GENERALIZATIONS OF EULER NUMBERS AND POLYNOMIALS

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Abstract. In this paper, the concepts of Euler numbers and Euler polynomials are generalized, and some basic properties are investigated.

1. Introduction

It is well-known that the Euler numbers and polynomials can be defined by the following definitions.

Definition 1.1 ([1])**.** The Euler numbers E_k are defined by the following expansion

$$\frac{2e^t}{e^{2t}+1}=\sum_{k=0}^{\infty}\frac{E_k}{k!}t^k, \quad |t|\le\pi. \tag{1.1}$$

In [4, p. 5], the Euler numbers is defined by

$$\frac{2e^{t/2}}{e^t+1}=\operatorname{sech}\frac{t}{2}=\sum_{n=0}^{\infty}\frac{(-1)^nE_n}{(2n)!}\left(\frac{t}{2}\right)^{2n}, \quad |t|\le\pi. \tag{1.2}$$

Definition 1.2 ([1, 4])**.** The Euler polynomials $E_k(x)$ for $x\in\mathbb{R}$ are defined by

$$\frac{2e^{xt}}{e^t+1}=\sum_{k=0}^{\infty}\frac{E_k(x)}{k!}t^k, \quad |t|\le\pi. \tag{1.3}$$

It can also be shown that the polynomials $E_i(t)$, $i\in\mathbb{N}$, are uniquely determined by the following two properties

$$E_i'(t)=iE_{i-1}(t), \quad E_0(t)=1; \tag{1.4}$$

$$E_i(t+1)+E_i(t)=2t^i. \tag{1.5}$$

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Euler polynomials are related to the Bernoulli numbers. For information about Bernoulli numbers and polynomials, please refer to [1, 2, 3, 4].

In this paper, we will give some generalizations of the concepts of Euler numbers and Euler polynomials, and research their basic properties.

2. Generalizations of Euler numbers and polynomials

In this section, we give two definitions, the generalized Euler number and the generalized Euler polynomial, which generalize the concepts of Euler number and Euler polynomial.

Definition 2.1. For positive numbers a, b, c, the generalized Euler numbers $E_k(a, b, c)$ are defined by

$$\frac{2c^t}{b^{2t} + a^{2t}} = \sum_{k=0}^{\infty} \frac{E_k(a, b, c)}{k!} t^k. \tag{2.1}$$

Definition 2.2. For any given positive numbers a, b, c and $x \in \mathbb{R}$, the generalized Euler polynomials $E_k(x; a, b, c)$ are defined by

$$\frac{2c^{xt}}{b^t + a^t} = \sum_{k=0}^{\infty} \frac{E_k(x; a, b, c)}{k!} t^k. \tag{2.2}$$

If taking $a = 1$, $b = c = e$, then Definition 1.1 and Definition 1.2 can be deduced from Definition 2.1 and Definition 2.2 respectively. Thus, the Definition 2.1 and Definition 2.2 generalize the concepts of Euler numbers and polynomials.

3. Some properties of the generalized Euler numbers

In this section, we study some basic properties of the generalized Euler numbers defined in Definition 2.1.

Theorem 3.1. *For positive numbers a, b, c and real number $x \in \mathbb{R}$, we have*

$$E_0(a, b, c) = 1, \quad E_k(1, e, e) = E_k, \quad E_k(1, e^{1/2}, e^x) = E_k(x), \tag{3.1}$$

$$E_k(a, b, c) = 2^k (\ln b - \ln a)^k E_k \left(\frac{\ln c - 2\ln a}{2(\ln b - \ln a)} \right), \tag{3.2}$$

$$E_k(a, b, c) = \sum_{j=0}^{k} \binom{k}{j} (\ln b - \ln a)^j (\ln c - \ln a - \ln b)^{k-j} E_j. \tag{3.3}$$

Proof. The formulae in (3.1) follow from Definition 1.1, Definition 1.2, and Definition 2.1 easily.

By Definition 1.2 and Definition 2.1 and direct computation, we have

$$\begin{aligned}\frac{2c^t}{b^{2t}-a^{2t}}&=\frac{2\exp\left(\frac{\ln c-2\ln a}{2(\ln b-\ln a)}\cdot 2t(\ln b-\ln a)\right)}{\exp\bigl(2t(\ln b-\ln a)\bigr)+1}\\&=\sum_{k=0}^{\infty}2^k(\ln b-\ln a)^kE_k\left(\frac{\ln c-2\ln a}{2(\ln b-\ln a)}\right)\frac{t^k}{k!}.\end{aligned}\tag{3.4}$$

Then, the formula (3.2) follows.

Substituting $E_k(x)=\sum_{j=0}^{k}2^{-j}\binom{k}{j}(x-\frac{1}{2})^{k-j}E_j$ into the formula (3.2) yields the formula (3.3). □

Theorem 3.2. *For $k\in\mathbb{N}$, we have*

$$E_k(a,b,c)=-\frac{1}{2}\sum_{j=0}^{k-1}\binom{k}{j}\left[(2\ln b-\ln c)^{k-j}+(2\ln a-\ln c)^{k-j}\right]E_j(a,b,c),\tag{3.5}$$

$$E_k(a,b,c)=E_k(b,a,c),\tag{3.6}$$

$$E_k(a^\alpha,b^\alpha,c^\alpha)=\alpha^kE_k(a,b,c).\tag{3.7}$$

Proof. By Definition 2.1, direct calculation yields

$$\begin{aligned}1&=\frac{1}{2}\left[\left(\frac{b^2}{c}\right)^t+\left(\frac{a^2}{c}\right)^t\right]\sum_{k=0}^{\infty}\frac{t^k}{k!}E_k(a,b,c)\\&=\frac{1}{2}\sum_{k=0}^{\infty}\frac{t^k}{k!}\left[\left(\ln\frac{b^2}{c}\right)^k+\left(\ln\frac{a^2}{c}\right)^k\right]\sum_{k=0}^{\infty}\frac{t^k}{k!}E_k(a,b,c)\\&=\frac{1}{2}\sum_{k=0}^{\infty}\left(\sum_{j=0}^{k}\binom{k}{j}\left[\left(\ln\frac{b^2}{c}\right)^{k-j}+\left(\ln\frac{a^2}{c}\right)^{k-j}\right]E_j(a,b,c)\right)\frac{t^k}{k!}.\end{aligned}\tag{3.8}$$

Equating coefficients of t^k in (3.8) gives us

$$\sum_{j=0}^{k}\binom{k}{j}\left[\left(\ln\frac{b^2}{c}\right)^{k-j}+\left(\ln\frac{a^2}{c}\right)^{k-j}\right]E_j(a,b,c)=0.\tag{3.9}$$

Formula (3.5) follows.

The rest formulae follow from Definition 2.1 and formula (3.2). □

Remark 3.1. For positive numbers a, b, and c, we have the following

$$\begin{aligned}&E_0(a,b,c)=1,\\&E_1(a,b,c)=\ln c-\ln a-\ln b,\\&E_2(a,b,c)=(\ln c-2\ln a)(\ln c-2\ln b),\\&E_3(a,b,c)=[(\ln c-\ln a-\ln b)^2-3(\ln b-\ln a)^2](\ln c-\ln a-\ln b).\end{aligned}\tag{3.10}$$

4. Some properties of the generalized Euler polynomials

In this section, we will investigate properties of the generalized Euler polynomials defined by Definition 2.2.

Theorem 4.1. *For any given positive numbers a,b,c and $x\in\mathbb{R}$, we have*

$$E_k(x;a,b,c)=\sum_{j=0}^{k}\binom{k}{j}\frac{(\ln c)^{k-j}}{2^j}\left(x-\frac{1}{2}\right)^{k-j}E_j(a,b,c), \tag{4.1}$$

$$E_k(x;a,b,c)=\sum_{j=0}^{k}\binom{k}{j}(\ln c)^{k-j}\left(\ln\frac{b}{a}\right)^{j}\left(x-\frac{1}{2}\right)^{k-j}E_j\left(\frac{\ln c-2\ln a}{2(\ln b-\ln a)}\right), \tag{4.2}$$

$$E_k(x;a,b,c)=\sum_{j=0}^{k}\sum_{\ell=0}^{j}\binom{k}{j}\binom{j}{\ell}\frac{(\ln c)^{k-j}}{2^j}\left[\ln\frac{b}{a}\right]^{\ell}\left[\ln\frac{c}{ab}\right]^{j-\ell}\left[x-\frac{1}{2}\right]^{k-j}E_\ell, \tag{4.3}$$

$$E_k(a,b,c)=2^kE_k\left(\frac{1}{2};a,b,c\right), \tag{4.4}$$

$$E_k(x)=E_k(x;1,e,e). \tag{4.5}$$

Proof. By Definition 2.1 and Definition 2.2, we have

$$\frac{2c^{2xt}}{b^{2t}+a^{2t}}=\sum_{k=0}^{\infty}2^kE_k(x;a,b,c)\frac{t^k}{k!} \tag{4.6}$$

and

$$\begin{aligned}\frac{2c^{2xt}}{b^{2t}+a^{2t}}&=\frac{2c^{t}}{b^{2t}+a^{2t}}\cdot c^{(2x-1)t}\\&=\left(\sum_{k=0}^{\infty}\frac{t^k}{k!}E_k(a,b,c)\right)\left(\sum_{k=0}^{\infty}\frac{t^k}{k!}(2x-1)^k(\ln c)^k\right)\\&=\sum_{k=0}^{\infty}\left(\sum_{j=0}^{k}\binom{k}{j}(\ln c)^{k-j}(2x-1)^{k-j}E_j(a,b,c)\right)\frac{t^k}{k!}.\end{aligned} \tag{4.7}$$

Equating the coefficients of $\frac{t^k}{k!}$ in (4.6) and (4.7) yields

$$2^kE_k(x;a,b,c)=\sum_{j=0}^{k}\binom{k}{j}(\ln c)^{k-j}(2x-1)^{k-j}E_j(a,b,c).$$

Formula (4.1) follows.

The rest formulae follow directly from substituting formulae (3.2) and (3.3) into (4.1) and taking $x=\frac{1}{2}$ in (4.1), respectively. $\square$

Theorem 4.2. *For positive integer $1 \le p \le k$, we have*

$$\frac{\partial^p}{\partial x^p} E_k(x;a,b,c) = \frac{k!}{(k-p)!}(\ln c)^p E_{k-p}(x;a,b,c), \tag{4.8}$$

$$\int_\beta^x E_k(t;a,b,c)\mathrm{d}t = \frac{1}{(k+1)\ln c}\left[E_{k+1}(x;a,b,c) - E_{k+1}(\beta;a,b,c)\right]. \tag{4.9}$$

Proof. Differentiating on both sides of formula (4.1) yields

$$\frac{\partial}{\partial x} E_k(x;a,b,c) = k(\ln c)E_{k-1}(x;a,b,c). \tag{4.10}$$

Using formula (4.10) and by mathematical induction, the formulae (4.8) follows.

Rearranging formula (4.10) produces

$$E_k(x;a,b,c) = \frac{1}{(k+1)\ln c}\frac{\partial}{\partial x} E_{k+1}(x;a,b,c). \tag{4.11}$$

Formula (4.9) follows from integrating on both sides of formula (4.11). □

Theorem 4.3. *For positive numbers a, b, c and $x \in \mathbb{R}$, we have*

$$E_k(x+1;a,b,c) = \sum_{j=0}^{k}\binom{k}{j}(\ln c)^{k-j}E_j(x;a,b,c), \tag{4.12}$$

$$\begin{aligned} E_k(x+1;a,b,c) &= 2x^k(\ln c)^k \\ &+ \sum_{j=0}^{k}\binom{k}{j}\left[(\ln c)^{k-j} - (\ln b)^{k-j} - (\ln a)^{k-j}\right]E_j(x;a,b,c), \end{aligned} \tag{4.13}$$

$$E_k(x+1;a,b,c) = E_k\left(x;\frac{a}{c},\frac{b}{c},c\right). \tag{4.14}$$

Proof. From Definition 2.2 and straightforward calculation, we have

$$\begin{aligned} \frac{2c^{xt}}{b^t+a^t}\cdot c^t &= \left[\sum_{k=0}^{\infty}\frac{t^k}{k!}E_k(x;a,b,c)\right]\left[\sum_{k=0}^{\infty}\frac{t^k}{k!}(\ln c)^k\right] \\ &= \sum_{k=0}^{\infty}\left[\sum_{j=0}^{k}\binom{k}{j}(\ln c)^{k-j}E_j(x;a,b,c)\right]\frac{t^k}{k!} \end{aligned} \tag{4.15}$$

and

$$\frac{2c^{xt}}{b^t+a^t}\cdot c^t = \frac{2c^{(x+1)t}}{b^t+a^t} = \sum_{k=0}^{\infty}\frac{t^k}{k!}E_k(x+1;a,b,c). \tag{4.16}$$

Therefore, from equating the coefficients of $\frac{t^k}{k!}$ in (4.15) and (4.16), formula (4.12) follows.

Similarly, we obtain

$$\frac{2c^{(x+1)t}}{b^t+a^t}=\sum_{k=0}^{\infty}\frac{t^k}{k!}E_k(x+1;a,b,c)=2c^{xt}+\frac{2c^{xt}}{b^t+a^t}(c^t-b^t-a^t)$$

$$=2\sum_{k=0}^{\infty}\frac{t^k}{k!}x^k(\ln c)^k+\left[\sum_{k=0}^{\infty}\frac{t^k}{k!}E_k(x;a,b,c)\right]\left[\sum_{k=0}^{\infty}\left((\ln c)^k-(\ln b)^k-(\ln a)^k\right)\frac{t^k}{k!}\right]$$

$$=\sum_{k=0}^{\infty}\left[2x^k(\ln c)^k+\sum_{j=0}^{k}\binom{k}{j}\left[(\ln c)^{k-j}-(\ln b)^{k-j}-(\ln a)^{k-j}\right]E_j(x;a,b,c)\right]\frac{t^k}{k!}.$$

By equating coefficients of $\frac{t^k}{k!}$, we obain formula (4.13).

Since

$$\sum_{k=0}^{\infty}\frac{t^k}{k!}E_k(x+1;a,b,c)=\frac{2c^{(x+1)t}}{b^t+a^t}$$

$$=\frac{2c^{xt}}{\left(\frac{b}{c}\right)^t+\left(\frac{a}{c}\right)^t}=\sum_{k=0}^{\infty}\frac{t^k}{k!}E_k\left(x;\frac{a}{c},\frac{b}{c},c\right), \quad (4.17)$$

By equating coefficients, we obtain formula (4.14). The proof is complete. □

Corollary 4.3.1. *The following formulae are valid for positive numbers a, b, c and real number x:*

$$E_k(x+1)+E_k(x)=2x^k, \quad (4.18)$$

$$E_k(x+1)=\sum_{j=0}^{k}\binom{k}{j}E_j(x), \quad (4.19)$$

$$E_k(x+1;1,b,b)+E_k(x;1,b,b)=2x^k(\ln b)^k, \quad (4.20)$$

$$E_k(x+1;1,b,b)=\sum_{j=0}^{k}\binom{k}{j}E_j(x;1,b,b)(\ln b)^{k-j}, \quad (4.21)$$

$$\sum_{j=0}^{k-1}\binom{k}{j}E_j(x;1,b,b)(\ln b)^{k-j}+2E_k(x;1,b,b)=2x^k(\ln b)^k, \quad (4.22)$$

$$\int_x^{x+1}E_k(t;a,b,c)\mathrm{d}t=\frac{1}{(k+1)\ln c}\sum_{j=0}^{k}\binom{k+1}{j}(\ln c)^{k-j}E_j(x;a,b,c). \quad (4.23)$$

Theorem 4.4. *For positive numbers $a,b,c>0$, $x\in\mathbb{R}$, and nonnegative integer k, we have*

$$E_k(1-x;a,b,c)=(-1)^kE_k\left(x;\frac{c}{a},\frac{c}{b},c\right), \quad (4.24)$$

$$E_k(1-x;a,b,c)=E_k\left(-x;\frac{a}{c},\frac{b}{c},c\right). \quad (4.25)$$

Proof. From Definition 2.2 and easy computation, we have

$$\sum_{k=0}^{\infty} \frac{t^k}{k!} E_k(1-x;a,b,c) = \frac{2c^{(1-x)t}}{b^t+a^t} = \frac{2c^t \cdot c^{-xt}}{b^t+a^t}$$
$$= \frac{2c^{-xt}}{\left(\frac{c}{b}\right)^{-t}+\left(\frac{c}{a}\right)^{-t}} = \sum_{k=0}^{\infty} \frac{t^k}{k!}(-1)^k E_k\left(x;\frac{c}{a},\frac{c}{b},c\right), \quad (4.26)$$

equating coefficients of t^k above leads to formula (4.24).

By the same procedure, we can establish formula (4.25). □

Theorem 4.5. *For positive numbers $a, b, c > 0$, nonnegative natural number k, and $x, y \in \mathbb{R}$, we have*

$$E_k(x+y;a,b,c) = \sum_{j=0}^{k} \binom{k}{j} (\ln c)^{k-j} y^{k-j} E_j(x;a,b,c), \quad (4.27)$$

$$E_k(x+y;a,b,c) = \sum_{j=0}^{k} \binom{k}{j} (\ln c)^{k-j} x^{k-j} E_j(y;a,b,c). \quad (4.28)$$

Proof. These two formulae can be deduced from the following calculation and considering symmetry of x and y:

$$\sum_{k=0}^{\infty} \frac{t^k}{k!} E_k(x+y;a,b,c) = \frac{2c^{(x+y)t}}{b^t+a^t} = \frac{2c^{xt} \cdot c^{yt}}{b^t+a^t}$$
$$= \left[\sum_{k=0}^{\infty} \frac{t^k}{k!} E_k(x;a,b,c)\right]\left[\sum_{k=0}^{\infty} \frac{t^k}{k!}(\ln c)^k y^k\right]$$
$$= \sum_{k=0}^{\infty}\left[\sum_{j=0}^{k} \binom{k}{j} (\ln c)^{k-j} y^{k-j} E_j(x;a,b,c)\right]\frac{t^k}{k!}. \quad (4.29)$$

The proof is complete. □

Theorem 4.6. *For natural numbers k and m, and positive number b, we have*

$$\sum_{\ell=1}^{m}(-1)^{\ell}\ell^k = \frac{1}{2(\ln b)^k}\left[(-1)^m E_k(m+1;1,b,b) - E_k(1;1,b,b)\right]. \quad (4.30)$$

Proof. Rearraging formula (4.20) give us

$$x^k = \frac{1}{2(\ln b)^k}\left[E_k(x+1;1,b,b) + E_k(x;1,b,b)\right]. \quad (4.31)$$

Replacing x by $\ell \in \mathbb{N}$ and suming up ℓ from 1 to m yields

$$\begin{aligned}\sum_{\ell=1}^{m}(-1)^{\ell}\ell^{k} &= \frac{1}{2(\ln b)^{k}}\sum_{\ell=1}^{m}(-1)^{\ell}\left[E_k(\ell+1;1,b,b)+E_k(\ell;1,b,b)\right]\\ &= \frac{1}{2(\ln b)^{k}}\left[(-1)^{m}E_k(m+1;1,b,b)-E_k(1;1,b,b)\right].\end{aligned} \tag{4.32}$$

The proof is complete. □

Remark 4.1. In the final, we give several concrete formulae as follows:

$$E_0(x;a,b,c)=1, \tag{4.33}$$

$$E_1(x;a,b,c)=x\ln c-\frac{1}{2}(\ln a+\ln b), \tag{4.34}$$

$$E_2(x;a,b,c)=\left(x-\frac{1}{2}\right)^{2}(\ln c)^{2}+\left(x-\frac{1}{2}\right)(\ln c-\ln b-\ln a)\ln c \tag{4.35}$$

$$+\frac{1}{4}(\ln c-2\ln a)(\ln c-2\ln b). \tag{4.36}$$

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