A COMPANION OF OSTROWSKI'S INEQUALITY FOR FUNCTIONS OF BOUNDED VARIATION AND APPLICATIONS

S.S. DRAGOMIR

Abstract. A companion of Ostrowski's inequality for functions of bounded variation and applications are given.

1. Introduction

In [1], the author has proved the following inequality of Ostrowski type for functions of bounded variation.

Theorem 1. *Let $f:[a,b]\to\mathbb{R}$ be a function of bounded variation on $[a,b]$. Denote by $\bigvee_a^b(f)$ its total variation on $[a,b]$. Then, for any $x\in[a,b]$, one has the inequality:*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\le\left[\frac{1}{2}+\left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right]\bigvee_a^b(f). \tag{1.1}$$

The constant $\frac{1}{2}$ is best possible in the sense that it cannot be replaced by a smaller constant.

The above inequality (1.1) has as a remarkable particular case, the mid-point inequality.

The corresponding version for the generalised trapezoid inequality was obtained in [2].

Theorem 2. *With the assumptions in Theorem 1, one has the inequality*

$$\left|\frac{(x-a)f(a)+(b-x)f(b)}{b-a}-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\le\left[\frac{1}{2}+\left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right]\bigvee_a^b(f) \tag{1.2}$$

for any $x\in[a,b]$.

Here the constant $\frac{1}{2}$ is also best possible.

The above inequality (1.2) incorporates the trapezoid inequality.

Recently, Guessab and Schmeisser [3], in the effort of incorporating together the mid-point and trapezoid inequality, have proved amongst others, the following companion of Ostrowski's inequality.

Date: May 20, 2002.

1991 *Mathematics Subject Classification.* Primary 26D15; Secondary 41A55.

Key words and phrases. Ostrowski's inequality, Trapezoid rule, Midpoint rule.

Theorem 3. *Assume that the function* $f:[a,b]\to\mathbb{R}$ *is of* $M-r-$*Hölder type with* $r\in(0,1]$, *i.e.,*

$$|f(t)-f(s)|\leq M|t-s|^{r}\quad \text{for any } t,s\in[a,b]. \tag{1.3}$$

Then, for each $x\in\left[a,\frac{a+b}{2}\right]$, *one has the inequality*

$$\left|\frac{f(x)+f(a+b-x)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq \left[\frac{2^{r+1}(x-a)^{r+1}+(a+b-2x)^{r+1}}{2^{r}(r+1)(b-a)}\right]M. \tag{1.4}$$

This inequality is sharp for each admissable x. *Equality is obtained if and only if* $f=\pm Mf_{*}+c$, *with* $c\in\mathbb{R}$ *and*

$$f_{*}(t)=\begin{cases}(x-t)^{r}, & \text{for } a\leq t\leq x\\ (t-x)^{r}, & \text{for } x\leq t\leq\frac{1}{2}(a+b)\\ f_{*}(a+b-t), & \text{for } \frac{1}{2}(a+b)\leq t\leq b.\end{cases} \tag{1.5}$$

Remark 1. *For* $r=1$, *i.e.,* f *is Lipschitzian with the constant* $L>0$, *and since*

$$\frac{4(x-a)^{2}+(a+b-2x)^{2}}{4(b-a)}=\left[\frac{1}{8}+2\left(\frac{x-\frac{3a+b}{4}}{b-a}\right)^{2}\right](b-a)$$

then, by (1.4), we get the following companion of Ostrowski's inequality for Lipschitzian functions

$$\left|\frac{f(x)+f(a+b-x)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq \left[\frac{1}{8}+2\left(\frac{x-\frac{3a+b}{4}}{b-a}\right)^{2}\right](b-a)L, \tag{1.6}$$

for any $x\in\left[a,\frac{a+b}{2}\right]$.

The constant $\frac{1}{8}$ *is best possible in (1.6) in the sense that it cannot be replaced by a smaller constant.*

By substituting $x=\frac{3a+b}{4}$ *into the above inequality, we obtain the following trapezoid type inequality, which is the best in the class,*

$$\left|\frac{f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\leq\frac{1}{8}(b-a)L. \tag{1.7}$$

The constant $\frac{1}{8}$ *here is also best possible in the above sense.*

For a recent monograph devoted to Ostrowski type inequalities, see [6].

The main aim of this paper is to provide a sharp bound for the difference

$$\frac{f(x)+f(a+b-x)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt,$$

where f is assumed to be of bounded variation. Some applications are also pointed out.

2. Some Integral Inequalities

The following identity holds.

Lemma 1. *Assume that the function* $f : [a, b] \to \mathbb{R}$ *is of bounded variation on* $[a, b]$. *Then we have the equality*

$$\frac{1}{2}[f(x) + f(a+b-x)] - \frac{1}{b-a}\int_a^b f(t)\,dt = \frac{1}{b-a}\left[\int_a^x (t-a)\,df(t) + \int_x^{a+b-x}\left(t - \frac{a+b}{2}\right)df(t) + \int_{a+b-x}^b (t-b)\,df(t)\right] \tag{2.1}$$

for any $x \in \left[a, \frac{a+b}{2}\right]$.

Proof. Obviously, all the Riemann-Stieltjes integrals from the right hand side of (2.1) exist because the functions $(\cdot - a)$, $\left(\cdot - \frac{a+b}{2}\right)$ and $(\cdot - b)$ are continuous on these intervals and f is of bounded variation.

Using the integration by parts formula for Riemann-Stieltjes integrals, we have, for any $x \in \left[a, \frac{a+b}{2}\right]$, that

$$\int_a^x (t-a)\,df(t) = f(x)(x-a) - \int_a^x f(t)\,dt,$$

$$\int_x^{a+b-x}\left(t - \frac{a+b}{2}\right)df(t) = f(a+b-x)\left(\frac{a+b}{2} - x\right) - f(x)\left(x - \frac{a+b}{2}\right) - \int_x^{a+b-x} f(t)\,dt$$

and

$$\int_{a+b-x}^b (t-b)\,df(t) = (x-a)\,f(a+b-x) - \int_{a+b-x}^b f(t)\,dt.$$

Summing the above equalities we deduce (2.1). ∎

Remark 2. *A version of this identity for piecewise continuously differentiable functions has been obtained in* [3, Lemma 3.2].

The following companion of Ostrowski's inequality holds.

Theorem 4. *Assume that the function* $f : [a, b] \to \mathbb{R}$ *is of bounded variation on* $[a, b]$. *Then we have the inequalities:*

$$\left|\frac{1}{2}[f(x) + f(a+b-x)] - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \le \frac{1}{b-a}\left[(x-a)\bigvee_a^x(f) + \left(\frac{a+b}{2} - x\right)\bigvee_x^{a+b-x}(f) + (x-a)\bigvee_{a+b-x}^b(f)\right] \tag{2.2}$$

$$\leq \begin{cases} \left[\frac{1}{4}+\left|\frac{x-\frac{3a+b}{4}}{b-a}\right|\right]\bigvee_a^b(f) \\ \\ \left[2\left(\frac{x-a}{b-a}\right)^{\alpha}+\left(\frac{\frac{a+b}{2}-x}{b-a}\right)^{\alpha}\right]^{\frac{1}{\alpha}} \\ \times\left[\left[\bigvee_a^x(f)\right]^{\beta}+\left[\bigvee_x^{a+b-x}(f)\right]^{\beta}+\left[\bigvee_{a+b-x}^{b}(f)\right]^{\beta}\right]^{\frac{1}{\beta}}, \quad \text{if} \quad \alpha>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1 \\ \\ \left[\frac{x-a+\frac{b-a}{2}}{b-a}\right]\max\left\{\bigvee_a^x(f),\bigvee_x^{a+b-x}(f),\bigvee_{a+b-x}^{b}(f)\right\} \end{cases}$$

for any $x\in\left[a,\frac{a+b}{2}\right]$, *where* $\bigvee_c^d(f)$ *denotes the total variation of* f *on* $[c,d]$. *The constant* $\frac{1}{4}$ *is best possible in the first branch of the second inequality in (2.2).*

Proof. We use the fact that for a continuous function $p:[c,d]\to\mathbb{R}$ and a function $v:[a,b]\to\mathbb{R}$ of bounded variation, one has the inequality

$$\left|\int_c^d p(t)\,dv(t)\right|\leq\sup_{t\in[c,d]}|p(t)|\bigvee_c^d(v). \tag{2.3}$$

Taking the modulus in (2.1) we have

$$\begin{aligned} &\left|\frac{1}{2}\left[f(x)+f(a+b-x)\right]-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \\ \leq\ & \frac{1}{b-a}\left[\left|\int_a^x(t-a)\,df(t)\right|+\left|\int_x^{a+b-x}\left(t-\frac{a+b}{2}\right)df(t)\right|\right. \\ & \left.+\left|\int_{a+b-x}^{b}(t-b)\,df(t)\right|\right] \\ \leq\ & \frac{1}{b-a}\left[(x-a)\bigvee_a^x(f)+\left(\frac{a+b}{2}-x\right)\bigvee_x^{a+b-x}(f)+(x-a)\bigvee_{a+b-x}^{b}(f)\right]=:M(x) \end{aligned}$$

and the first inequality in (2.2) is obtained.

Now, observe that

$$\begin{aligned} M(x) &\leq \frac{1}{b-a}\max\left\{x-a,\frac{a+b}{2}-x\right\}\left[\bigvee_a^x(f)+\bigvee_x^{a+b-x}(f)+\bigvee_{a+b-x}^{b}(f)\right] \\ &= \frac{1}{b-a}\left[\frac{1}{4}(b-a)+\left|x-\frac{3a+b}{4}\right|\right]\bigvee_a^b(f) \end{aligned}$$

and the first branch in the second inequality in (2.2) is proved.

Using Hölder's discrete inequality we have (for $\alpha > 1$, $\frac{1}{\alpha} + \frac{1}{\beta} = 1$) that

$$M(x) \leq \frac{1}{b-a}\left[(x-a)^{\alpha} + \left(\frac{a+b}{2} - x\right)^{\alpha} + (x-a)^{\alpha}\right]^{\frac{1}{\alpha}} \times \left[\left[\bigvee_{a}^{x}(f)\right]^{\beta} + \left[\bigvee_{x}^{a+b-x}(f)\right]^{\beta} + \left[\bigvee_{a+b-x}^{b}(f)\right]^{\beta}\right]^{\frac{1}{\beta}}$$

giving the second branch in the second inequality.

Finally, we have

$$M(x) \leq \frac{1}{b-a}\max\left\{\bigvee_{a}^{x}(f), \bigvee_{x}^{a+b-x}(f), \bigvee_{a+b-x}^{b}(f)\right\} \times \left[(x-a) + \left(\frac{a+b}{2} - x\right) + (x-a)\right],$$

which is equivalent with the last inequality in (2.2).

The sharpness of the constant $\frac{1}{4}$ in the first branch of the second inequality in (2.2) will be proved in a particular case later. ∎

Corollary 1. *With the assumptions in Theorem 4, one has the trapezoid inequality*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq \frac{1}{2}\bigvee_{a}^{b}(f). \tag{2.4}$$

The constant $\frac{1}{2}$ is best possible in (2.4).

Proof. Follows from the first inequality in (2.2) on choosing $x = a$. For the sharpness of the constant, assume that (2.4) holds with a constant $A > 0$, i.e.,

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq A\bigvee_{a}^{b}(f). \tag{2.5}$$

If we choose $f : [a, b] \to \mathbb{R}$ with

$$f(x) = \begin{cases} 1 & \text{if } x = a, \\ 0 & \text{if } x \in (a, b), \\ 1 & \text{if } x = b, \end{cases}$$

then f is of bounded variation on $[a, b]$ and

$$\frac{f(a)+f(b)}{2} = 1, \quad \int_a^b f(t)\,dt = 0, \quad \text{and} \quad \bigvee_{a}^{b}(f) = 2,$$

giving in (2.5) $1 \leq 2A$, thus $A \geq \frac{1}{2}$ and the corollary is proved. ∎

Remark 3. *The inequality (2.4) was first proved in a different manner in* [4].

Corollary 2. *With the assumptions in Theorem 4, one has the midpoint inequality*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\leq\frac{1}{2}\bigvee_a^b(f)\,. \tag{2.6}$$

The constant $\frac{1}{2}$ *is best possible in (2.6).*

Proof. Follows from the first inequality in (2.2) on choosing $x=\frac{a+b}{2}$. For the sharpness of the constant, assume that (2.6) holds with a constant $B>0$, i.e.,

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\leq B\bigvee_a^b(f)\,. \tag{2.7}$$

If we choose $f:[a,b]\to\mathbb{R}$ with

$$f(x)=\begin{cases}0 & \text{if} \quad x\in\left[a,\frac{a+b}{2}\right),\\ 1 & \text{if} \quad x=\frac{a+b}{2},\\ 0 & \text{if} \quad x\in\left(\frac{a+b}{2},b\right],\end{cases}$$

then f is of bounded variation on $[a,b]$, and

$$f\left(\frac{a+b}{2}\right)=1,\quad \int_a^b f(t)\,dt=0,\quad \text{and} \quad \bigvee_a^b(f)=2,$$

giving in (2.7), $1\leq 2B$, thus $B\geq\frac{1}{2}$. ∎

Remark 4. *The inequality (2.6) was firstly proved in a different manner in* [5].

The best inequality we may get from Theorem 4 on using the bound provided by the first branch in the second inequality in (2.2) is incorporated in the following corollary.

Corollary 3. *With the assumptions in Theorem 4, one has the inequality:*

$$\left|\frac{f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\leq\frac{1}{4}\bigvee_a^b(f)\,. \tag{2.8}$$

The constant $\frac{1}{4}$ *is best possible.*

Proof. Follows by Theorem 4 on choosing $x=\frac{3a+b}{4}$.

To prove the sharpness of the constant $\frac{1}{4}$, assume that (2.8) holds with a constant $C>0$, i.e.,

$$\left|\frac{f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\leq C\bigvee_a^b(f)\,. \tag{2.9}$$

Consider the function $f:[a,b]\to\mathbb{R}$, given by

$$f(x)=\begin{cases}1 & \text{if} \quad x\in\left\{\frac{3a+b}{4},\frac{a+3b}{4}\right\},\\ 0 & \text{if} \quad x\in[a,b]\setminus\left\{\frac{3a+b}{4},\frac{a+3b}{4}\right\}.\end{cases}$$

Then f is of bounded variation on $[a,b]$,

$$\frac{f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)}{2}=1,\quad \int_a^b f(t)\,dt=0$$

and

$$\bigvee_a^b(f)=4,$$

giving in (2.9) $4C\geq 1$, thus $C\geq\frac{1}{4}$.

This example can be used to prove the sharpness of the constant $\frac{1}{4}$ in (2.2) as well. ∎

3. Applications for P.D.F.'s

Let X be a random variable taking values in the finite interval $[a,b]$, with the probability density function $f:[a,b]\to[0,\infty)$ and with the cumulative distribution function $F(x)=\Pr(X\leq x)=\int_a^x f(t)\,dt$.

We may state the following theorem.

Theorem 5. *With the above assumptions, we have the inequality*

$$\begin{aligned}(3.1)\qquad &\left|\frac{1}{2}\left[F(x)+F(a+b-x)\right]-\frac{b-E(X)}{b-a}\right|\\ &\leq \frac{1}{b-a}\left\{\left(2x-\frac{3a+b}{4}\right)\left[F(x)-F(a+b-x)\right]+(x-a)\right\}\\ &\leq \frac{1}{4}+\left|\frac{x-\frac{3a+b}{4}}{b-a}\right|,\end{aligned}$$

for any $x\in\left[a,\frac{a+b}{2}\right]$, where $E(X)$ denotes the expectation of X.

Proof. If we apply Theorem 4 for F, which is monotonic nondecreasing, we get

$$\begin{aligned}(3.2)\qquad &\left|\frac{1}{2}\left[F(x)+F(a+b-x)\right]-\frac{1}{b-a}\int_a^b F(t)\,dt\right|\\ &\leq \frac{1}{b-a}\left[(x-a)F(x)+\left(\frac{a+b}{2}-x\right)(F(a+b-x)-F(x))\right.\\ &\qquad\left.+(x-a)(1-F(a+b-x))\right]\\ &\leq \frac{1}{4}+\left|\frac{x-\frac{3a+b}{4}}{b-a}\right|.\end{aligned}$$

Since

$$E(X)=\int_a^b t\,dF(t)=b-\int_a^b F(t)\,dt,$$

then by (3.2) we get (3.1) and the theorem is proved. ∎

In particular, we have:

Corollary 4. *With the above assumptions, we have:*

$$\left|\frac{1}{2}\left[F\left(\frac{3a+b}{4}\right)+F\left(\frac{a+3b}{4}\right)\right]-\frac{b-E(X)}{b-a}\right|\leq\frac{1}{4}.$$

4. A Composite Quadrature Formula

Let $I_n : a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$ be a division of the interval $[a,b]$ and $h_i := x_{i+1} - x_i$ $(i = 0, \dots, n-1)$ and $\nu(I_n) := \max\{h_i | i = 0, \dots, n-1\}$.

Consider the composite quadrature rule

$$Q_n(I_n, f) = \frac{1}{2}\sum_{i=0}^{n-1}\left[f\left(\frac{3x_i + x_{i+1}}{4}\right) + f\left(\frac{x_i + 3x_{i+1}}{4}\right)\right]h_i. \tag{4.1}$$

The following result holds.

Theorem 6. *Let $f : [a,b] \to \mathbb{R}$ be a function of bounded variation on $[a,b]$. Then we have*

$$\int_a^b f(t)\,dt = Q_n(I_n, f) + R_n(I_n, f) \tag{4.2}$$

where $Q_n(I_n, f)$ is defined in formula (4.1), and the remainder $R_n(I_n, f)$ satisfies the estimate

$$|R_n(I_n, f)| \le \frac{1}{4}\nu(I_n)\bigvee_a^b(f). \tag{4.3}$$

The constant $\frac{1}{4}$ is best possible.

Proof. Applying Corollary 3 on the interval $[x_i, x_{i+1}]$ we may state that

$$\left|\int_{x_i}^{x_{i+1}} f(t)\,dt - \frac{1}{2}\left[f\left(\frac{3x_i + x_{i+1}}{4}\right) + f\left(\frac{x_i + 3x_{i+1}}{4}\right)\right]h_i\right| \le \frac{1}{4}h_i\bigvee_{x_i}^{x_{i+1}}(f), \tag{4.4}$$

for any $i \in \{0, \dots, n-1\}$.

Summing the inequality (4.4) over i from 0 to $n-1$, and using the generalised triangle inequality we get

$$|R_n(I_n, f)| \le \frac{1}{4}\sum_{i=0}^{n-1}h_i\bigvee_{x_i}^{x_{i+1}}(f) \le \frac{1}{4}\nu(I_n)\sum_{i=0}^{n-1}\bigvee_{x_i}^{x_{i+1}}(f) = \frac{1}{4}\nu(I_n)\bigvee_a^b(f),$$

and the proof is completed. ∎

For the particular case when the division I_n is equidistant, i.e.,

$$I_n : x_i = a + i \cdot \frac{b-a}{n}, \quad i = 0, \dots, n,$$

we may consider the quadrature rule:

$$Q_n(f) := \frac{b-a}{2n}\sum_{i=0}^{n-1}\left\{f\left[a + \left(\frac{4i+1}{4n}\right)(b-a)\right] + f\left[a + \left(\frac{4i+3}{4n}\right)(b-a)\right]\right\}. \tag{4.5}$$

The following corollary will be more useful in practice.

Corollary 5. *With the assumption of Theorem 6, we have*

$$\int_a^b f(t)\,dt = Q_n(f) + R_n(f), \tag{4.6}$$

where $Q_n(f)$ *is defined by (4.5) and the remainder* $R_n(f)$ *satisfies the estimate*

$$|R_n(f)| \leq \frac{1}{4} \cdot \frac{b-a}{n} \bigvee_a^b (f) . \tag{4.7}$$

The constant $\frac{1}{4}$ *is sharp.*

Remark 5. *If one is interested in finding the minimal number of points for the equidistant partition* I_n *so that the theoretical error in (4.7) is smaller that* $\varepsilon > 0$, *then this number* n_ε *is given by*

$$n_\varepsilon := \left[\frac{1}{4} \cdot \frac{b-a}{\varepsilon} \bigvee_a^b (f) \right] + 1. \tag{4.8}$$

References

[1] S.S. DRAGOMIR, On the Ostrowski's integral inequality for mappings with bounded variation and applications, *Math. Ineq. & Appl.,* **4**(1) (2001), 59-66.

[2] P. CERONE, S.S. DRAGOMIR and C.E.M. PEARCE, A generalised trapezoid inequality for functions of bounded variation, *Turkish J. Math.,* **24**(2) (2000), 147-163.

[3] A. GUESSAB and G. SCHMEISSER, Sharp integral inequalities of the Hermite-Hadamard type, *J. Approx. Th.,* **115** (2002), 260-288.

[4] S.S. DRAGOMIR, The Ostrowski integral inequality for mappings of bounded variation, *Bull. Austral. Math. Soc.,* **60**(1) (1999), 495-508.

[5] S.S. DRAGOMIR, On the midpoint quadrature formula for mappings with bounded variation and applications, *Kragujevac J. Math.,* **22** (2000), 13-18.

[6] S.S. DRAGOMIR and Th.M. RASSIAS (Ed.), *Ostrowski Type Inequalities and Applications in Numerical Integration,* Kluwer Academic Publishers, Dordrechy/Boston/London, 2002.

School of Communications and Informatics, Victoria University of Technology, PO Box 14428, MCMC 8001, Victoria, Australia.

E-mail address: sever@matilda.vu.edu.au

URL: http://rgmia.vu.edu.au/SSDragomirWeb.html