MEANS, g–CONVEX DOMINATED FUNCTIONS & HADAMARD–TYPE INEQUALITIES

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Abstract. Hadamard–type inequalities are derived for g–convex dominated maps. Applications are given involving two functionals and some common means.

1. Introduction

The Hermite–Hadamard inequality

$$g\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b g(x)dx \leq \frac{g(a)+g(b)}{2}$$

for a convex real–valued function g on a finite interval $[a,b]$ is central to mathematical analysis and is the subject of a huge literature dealing with various generalisations and refinements. In this note we connect together some disparate threads through a Hermite–Hadamard motif. The first of these threads is the unifying concept of a g–convex dominated function (see [6]).

Definition 1. *Let $g : I \to \mathbb{R}$ be a given convex function. The real function $f : I \to \mathbb{R}$ is called $g-$convex dominated on I if*

$$\begin{aligned} &|\lambda f(x) + (1-\lambda) f(y) - f(\lambda x + (1-\lambda) y)| \\ &\leq \lambda g(x) + (1-\lambda) g(y) - g(\lambda x + (1-\lambda) y) \end{aligned} \tag{1.1}$$

for all $x, y \in I$ and $\lambda \in [0,1]$.

The class of g–convex dominated functions on an interval I is manifestly nonempty. If g is convex on I and $e : I \to \mathbb{R}$ is defined by

$$e(x) := x,$$

then e and g are both g–convex dominated on I. Indeed there are concave functions which are g–convex dominated (for example $-g$) as well as functions which are neither convex nor concave. The concept of g–convex dominated functions draws together functions with some convex–like properties. We aim to elucidate some of these properties.

A second thread involves several means in common use for a pair x, y of positive numbers, namely the following. For $x \neq y$ and $p \in \mathbb{R} \setminus \{-1, 0\}$, we define the p–logarithmic mean (generalised logarithmic mean) $L_p(x,y)$ by

$$L_p(x,y) := \left[\frac{y^{p+1} - x^{p+1}}{(p+1)(y-x)}\right]^{1/p}.$$

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In fact the singularities at $p = -1, 0$ are removable and L_p can be defined for $p = -1, 0$ so as to make $L_p(x, y)$ a continuous function of p. In the limit as $p \to 0$ we obtain the identric mean $I(x, y)$, given by

$$I(x,y) := \frac{1}{e} \left(\frac{y^y}{x^x} \right)^{1/(y-x)},$$

and in the case $p \to -1$ the logarithmic mean $L(x, y)$, given by

$$L(x,y) := \frac{y-x}{\ln y - \ln x}.$$

In each case we define the mean as x when $y = x$, which occurs as the limiting value of $L_p(x, y)$ for $y \to x$. See [1] Chapter 6, Section 3 for more detail on these means.

In addition we have the arithmetic, geometric and harmonic means, defined respectively by

$$A(x,y) := \frac{x+y}{2}, \qquad G(x,y) := \sqrt{xy} \quad \text{and} \quad H(x,y) := \frac{2xy}{x+y}.$$

The first two arise from $L_p(x, y)$ in the respective cases $p = 1$, $p = -2$. Remarkably there is no value of p for which $L_p = H$ (see [1] p. 347). However H is connected with the generalised logarithmic–mean canon by

$$H(x,y) = \left[A\left(x^{-1}, y^{-1} \right) \right]^{-1}. \tag{1.2}$$

The final thread involves two functionals which interpolate between $f((a+b)/2)$ and $\int_a^b f(x)dx/(b-a)$. If $f : [a, b] \to \mathbb{R}$ with $f \in L_1[a, b]$, we define the induced mapping $H_f : [0, 1] \to \mathbb{R}$ by

$$H_f(t) := \frac{1}{b-a} \int_a^b f\left(tx + (1-t)\frac{a+b}{2} \right) dx.$$

Similarly for $f : [a, b] \to \mathbb{R}$ integrable on $[a, b]$ we may define $F_f : [0, 1] \to \mathbb{R}$ by

$$F_f(t) := \frac{1}{(b-a)^2} \int_a^b \int_a^b f(tx + (1-t)y)\, dxdy.$$

For treatments of these functionals see [2]–[6].

In Section 2 we present some general results for g–convex dominated functions. In Section 3 a number of simple particular examples are given in which the means L_p and H appear naturally. Finally in Section 4 we show that the two functionals defined above inherit g–convex dominated properties.

2. g–convex dominated maps

We shall make use of the following characterisation of convex–dominated functions established in [6].

Lemma 1. *Let g be a convex function on I and $f : I \to \mathbb{R}$. Then the following statements are equivalent:*

(i) *f is g–convex dominated on I;*
(ii) *the mappings $g - f$ and $g + f$ are convex on I;*

(iii) there exist two convex mappings h, k defined on I such that

$$f=\frac{1}{2}(h-k) \text{ and } g=\frac{1}{2}(h+k).$$

Proof. "$(i) \Longleftrightarrow (ii)$". Condition (1.1) is equivalent to

$$\begin{aligned} & g(\lambda x+(1-\lambda) y)-\lambda g(x)-(1-\lambda) g(y) \\ & \quad \leq \lambda f(x)+(1-\lambda) f(y)-f(\lambda x+(1-\lambda) y) \\ & \quad \leq \lambda g(x)+(1-\lambda) g(y)-g(\lambda x+(1-\lambda) y)\end{aligned}$$

for all $x, y \in I$ and $\lambda \in [0,1]$. The two inequalities may be rearranged as

$$\begin{aligned} & \lambda[f(x)+g(x)]+(1-\lambda)[f(y)+g(y)) \\ & \quad \geq f(\lambda x+(1-\lambda) y)+g(\lambda x+(1-\lambda) y)\end{aligned}$$

and

$$\begin{aligned} & \lambda[g(x)-f(x)]+(1-\lambda)[g(y)-f(y)] \\ & \quad \geq g(\lambda x+(1-\lambda) y)-f(\lambda x+(1-\lambda) y)\end{aligned}$$

for all $x, y \in I$ and $\lambda \in [0,1]$, which are equivalent to the convexity of $g+f$ and $g-f$ respectively.

The equivalence "$(ii) \Longleftrightarrow (iii)$" is immediate. □

Proposition 1. *Suppose f'', g'' exist and satisfy*

$$|f''(x)| \leq g''(x)$$

on an interval I. Then f is g–convex dominated on I.

Proof. By the given condition

$$g''(x), \quad g''(x)-f''(x), \quad g''(x)+f''(x)$$

are all nonnegative on I, so g, $g-f$, $g+f$ are all convex on I, whence the stated result follows by Lemma 1. □

Theorem 1. *Let $g: I \rightarrow \mathbb{R}$ be a convex function and $f: I \rightarrow \mathbb{R}$ a g–convex dominated mapping. Then for all $a, b \in I$ with $a<b$,*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a} \int_a^b f(x) d x\right| \leq \frac{1}{b-a} \int_a^b g(x) d x-g\left(\frac{a+b}{2}\right)$$

and

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a} \int_a^b f(x) d x\right| \leq \frac{g(a)+g(b)}{2}-\frac{1}{b-a} \int_a^b g(x) d x.$$

Proof. Since f is g–convex dominated, we have by Lemma 1 that $g+f$ and $g-f$ are convex on $[a,b]$, and so by the classical Hermite–Hadamard inequality

$$\begin{aligned} (f+g)\left(\frac{a+b}{2}\right) & \leq \frac{1}{b-a} \int_a^b(f+g)(x) d x \\ & \leq \frac{(f+g)(a)+(f+g)(b)}{2}\end{aligned}$$

and

$$\begin{aligned}(g-f)\left(\frac{a+b}{2}\right) &\leq \frac{1}{b-a}\int_a^b (g-f)(x)\,dx \\ &\leq \frac{(g-f)(a)+(g-f)(b)}{2}.\end{aligned}$$

These inequalities are equivalent to those in the enunciation. □

3. Means

We now give several corollaries that provide examples of convex dominated functions and involve generalised logarithmic means.

Corollary 1. *Suppose $[a,b]\subset(0,\infty)$ and $p\in\mathbb{R}\backslash\{-2,-1\}$. Let $f:[a,b]\to\mathbb{R}$ be a twice differentiable mapping such that $|f''(x)|\leq Mx^p$ $(M>0)$ for $x\in[a,b]$. Then*

$$\begin{aligned}&\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \\ &\qquad\leq \frac{M}{(p+1)(p+2)}\left[[L_{p+2}(a,b)]^{p+2}-[A(a,b)]^{p+2}\right]\end{aligned}$$

and

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \\ &\qquad\leq \frac{M}{(p+1)(p+2)}\left[A\left(a^{p+2},b^{p+2}\right)-[L_{p+2}(a,b)]^{p+2}\right].\end{aligned}$$

Proof. Define the mapping $g:[a,b]\to\mathbb{R}$ by

$$g(x)=\frac{Mx^{p+2}}{(p+1)(p+2)}.$$

Then

$$g''(x)=Mx^p\geq|f''(x)|$$

on $[a,b]$. The stated results follow from Proposition 1 and Theorem 1. □

In particular we derive the following in the case $p=0$.

Remark 1. *Let f be twice differentiable on $[a,b]$ and suppose that*

$$M:=\sup_{x\in[a,b]}|f''(x)|<\infty.$$

Then

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\leq\frac{M}{24}(b-a)^2$$

and

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\leq\frac{M}{12}(b-a)^2.$$

Remark 2. *In the case $p=-3$, the enunciation yields via (1.2) that*

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\,dx \right| \le \frac{M}{2}\cdot\left[\frac{A(a,b)-L(a,b)}{A(a,b)\,L(a,b)}\right]$$

and

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx \right| \le \frac{M}{2}\cdot\left[\frac{L(a,b)-H(a,b)}{H(a,b)\,L(a,b)}\right].$$

We now examine the two cases $p=-2,-1$ excluded in the preceding corollary. First we take $p=-2$.

Corollary 2. *Suppose $[a,b]\subset(0,\infty)$ and let $f:[a,b]\to\mathbb{R}$ be twice differentiable and such that $|f''(x)|\le M/x^2$ for all $x\in(a,b)$. Then*

$$\exp\left[\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\,dx \right|\right] \le \left[\frac{A(a,b)}{I(a,b)}\right]^M$$

and

$$\exp\left[\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx \right|\right] \le \left[\frac{I(a,b)}{G(a,b)}\right]^M.$$

Proof. Define $g:[a,b]\to\mathbb{R}$ by $g(x)=-M\ln x$. Then $g''(x)=M/x^2$. Proposition 1 and Theorem 1 provide

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\,dx \right| \le M\left[\ln\frac{a+b}{2} - \frac{\int_a^b \ln x dx}{b-a}\right]$$

and

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx \right| \le M\left[\frac{\int_a^b \ln x dx}{b-a} - \frac{\ln a+\ln b}{2}\right].$$

The stated inequalities follow from

$$\int_a^b \ln x\ dx = b\ln b - a\ln a - (b-a) = (b-a)\ln\ I(a,b).$$

□

For $p=-1$ we obtain the following.

Corollary 3. *Suppose $[a,b]\subset(0,\infty)$ and let $f:[a,b]\to\mathbb{R}$ be twice differentiable with $|f''(x)|\le M/x$ for all $x\in(a,b)$. Then*

$$\exp\left[\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\,dx \right|\right] \le \frac{\left[\left(\frac{b^{\frac{b^2}{2}}}{a^{\frac{a^2}{2}}}\right)e^{-\frac{3}{4}\left(b^2-a^2\right)}\right]^{\frac{M}{b-a}}}{\left[\left(\frac{a+b}{2}\right)^{\frac{a+b}{2}}e^{-\frac{a+b}{2}}\right]^M},$$

$$\exp\left[\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx \right|\right] \le \frac{\left[\left(a^a b^b\right)^{\frac{1}{2}}e^{-\frac{a+b}{2}}\right]^M}{\left[\left(\frac{b^{\frac{b^2}{2}}}{a^{\frac{a^2}{2}}}\right)e^{-\frac{3}{4}\left(b^2-a^2\right)}\right]^{\frac{M}{b-a}}}.$$

Proof. Consider the mapping $g(x) = Mx\ln x - Mx$. Then $g''(x) = M/x$. Proposition 1 and Theorem 1 provide

$$\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \le \frac{1}{b-a}\int_a^b g(x)\,dx - g\left(\frac{a+b}{2}\right)$$

and

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \le \frac{g(a)+g(b)}{2} - \frac{1}{b-a}\int_a^b g(x)\,dx.$$

We have

$$\int_a^b (x\ln x - x)\,dx = \ln\left[\left(\frac{b^{\frac{b^2}{2}}}{a^{\frac{a^2}{2}}}\right)e^{-\frac{3}{4}\left(b^2-a^2\right)}\right]^M,$$

$$g\left(\frac{a+b}{2}\right) = \ln\left[\left(\frac{a+b}{2}\right)^{\frac{a+b}{2}} e^{-\frac{a+b}{2}}\right]^M,$$

$$\frac{1}{b-a}\int_a^b g(x)\,dx = \ln\left[\left(\frac{b^{\frac{b^2}{2}}}{a^{\frac{a^2}{2}}}\right)e^{-\frac{3}{4}\left(b^2-a^2\right)}\right]^{\frac{M}{b-a}}$$

and

$$\frac{g(a)+g(b)}{2} = \ln\left[\left(a^a b^b\right)^{\frac{1}{2}} e^{-\frac{a+b}{2}}\right]^M.$$

These yield

$$\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \le \ln\left\{\frac{\left[\left(\frac{b^{\frac{b^2}{2}}}{a^{\frac{a^2}{2}}}\right)e^{-\frac{3}{4}\left(b^2-a^2\right)}\right]^{\frac{M}{b-a}}}{\left[\left(\frac{a+b}{2}\right)^{\frac{a+b}{2}} e^{-\frac{a+b}{2}}\right]^M}\right\}$$

and

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \le \ln\left\{\frac{\left[\left(a^a b^b\right)^{\frac{1}{2}} e^{-\frac{a+b}{2}}\right]^M}{\left[\left(\frac{b^{\frac{b^2}{2}}}{a^{\frac{a^2}{2}}}\right)e^{-\frac{3}{4}\left(b^2-a^2\right)}\right]^{\frac{M}{b-a}}}\right\},$$

whence the desired results. □

The remarks in the introduction suggest that the results of Corollaries 2 and 3 may derive in the limit from those of Corollary 1 by letting $p \to -2$ and $p \to -1$ respectively.

We shall have the first inequality in Corollary 2, or rather the version of it obtained by taking natural logarithms of both sides, as a limiting form of the first inequality in Corollary 1 if we can show that

$$\frac{1}{(p+1)(p+2)}\left[\left[L_{p+2}(a,b)\right]^{p+2} - \left[A(a,b)\right]^{p+2}\right] \to \ln\left[\frac{A(a,b)}{I(a,b)}\right]$$

as $p \to -2$. *In extenso* this reads as

$$\frac{1}{(p+1)(p+2)}\left[\frac{b^{p+3}-a^{p+3}}{(p+3)(b-a)} - \left(\frac{a+b}{2}\right)^{p+2}\right] \to \ln\frac{a+b}{2} - \left[\frac{b\ln b - a\ln a}{b-a} - 1\right].$$

Set $p = -2 + h$. It suffices to show that

$$\frac{1}{h}\left[\left(\frac{a+b}{2}\right)^h - \frac{b^{1+h} - a^{1+h}}{(1+h)(b-a)}\right] \to \ \ln\frac{a+b}{2} - \left[\frac{b\ln b - a\ln a}{b-a} - 1\right] \tag{3.1}$$

as $h \to 0$. By L'Hôpital's rule, the left–hand side has limit equal to

$$\frac{d}{dx}\ \left(\frac{a+b}{2}\right)^x\Bigg|_{x=0} - \frac{d}{dx}\ \frac{b^x - a^x}{x(b-a)}\Bigg|_{x=1},$$

which is readily verified to reduce to the right–hand side of (3.1).

The second inequality of Corollary 2 and the two inequalities of Corollary 3 may be derived similarly.

4. Functionals

We now examine how the two induced maps defined in the introduction inherit g–convex dominated properties. We make use of the following result (see [2] and [3]).

Proposition 2. *If g is convex, then*

(*a*) *H_g is convex;*
(*b*) *H_g is monotone nondecreasing.*

Theorem 2. *Let $g : [a, b] \to \mathbb{R}$ be convex and $f : [a, b] \to \mathbb{R}$ a g–convex dominated mapping on $[a, b]$. Then*

(*i*) *H_f is H_g–convex dominated on $[0, 1]$;*
(*ii*) *for all $0 \le t_1 < t_2 \le 1$ we have*

$$0 \le |H_f(t_2) - H_f(t_1)| \le H_g(t_2) - H_g(t_1); \tag{4.1}$$

(*iii*) *for all $t \in [0, 1]$*

$$0 \le \left|f\left(\frac{a+b}{2}\right) - H_f(t)\right| \le H_g(t) - g\left(\frac{a+b}{2}\right)$$

and

$$0 \le \left|\frac{1}{b-a}\int_a^b f(x)\,dx - H_f(t)\right| \le \frac{1}{b-a}\int_a^b g(x)\,dx - H_g(t).$$

Proof. (*i*) Since f is g–convex dominated on $[a, b]$, it follows from Lemma 1 that $g - f$ and $g + f$ are convex on $[a, b]$ and so by Proposition 2 H_{g-f} and H_{g+f} are convex on $[0, 1]$. By the linearity of the mapping $f \mapsto H_f$, we have $H_{g-f} = H_g - H_f$ and $H_{g+f} = H_g + H_f$. Since H_g is convex, Lemma 1 yields that H_f is H_g–dominated on $[0, 1]$.

(ii) By Proposition 2 H_{g-f} and H_{g+f} are monotone nondecreasing on $[0,1]$ and thus

$$H_g(t_1) - H_f(t_1) = H_{g-f}(t_1) \le H_{g-f}(t_2) = H_g(t_2) - H_f(t_2)$$

and

$$H_g(t_1) + H_f(t_1) = H_{g+f}(t_1) \le H_{g+f}(t_2) = H_g(t_2) + H_f(t_2).$$

Therefore

$$H_f(t_2) - H_f(t_1) \le H_g(t_2) - H_g(t_1)$$

and

$$H_g(t_2) - H_g(t_1) \ge -[H_f(t_2) - H_f(t_1)],$$

which are equivalent to (4.1).

(iii) Since

$$H_g(0) = g\left(\frac{a+b}{2}\right) \text{ and } H_f(0) = f\left(\frac{a+b}{2}\right),$$

the first inequality in *(iii)* occurs as a special case of (4.1). Likewise the second arises with $t = 1$.

□

Similarly we have the following result for the second functional, F_f (see [3]).

Proposition 3. *If g is convex, then*

(a) *F_g is convex;*
(b) *F_g is monotone nonincreasing on $[0,1/2]$ and monotone nondecreasing on $[1/2,1]$.*

This leads to the following result.

Theorem 3. *Let $g : [a,b] \to \mathbb{R}$ be convex and $f : [a,b] \to \mathbb{R}$ a g–convex dominated function on $[a,b]$. Then*

(i) *F_f is F_g–convex dominated on $[0,1]$;*
(ii) *we have*

$$0 \le |F_f(t_2) - F_f(t_1)| \le F_g(t_2) - F_g(t_1) \quad \text{for} \quad \frac{1}{2} \le t_1 < t_2 \le 1$$

and

$$0 \le |F_f(t_2) - F_f(t_1)| \le F_g(t_1) - F_g(t_2) \quad \text{for} \quad 0 \le t_1 < t_2 \le \frac{1}{2};$$

(iii) *for all $t \in [0,1]$ we have*

$$0 \le \left| \frac{1}{b-a}\int_a^b f(x)\,dx - F_f(t) \right| \le \frac{1}{b-a}\int_a^b g(x)\,dx - F_g(t),$$

$$\begin{aligned} 0 &\le \left| \frac{1}{(b-a)^2}\int_a^b\int_a^b f\left(\frac{x+y}{2}\right)dxdy - F_f(t) \right| \\ &\le F_g(t) - \frac{1}{(b-a)^2}\int_a^b\int_a^b f\left(\frac{x+y}{2}\right)dxdy \end{aligned}$$

and

$$0 \leq |F_f(t) - H_f(t)| \leq F_g(t) - H_g(t).$$

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