \frac{\omega}{2\pi\sqrt{3}} \cdot \|P'\|_{[a,b],q} \sum_{i=0}^{n-1} h_i^{\frac{5}{2}} & \text{if} \quad P' \in L_2[a,b]; \end{cases}$$

$$\leq \begin{cases} \frac{\omega}{4\sqrt{3}} (M - m) \nu(h)(b-a) & \text{if} \quad m \leq P(s) \leq M, \ s \in [a,b], \\ \frac{\omega}{12} \cdot \|P'\|_{[a,b],\infty} \nu^2(h)(b-a) & \text{if} \quad P' \in L_\infty[a,b]; \\ \frac{\omega}{2\pi\sqrt{3}} \cdot \|P'\|_{[a,b],2} \nu^{\frac{3}{2}}(h)(b-a) & \text{if} \quad P' \in L_2[a,b]. \end{cases}$$

Remark 3. *We note that all the bounds go to zero if the norm of the division* I_n *is small, i.e.,* $\nu(h) \to 0$. *It is obvious that the higher order of accuracy is provided by the second bound.*

In practical considerations, it is useful to use the equidistant partitioning

$$E_n : x_i = a + \frac{i}{n}(b-a), \quad i = 0, \ldots, n.$$

In this partitioning, we consider the quadrature formula:

$$A_n(P, \omega, t) := \sum_{i=0}^{n-1} COS\left[\omega\left(t - a - \frac{i+1}{n}(b-a)\right), \omega\left(t - a - \frac{i}{n}(b-a)\right)\right] \times \int_{a+\frac{i}{n}(b-a)}^{a+\frac{i+1}{n}(b-a)} P(s)\,ds. \tag{3.6}$$

Then, we may state the following result in estimating the error of approximating the integral operator $A(a, b, P, \omega, t)$ by the use of quadrature formula $A_n(P, \omega, t)$.

Corollary 5. *Assume that* P *is as in Theorem 2. Then*

$$A(a, b, P, \omega, t) = A_n(P, \omega, t) + E_n(P, \omega, t), \tag{3.7}$$

where $A_n(P,\omega,t)$ *is given by (3.6) and the error* $E_n(P,\omega,t)$ *satisfies the bounds:*

$$|E_n(P,\omega,t)| \leq \begin{cases} \dfrac{\omega(M-m)(b-a)^2}{4n\sqrt{3}} & \text{if} \quad m \leq P(s) \leq M,\ s\in[a,b], \\[2ex] \dfrac{\omega\|P'\|_{[a,b],\infty}(b-a)^3}{12n^2} & \text{if} \quad P' \in L_\infty[a,b]; \\[2ex] \dfrac{\omega\|P'\|_{[a,b],2}(b-a)^{\frac{5}{2}}}{2\pi n^{\frac{3}{2}}\sqrt{3}} & \text{if} \quad P' \in L_2[a,b]. \end{cases} \tag{3.8}$$

Now, for a fixed $t>0$, let us consider the uniform partitioning of the interval $[0,t]$:

$$U_n : x_i := \frac{i}{n}t, \quad i=0,\dots,n.$$

We can consider the quadrature formula

$$Q_n(P,\omega,t) := -\sum_{i=0}^{n-1} COS\left(\omega\cdot\frac{n-i-1}{n}t, \omega\cdot\frac{n-i}{n}t\right)\int_{\frac{i}{n}t}^{\frac{i+1}{n}t} P(s)\,ds. \tag{3.9}$$

Then we may state the following result in approximating the solution of equation (1.2) with $R=0$ and the initial condition (1.3).

Theorem 4. *Assume that* P *is absolutely continuous on any compact interval* $[0,t]$, $t\in\mathbb{R}^+$. *If* $q(\cdot)$ *is the solution of equation (1.2) with* $R=0$ *and the intial conditions (1.3), then we have:*

$$q(t) = Q\cos\omega t + \frac{I}{\omega}\sin\omega t + \frac{E_0}{\omega L}Q_n(P,\omega,t) + R_n(P,\omega,t), \tag{3.10}$$

where the quadrature formula $Q_n(P,\omega,t)$ *is given in (3.9) and the remainder* $R_n(P,\omega,f)$ *in (3.10) satisfies the estimate*

$$|R_n(P,\omega,t)| \leq \begin{cases} \dfrac{E_0}{4Ln\sqrt{3}}(M_t-m_t)t^2 & \text{where} \quad m_t \leq P(s) \leq M_t,\ s\in[0,t]; \\[2ex] \dfrac{E_0}{12Ln^2}\|P'\|_{[0,t],\infty}t^3 & \text{if} \quad P'\in L_\infty[0,t]; \\[2ex] \dfrac{E_0}{2L\pi n^{\frac{3}{2}}\sqrt{3}}\|P'\|_{[0,t],2}t^{\frac{3}{2}} & \text{if} \quad P'\in L_2[0,t]. \end{cases} \tag{3.11}$$

Proof. If we apply Corollary 5 for $a=0$, $b=t$, then we get

$$\begin{aligned} A(0,t;P,\omega,t) \quad : &= \int_0^t P(s)\sin[\omega(t-s)]\,ds \\ &= Q(P,\omega,t) + S_n(P,\omega,t) \end{aligned} \tag{3.12}$$

where

$$(3.13)\quad |S_n(P,\omega,t)| \leq \begin{cases} \dfrac{\omega(M_t - m_t)t^2}{4n\sqrt{3}} & \text{where} \quad m_t \leq P(s) \leq M_t,\ s \in [0,t]; \\ \\ \dfrac{\omega \|P'\|_{[0,t],\infty} t^3}{12n^2} & \text{if} \quad P' \in L_\infty[0,t]; \\ \\ \dfrac{\omega \|P'\|_{[0,t],2} t^{\frac{3}{2}}}{2\pi n^{\frac{3}{2}}\sqrt{3}} & \text{if} \quad P' \in L_2[0,t]. \end{cases}$$

Using (3.12), (3.13) and (1.4), we deduce (3.10) and (3.11). ∎

Remark 4. *If one would like to numerically approximate $q(\cdot)$ on the given interval $[0,t]$ with a theoretical accuracy better than a given $\varepsilon > 0$ (ε-small), then the minimal number of nodes n_0 to reach that accuracy as given by the second bound in (3.11)) will be*

$$n_0 := \left\lceil \left(\frac{E_0}{12L\varepsilon} \|P'\|_{[0,t],\infty} t^3 \right)^{\frac{1}{2}} \right\rceil + 1,$$

where $\lceil a \rceil$ denotes the integer part of $a \in \mathbb{R}$.

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School of Communications and Informatics, Victoria University of Technology, PO Box 14428, Melbourne City MC, Victoria 8001, Australia.

E-mail address: `sever.dragomir@vu.edu.au`

URL: `http://rgmia.vu.edu.au/SSDragomirWeb.html`

E-mail address: `akhtar.kalam@vu.edu.au`