SOME INEQUALITIES FOR THE CHEBYSHEV FUNCTIONAL

Sever S. Dragomir
School of Communications & Informatics,
Victoria University, PO Box 14428, MC 8001,
Melbourne, Victoria, Australia

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1. Introduction

For two real n-tuples $\overline{\mathbf{a}}=(a_1,...,a_n)\,,\overline{\mathbf{b}}=(b_1,...,b_n)$ and $\overline{\mathbf{p}}=(p_1,...,p_n)$ with $p_i\geq 0, i\in\{1,...,n\}$ and $\sum_{i=1}^n p_i=1$, consider the Chebyshev functional:

$$T\left(\overline{\mathbf{p}},\overline{\mathbf{a}},\overline{\mathbf{b}}\right):=\sum_{i=1}^n p_ia_ib_i-\sum_{i=1}^n p_ia_i\sum_{i=1}^n p_ib_i. \tag{1}$$

By Korkine's identity (obtained in 1882, [1], see also [2] or [3, p. 242]) one has the representation

$$\begin{aligned} T\left(\overline{\mathbf{p}},\overline{\mathbf{a}},\overline{\mathbf{b}}\right) &= \frac{1}{2}\sum_{i,j=1}^n p_ip_j\left(a_j-a_i\right)\left(b_j-b_i\right) \qquad (2)\\ &= \sum_{1\leq i<j\leq n}^n p_ip_j\left(a_j-a_i\right)\left(b_j-b_i\right). \end{aligned}$$

By Sonin's identity (obtained in 1898, see [4], or [3, p. 246]), we also have the representation

$$T\left(\overline{\mathbf{p}},\overline{\mathbf{a}},\overline{\mathbf{b}}\right)=\sum_{i=1}^n\left[p_ia_i-A_n\left(\overline{\mathbf{p}},\overline{\mathbf{a}}\right)\right]\left[b_i-A_n\left(\overline{\mathbf{p}},\overline{\mathbf{b}}\right)\right] \tag{3}$$

where

$$A_n\left(\overline{\mathbf{p}},\overline{\mathbf{a}}\right):=\sum_{i=1}^n p_ia_i.$$

We note that, the above identities may be proved by direct computation. The proof is left as an exercise to the reader.

Observe that if the n-tuples $\overline{\mathbf{a}},\overline{\mathbf{b}}$ are synchronous, i.e.,

$$\left(a_j-a_i\right)\left(b_j-b_i\right)\geq 0 \text{ for any } i,j\in\{1,...,n\}\,,$$

then obviously

$$T\left(\overline{\mathbf{p}},\overline{\mathbf{a}},\overline{\mathbf{b}}\right)\geq 0 \tag{4}$$

which is well known in the litarature as *Chebyshev's inequality.*

Another sufficient condition for the positivity of the Chebyshev functional that easily follows from the identity (3) is

$$\left[a_i-A_n\left(\overline{\mathbf{p}},\overline{\mathbf{a}}\right)\right]\left[b_i-A_n\left(\overline{\mathbf{p}},\overline{\mathbf{b}}\right)\right]\geq 0 \text{ for any } i\in\{1,...,n\}\,.$$

The following result for the Chebyshev functional holds:

$$\left[T\left(\overline{\mathbf{p}},\overline{\mathbf{a}},\overline{\mathbf{b}}\right)\right]^2\leq T\left(\overline{\mathbf{p}},\overline{\mathbf{a}},\overline{\mathbf{a}}\right)T\left(\overline{\mathbf{p}},\overline{\mathbf{b}},\overline{\mathbf{b}}\right). \tag{5}$$

It is the main aim of this note to provide some reverse inequalities for (5).

2. The Results

The following lemma that is of interest in itself holds.

Lemma 1. *Assume that* $\overline{\mathbf{a}} = (a_1, ..., a_n)$, $\overline{\mathbf{b}} = (b_1, ..., b_n)$ *are real* n*-tuples so that for any* $i, j \in \{1, ..., n\}$ *with* $i < j$, *one has*

$$m(b_j - b_i) \leq a_j - a_i \leq M(b_j - b_i), \tag{6}$$

where m, M *are given real numbers. If* $\overline{\mathbf{p}} = (p_1, ..., p_n)$ *with* $p_i \geq 0, i \in \{1, ..., n\}$ *and* $\sum_{i=1}^{n} p_i = 1$, *then one has the inequality*

$$(m + M) T(\overline{\mathbf{p}}, \overline{\mathbf{a}}, \overline{\mathbf{b}}) \geq T(\overline{\mathbf{p}}, \overline{\mathbf{a}}, \overline{\mathbf{a}}) + mMT(\overline{\mathbf{p}}, \overline{\mathbf{b}}, \overline{\mathbf{b}}). \tag{7}$$

Proof. Using the assumption (6) we obviously have

$$[M(b_j - b_i) - (a_j - a_i)][(a_j - a_i) - m(b_j - b_i)] \geq 0 \tag{8}$$

for any $i, j \in \{1, ..., n\}$ with $i < j$. This relation is equivalent, after multiplication, to

$$(a_j - a_i)^2 + mM(b_j - b_i)^2 \leq (m + M)(a_j - a_i)(b_j - b_i), \tag{9}$$

for $i, j \in \{1, ..., n\}$ with $i < j$. Multiplying (9) by $p_i p_j \geq 0$, summing over i, j with $1 \leq i < j \leq n$ and using Korkine's identity we deduce the required inequality (7). ■

Remark 2. *A sufficient condition for* (6) *to hold is*

$$m\Delta b_l \leq \Delta a_l \leq M\Delta b_l \tag{10}$$

for any $l \in \{1, ..., n-1\}$, *where* $\Delta b_l := b_{l+1} - b_l$ *denotes the forward difference.*

The following result provides a reverse inequality for (5).

Theorem 3. *If* $\overline{\mathbf{p}}, \overline{\mathbf{a}}, \overline{\mathbf{b}}$ *are as in Lemma* 1 *and, in addition, we assume that* $M \geq m > 0$, *then the following inequality*

$$\left[T(\overline{\mathbf{p}}, \overline{\mathbf{a}}, \overline{\mathbf{b}})\right]^2 \geq \frac{4mM}{(m + M)^2} T(\overline{\mathbf{p}}, \overline{\mathbf{a}}, \overline{\mathbf{a}}) T(\overline{\mathbf{p}}, \overline{\mathbf{b}}, \overline{\mathbf{b}}). \tag{11}$$

holds. The constant 4 *in* (11) *is best in the sense that it cannot be replaced by a larger constant.*

Proof. We use the following elementary inequality

$$\alpha x^2 + \frac{1}{\alpha} y^2 \geq 2xy, x, y \geq 0, \alpha > 0;$$

to get, for the choices

$$\begin{aligned} \alpha &= \sqrt{mM} > 0, x = \left[T(\overline{\mathbf{p}}, \overline{\mathbf{b}}, \overline{\mathbf{b}})\right]^{1/2} \geq 0, \\ \text{and } y &= \left[T(\overline{\mathbf{p}}, \overline{\mathbf{a}}, \overline{\mathbf{a}})\right]^{1/2} \geq 0 \end{aligned}$$

the inequality

$$\begin{aligned} & \sqrt{mM} T\left(\overline{\mathbf{p}}, \overline{\mathbf{b}}, \overline{\mathbf{b}}\right) + \frac{1}{\sqrt{mM}} T\left(\overline{\mathbf{p}}, \overline{\mathbf{a}}, \overline{\mathbf{a}}\right) \\ \geq \; & 2\left[T\left(\overline{\mathbf{p}}, \overline{\mathbf{a}}, \overline{\mathbf{a}}\right)\right]^{1/2}\left[T\left(\overline{\mathbf{p}}, \overline{\mathbf{b}}, \overline{\mathbf{b}}\right)\right]^{1/2}. \end{aligned} \tag{12}$$

Now, making use of (7) and (12) we deduce the desired inequality (11).

To prove the sharpness of the constant in the sense mentioned above, assume that (11) holds with a constant $C > 0$, *i.e.*,

$$\left[T\left(\overline{\mathbf{p}}, \overline{\mathbf{a}}, \overline{\mathbf{b}}\right)\right]^2 \geq \frac{CmM}{(m+M)^2} T\left(\overline{\mathbf{p}}, \overline{\mathbf{a}}, \overline{\mathbf{a}}\right) T\left(\overline{\mathbf{p}}, \overline{\mathbf{b}}, \overline{\mathbf{b}}\right). \tag{13}$$

Choose $n = 2$, and assume that $b_2 > b_1, a_2 > a_1, p_1 > 0, p_2 > 0$ and $p_1 + p_2 = 1$. Also suppose that $m(b_2 - b_1) \leq a_2 - a_1 \leq M(b_2 - b_1)$ with $M > m > 0$.

Since, in this case, we obviously have

$$\begin{aligned} T\left(\overline{\mathbf{p}}, \overline{\mathbf{a}}, \overline{\mathbf{b}}\right) &= p_1 p_2 (a_2 - a_1)(b_2 - b_1) \\ T\left(\overline{\mathbf{p}}, \overline{\mathbf{a}}, \overline{\mathbf{a}}\right) &= p_1 p_2 (a_2 - a_1)^2 \\ T\left(\overline{\mathbf{p}}, \overline{\mathbf{b}}, \overline{\mathbf{b}}\right) &= p_1 p_2 (b_2 - b_1)^2 \end{aligned}$$

then by (11) we get

$$\begin{aligned} & \left[p_1 p_2 (a_2 - a_1)(b_2 - b_1)\right]^2 \\ \geq \; & \frac{CmM}{(m+M)^2} p_1 p_2 (a_2 - a_1)^2 p_1 p_2 (b_2 - b_1)^2 \end{aligned}$$

giving

$$(m+M)^2 \geq CmM.$$

If in this last inequality we let $m := 1 - \varepsilon, M := 1 + \varepsilon, \varepsilon \in (0,1)$, then we obtain $4 \geq C\left(1 - \varepsilon^2\right)$ for any $\varepsilon \in (0,1)$ showing that $C \leq 4$. ■

The following corollary is a natural consequence of the above theorem and provides different counterparts for the inequality (5).

Corollary 4. *With the assumptions in Theorem 3 we have*

$$\begin{aligned} 0 &\leq \left[T\left(\overline{\mathbf{p}}, \overline{\mathbf{a}}, \overline{\mathbf{a}}\right)\right]^{1/2}\left[T\left(\overline{\mathbf{p}}, \overline{\mathbf{b}}, \overline{\mathbf{b}}\right)\right]^{1/2} - T\left(\overline{\mathbf{p}}, \overline{\mathbf{a}}, \overline{\mathbf{b}}\right) \\ &\leq \frac{\left(\sqrt{M} - \sqrt{m}\right)^2}{2\sqrt{mM}} T\left(\overline{\mathbf{p}}, \overline{\mathbf{a}}, \overline{\mathbf{b}}\right) \end{aligned}$$

and

$$\begin{aligned} 0 &\leq T\left(\overline{\mathbf{p}}, \overline{\mathbf{a}}, \overline{\mathbf{a}}\right) T\left(\overline{\mathbf{p}}, \overline{\mathbf{b}}, \overline{\mathbf{b}}\right) - \left[T\left(\overline{\mathbf{p}}, \overline{\mathbf{a}}, \overline{\mathbf{b}}\right)\right]^2 \\ &\leq \frac{(M-m)^2}{4mM}\left[T\left(\overline{\mathbf{p}}, \overline{\mathbf{a}}, \overline{\mathbf{b}}\right)\right]^2. \end{aligned}$$

The following result also holds

Theorem 5. *Let $f,g:[\alpha,\beta]\to R$ be continuous on $[\alpha,\beta]$ and differentiable on (α,β) with $g'(x)\neq 0$ for $x\in(\alpha,\beta)$. Assume also that*

$$-\infty<\gamma:=\inf_{x\in(\alpha,\beta)}\frac{f'(x)}{g'(x)},\ \sup_{x\in(\alpha,\beta)}\frac{f'(x)}{g'(x)}=:\Gamma<\infty. \tag{14}$$

If $\overline{\mathbf{x}}:=(x_1,...,x_n)$ is a real n-tuple with $x_i\in[\alpha,\beta]$ and $x_i\neq x_j$ for $i\neq j$ and if we denote by $\mathbf{f}(\overline{\mathbf{x}})$ the n-tuple $(f(x_1),...,f(x_n))$, then we have the inequality

$$(\gamma+\Gamma)\,T(\overline{\mathbf{p}},\mathbf{f}(\overline{\mathbf{x}}),\mathbf{g}(\overline{\mathbf{x}}))\geq T(\overline{\mathbf{p}},\mathbf{f}(\overline{\mathbf{x}}),\mathbf{f}(\overline{\mathbf{x}}))+\gamma\Gamma T(\overline{\mathbf{p}},\mathbf{g}(\overline{\mathbf{x}}),\mathbf{g}(\overline{\mathbf{x}})) \tag{15}$$

for any $\overline{\mathbf{p}}=(p_1,...,p_n)$ with $p_i\geq 0,i\in\{1,...,n\}$ and $\sum_{i=1}^n p_i=1$.

Proof. Applying Cauchy Mean-Value Theorem, for any $i,j\in\{1,...,n\}$ with $i<j$ there exists $\xi_{ij}\in(\alpha,\beta)$ so that

$$\frac{f(x_j)-f(x_i)}{g(x_j)-g(x_i)}=\frac{f'\left(\xi_{ij}\right)}{g'\left(\xi_{ij}\right)}\in[\gamma,\Gamma].$$

Then obviously

$$\left[\frac{f(x_j)-f(x_i)}{g(x_j)-g(x_i)}-\gamma\right]\left[\Gamma-\frac{f(x_j)-f(x_i)}{g(x_j)-g(x_i)}\right]\geq 0$$

for any $i,j\in\{1,...,n\}$ with $i<j$. Using a similar argument to the one in Lemma 1 we deduce the desired result (15). ■

The following corollary is a natural consequence of the above theorem.

Corollary 6. *With the assumptions in Theorem 5 and if, in addition, $\Gamma\geq\gamma>0$, then one has the inequalities*

$$[T(\overline{\mathbf{p}},\mathbf{f}(\overline{\mathbf{x}}),\mathbf{g}(\overline{\mathbf{x}}))]^2\geq\frac{4\gamma\Gamma}{(\gamma+\Gamma)^2}T(\overline{\mathbf{p}},\mathbf{f}(\overline{\mathbf{x}}),\mathbf{f}(\overline{\mathbf{x}}))\,T(\overline{\mathbf{p}},\mathbf{g}(\overline{\mathbf{x}}),\mathbf{g}(\overline{\mathbf{x}})),$$

$$\begin{aligned}0&\leq[T(\overline{\mathbf{p}},\mathbf{f}(\overline{\mathbf{x}}),\mathbf{f}(\overline{\mathbf{x}}))]^{1/2}[T(\overline{\mathbf{p}},\mathbf{g}(\overline{\mathbf{x}}),\mathbf{g}(\overline{\mathbf{x}}))]^{1/2}-T(\overline{\mathbf{p}},\mathbf{f}(\overline{\mathbf{x}}),\mathbf{g}(\overline{\mathbf{x}}))\\&\leq\frac{\left(\sqrt{\Gamma}-\sqrt{\gamma}\right)^2}{2\sqrt{\Gamma\gamma}}T(\overline{\mathbf{p}},\mathbf{f}(\overline{\mathbf{x}}),\mathbf{g}(\overline{\mathbf{x}})),\end{aligned}$$

and

$$\begin{aligned}0&\leq T(\overline{\mathbf{p}},\mathbf{f}(\overline{\mathbf{x}}),\mathbf{f}(\overline{\mathbf{x}}))\,T(\overline{\mathbf{p}},\mathbf{g}(\overline{\mathbf{x}}),\mathbf{g}(\overline{\mathbf{x}}))-[T(\overline{\mathbf{p}},\mathbf{f}(\overline{\mathbf{x}}),\mathbf{g}(\overline{\mathbf{x}}))]^2\\&\leq\frac{(\Gamma-\gamma)^2}{4\Gamma\gamma}[T(\overline{\mathbf{p}},\mathbf{f}(\overline{\mathbf{x}}),\mathbf{g}(\overline{\mathbf{x}}))]^2.\end{aligned}$$

Now, we point out other sufficient conditions for the inequality (7) to hold.

Lemma 7. *Assume that $\overline{\mathbf{a}}=(a_1,...,a_n)$, $\overline{\mathbf{b}}=(b_1,...,b_n)$ are real n-tuples, $\overline{\mathbf{p}}=(p_1,...,p_n)$ with $p_i\geq 0,i\in\{1,...,n\}$ and $\sum_{i=1}^n p_i=1$ and $b_i\neq A_n\left(\overline{\mathbf{p}},\overline{\mathbf{b}}\right)$ for each $i\in\{1,...,n\}$. If*

$$-\infty<l\leq\frac{a_i-A_n(\overline{\mathbf{p}},\overline{\mathbf{a}})}{b_i-A_n\left(\overline{\mathbf{p}},\overline{\mathbf{b}}\right)}\leq L<\infty; \tag{16}$$

for any $i\in\{1,...,n\}$, where l,L are given real numbers, then one has the inequality

$$(l+L)\,T\left(\overline{\mathbf{p}},\overline{\mathbf{a}},\overline{\mathbf{b}}\right)\geq T(\overline{\mathbf{p}},\overline{\mathbf{a}},\overline{\mathbf{a}})+lLT\left(\overline{\mathbf{p}},\overline{\mathbf{b}},\overline{\mathbf{b}}\right). \tag{17}$$

Proof. Using (16) we obviously have

$$\left[L-\frac{a_i-A_n(\overline{\mathbf{p}},\overline{\mathbf{a}})}{b_i-A_n(\overline{\mathbf{p}},\overline{\mathbf{b}})}\right]\left[\frac{a_i-A_n(\overline{\mathbf{p}},\overline{\mathbf{a}})}{b_i-A_n(\overline{\mathbf{p}},\overline{\mathbf{b}})}-l\right]\geq 0 \tag{18}$$

for each $i\in\{1,...,n\}$.

If we multiply (18) by $\left[b_i-A_n(\overline{\mathbf{p}},\overline{\mathbf{b}})\right]^2\geq 0$, we get

$$\begin{aligned}&[a_i-A_n(\overline{\mathbf{p}},\overline{\mathbf{a}})]^2+lL\left[b_i-A_n(\overline{\mathbf{p}},\overline{\mathbf{b}})\right]^2\\ \leq\ &(l+L)\left[a_i-A_n(\overline{\mathbf{p}},\overline{\mathbf{a}})\right]\left[b_i-A_n(\overline{\mathbf{p}},\overline{\mathbf{b}})\right]\end{aligned} \tag{19}$$

for each $i\in\{1,...,n\}$.

Finally, if we multiply (19) by $p_i\geq 0$, sum over i from 1 to n and use Sonin's identity, then we obtain the desired inequality (17). ■

Using Lemma 7 and a similar argument to that in the Theorem 3 we may state the following result as well.

Theorem 8. *With the assumptions of Lemma 7 and, in addition, if $L\geq l>0$, then one has the inequality*

$$\left[T(\overline{\mathbf{p}},\overline{\mathbf{a}},\overline{\mathbf{b}})\right]^2\geq\frac{4lL}{(l+L)^2}T(\overline{\mathbf{p}},\overline{\mathbf{a}},\overline{\mathbf{a}})\,T(\overline{\mathbf{p}},\overline{\mathbf{b}},\overline{\mathbf{b}}).$$

The constant 4 cannot be replaced by a larger constant.

The following corollary also holds

Corollary 9. *With the assumptions of Lemma 7 and, in addition, if $L\geq l>0$, then one has the inequalities*

$$\begin{aligned}0\ &\leq\ [T(\overline{\mathbf{p}},\overline{\mathbf{a}},\overline{\mathbf{a}})]^{1/2}\left[T(\overline{\mathbf{p}},\overline{\mathbf{b}},\overline{\mathbf{b}})\right]^{1/2}-T(\overline{\mathbf{p}},\overline{\mathbf{a}},\overline{\mathbf{b}})\\ &\leq\ \frac{\left(\sqrt{L}-\sqrt{l}\right)^2}{2\sqrt{lL}}T(\overline{\mathbf{p}},\overline{\mathbf{a}},\overline{\mathbf{b}})\end{aligned}$$

and

$$\begin{aligned}0\ &\leq\ T(\overline{\mathbf{p}},\overline{\mathbf{a}},\overline{\mathbf{a}})\,T(\overline{\mathbf{p}},\overline{\mathbf{b}},\overline{\mathbf{b}})-\left[T(\overline{\mathbf{p}},\overline{\mathbf{a}},\overline{\mathbf{b}})\right]^2\\ &\leq\ \frac{(L-l)^2}{4lL}\left[T(\overline{\mathbf{p}},\overline{\mathbf{a}},\overline{\mathbf{b}})\right]^2.\end{aligned}$$

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