AN APPROXIMATION FOR THE FOURIER TRANSFORM OF ABSOLUTELY CONTINUOUS MAPPINGS

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Abstract

An approximation for the Fourier transforms of absolutely continuous mappings in terms of the exponential mean, and the study of error bounds when $g' \in L_p[a,b]$ $(1 \le p \le \infty)$ are given.

I. INTRODUCTION

The *Fourier Transform* is an important mathematical tool in a wide variety of fields of science and engineering [1, p. XI].

In this paper, by the use of some integral identities and inequalities developed in [2](see also [3]), we point out some approximations of the Fourier transform in terms of the complex exponential mean, $E(z,w)$ (see Section 2) and study the error of approximation for different classes of absolutely continuous mappings defined on finite intervals.

Throughout this paper $g:[a,b] \to \mathbb{R}$ is an absolutely continuous mapping defined on the finite interval $[a,b]$ and $\mathcal{F}(g)$ is its Fourier transform. That is,

$$\mathcal{F}(g)(t) := \int_a^b g(s)\, e^{-2\pi i t s} ds.$$

The inverse Fourier transform of g will also be considered, being defined by

$$\mathcal{F}^{-1}(g)(t) := \int_a^b g(s)\, e^{2\pi i t s} ds.$$

II. SOME INTEGRAL INEQUALITIES

The following inequality holds:

Theorem 1: We have

$$\left| \mathcal{F}(g)(x) - E(-2\pi i x a, -2\pi i x b) \int_a^b g(t)\, dt \right| \quad (1)$$

$$\le \begin{cases} \frac{1}{3} \|g'\|_\infty (b-a)^2 \text{ if } g' \in L_\infty[a,b], \\ \frac{2^{\frac{1}{q}}}{[(q+1)(q+2)]^{\frac{1}{q}}} (b-a)^{1+\frac{1}{q}} \|g'\|_p \\ \text{if } g' \in L_p[a,b]; \frac{1}{p} + \frac{1}{q} = 1, p > 1, \\ (b-a) \|g'\|_1 \end{cases};$$

for all $x \in [a,b], x \neq 0$, where E is the exponential mean defined as:

$$E(z,w) := \begin{cases} \frac{e^z - e^w}{z-w} \text{ if } z \neq w \\ w \text{ if } z = w \end{cases}, z, w \in \mathbb{C}.$$

Proof: Using the integration by parts formula for absolutely continuous mappings on $[a,b]$, we have

$$\int_a^x (s-a)\, g'(s)\, ds = (x-a)\, g(x) - \int_a^x g(s)\, ds \quad (2)$$

and

$$\int_x^b (s-b)\, g'(s)\, ds = (b-x)\, g(x) - \int_x^b g(s)\, ds \quad (3)$$

for all $x \in [a,b]$.
Adding (2) and (3), we get

$$(b-a)\, g(x) - \int_a^b g(s)\, ds = \int_a^x (s-a)\, g'(s)\, ds + \int_x^b (s-b)\, g'(s)\, ds. \quad (4)$$

Define the mapping $k : [a,b]^2 \to \mathbb{R}$ given by

$$k(u,v) := \begin{cases} v-a \text{ if } v \in [a,u] \\ v-b \text{ if } v \in (u,b] \end{cases}.$$

Using this mapping, we can write (4) as (see also [2])

$$g(x) = \frac{1}{b-a}\int_a^b g(s)\,ds + \frac{1}{b-a}\int_a^b k(x,s)\,g'(s)\,ds, x \in [a,b]. \quad (5)$$

Assume that $x \in [a,b]$ and $x \neq 0$. Then,

$$\begin{aligned}\mathcal{F}(g)(x) &= \int_a^b g(t)\,e^{-2\pi ixt}dt \quad (6)\\ &= \int_a^b \left[\frac{1}{b-a}\int_a^b g(s)\,ds + \frac{1}{b-a}\int_a^b k(t,s)\,g'(s)\,ds\right] e^{-2\pi ixt}dt\\ &= \frac{1}{b-a}\int_a^b g(s)\,ds \int_a^b e^{-2\pi ixt}dt + \frac{1}{b-a}\int_a^b\int_a^b k(t,s)\,g'(s)\,e^{-2\pi ixt}dsdt\\ &= E(-2\pi ixb, -2\pi ixa)\int_a^b g(s)\,ds + \frac{1}{b-a}\int_a^b\int_a^b k(t,s)\,g'(s)\,e^{-2\pi ixt}dsdt.\end{aligned}$$

Using the properties of modulus, we have, by (6), that

$$\begin{aligned}&\left|\mathcal{F}(g)(x) - \int_a^b g(t)\,dtE(-2\pi ixb, -2\pi ixa)\right| \quad (7)\\ &= \frac{1}{b-a}\left|\int_a^b\int_a^b k(t,s)\,g'(s)\,e^{-2\pi ixt}dsdt\right|\\ &\leq \frac{1}{b-a}\int_a^b\int_a^b |k(t,s)|\,|g'(s)|\,\left|e^{-2\pi ixt}\right| dsdt\\ &= \frac{1}{b-a}\int_a^b\int_a^b |k(t,s)|\,|g'(s)|\,dsdt.\end{aligned}$$

Observe that

$$\begin{aligned}&\int_a^b\int_a^b |k(t,s)|\,|g'(s)|\,dsdt\\ &\leq \|g'\|_\infty \int_a^b\int_a^b |k(t,s)|\,dsdt\\ &= \|g'\|_\infty \int_a^b \left[\int_a^t |s-a|\,ds + \int_t^b |s-b|\,ds\right] dt\\ &= \|g'\|_\infty \int_a^b \left[\frac{(t-a)^2}{2} + \frac{(b-t)^2}{2}\right] dt\\ &= \|g'\|_\infty \left[\frac{(b-a)^3}{6} + \frac{(b-a)^3}{6}\right]\\ &= \|g'\|_\infty \frac{(b-a)^3}{3}.\end{aligned}$$

Using (7), we get the first inequality in (1). Applying Hölder's integral inequality for double integrals, we get

$$\begin{aligned}&\int_a^b\int_a^b |k(t,s)|\,|g'(s)|\,dsdt\\ &\leq \left(\int_a^b\int_a^b |k(t,s)|^q\,dsdt\right)^{\frac{1}{q}} \times \left(\int_a^b\int_a^b |g'(s)|^p\,dsdt\right)^{\frac{1}{p}}\\ &= (b-a)^{\frac{1}{p}}\,\|g'\|_p \times \left[\int_a^b \left[\int_a^t |s-a|^q\,ds + \int_t^b |s-b|^q\,ds\right] dt\right]^{\frac{1}{q}}\\ &= (b-a)^{\frac{1}{p}}\,\|g'\|_p \times \left(\int_a^b \left[\frac{(t-a)^{q+1} + (b-t)^{q+1}}{q+1}\right] dt\right)^{\frac{1}{q}}\\ &= (b-a)^{\frac{1}{p}}\,\|g'\|_p \left[\frac{(b-a)^{q+1} + (b-a)^{q+1}}{(q+1)(q+2)}\right]^{\frac{1}{q}}\\ &= 2^{\frac{1}{q}}\frac{(b-a)^{\frac{1}{p}}(b-a)^{1+\frac{2}{q}}}{[(q+1)(q+2)]^{\frac{1}{q}}}\|g'\|_p\end{aligned}$$

Using (7), we get the second part of (1). Finally, observe that,

$$\begin{aligned}&\int_a^b\int_a^b |k(t,s)|\,|g'(s)|\,dsdt\\ &\leq \sup_{(t,s)\in[a,b]^2} |k(s,t)| \int_a^b\int_a^b |g'(s)|\,dsdt\\ &= (b-a)\,\|g'\|_1\,(b-a)\\ &= (b-a)^2\,\|g'\|_1\end{aligned}$$

and the theorem is thus proved. ■

Remark 2: By a similar argument, we can prove that

$$\left|\mathcal{F}^{-1}(g)(x)-E(2\pi ixa,2\pi ixb)\int_a^b g(t)\,dt\right| \le \begin{cases} \frac{1}{3}\|g'\|_\infty (b-a)^2 \text{ if } g'\in L_\infty[a,b], \\ \frac{2^{\frac{1}{q}}}{[(q+1)(q+2)]^{\frac{1}{q}}}(b-a)^{1+\frac{1}{q}}\|g'\|_p \\ \text{ if } g'\in L_p[a,b], \\ (b-a)\|g'\|_1 \end{cases};$$

for all $x\in[a,b], x\neq 0$.

III. A NUMERICAL QUADRATURE FORMULA

Let $I_n : a = x_0 < x_1 < ... < x_{n-1} < x_n = b$ be a division of the interval $[a,b]$, put $h_k := x_{k+1}-x_k$ $(k=0,...,n-1)$ and $\nu(h) := \max\{h_k | k=0,...,n-1\}$. Define the sum

$$\mathcal{E}(g,I_n,x) := \sum_{k=0}^{n-1} E(-2\pi ixx_k,-2\pi ixx_{k+1})\int_{x_k}^{x_{k+1}} g(t)\,dt \tag{8}$$

where $x\in[a,b], x\neq 0$.

The following approximation theorem holds.

Theorem 3: We have the quadrature rule,

$$\mathcal{F}(g)(x)=\mathcal{E}(g,I_n,x)+R(g,I_n,x), \tag{9}$$

where $\mathcal{E}(g,I_n,\cdot)$ approximates the Fourier transform $\mathcal{F}(g)$ at every point $x\in[a,b], x\neq 0$. The error of approximation $R(g,I_n,\cdot)$ satisfies the bound

$$|R(g,I_n,x)| \le \begin{cases} \frac{1}{3}\|g'\|_\infty \sum_{k=0}^{n-1} h_k^2 \text{ if } g'\in L_\infty[a,b], \\ \frac{2^{\frac{1}{q}}}{[(q+1)(q+2)]^{\frac{1}{q}}}\|g'\|_p\left(\sum_{k=0}^{n-1} h_k^{q+1}\right)^{\frac{1}{q}}, \\ \frac{1}{p}+\frac{1}{q}=1, p>1;\text{ if } g'\in L_p[a,b], \\ \nu(h)\|g'\|_1, \end{cases} \tag{10}$$

for all $x\in[a,b], x\neq 0$.

Proof: Applying Theorem 1 to every subinterval $[x_k,x_{k+1}]$, we can state that

$$\left|\int_{x_k}^{x_{k+1}} g(t)e^{-2\pi ixt}ds - E(-2\pi ixx_k,-2\pi ix_{k+1})\int_{x_k}^{x_{k+1}} g(t)\,dt\right| \le \begin{cases} \frac{1}{3}\sup_{t\in[x_k,x_{k+1}]}|g'(t)|(x_{k+1}-x_k)^2, \\ \frac{2^{\frac{1}{q}}}{[(q+1)(q+2)]^{\frac{1}{q}}}(x_{k+1}-x_k)^{1+\frac{1}{q}} \\ \times\left(\int_{x_k}^{x_{k+1}}|g'(t)|^p dt\right)^{\frac{1}{p}}, \\ (x_{k+1}-x_k)\int_{x_k}^{x_{k+1}}|g'(t)|\,dt, \end{cases}$$

where $x\neq 0$ can be seen as a real parameter.
Summing over k from 0 to $n-1$ and using the generalized triangle inequality, we can state that

$$|R(g,I_n,x)| = |\mathcal{F}(g)(x)-\mathcal{E}(g,I_n,x)| \le \begin{cases} \frac{1}{3}\sum_{k=0}^{n-1}\sup_{t\in[x_k,x_{k+1}]}|g'(t)|h_k^2, \\ \frac{2^{\frac{1}{q}}}{[(q+1)(q+2)]^{\frac{1}{q}}}\sum_{k=0}^{n-1}h_k^{1+\frac{1}{q}} \\ \times\left(\int_{x_k}^{x_{k+1}}|g'(t)|^p dt\right)^{\frac{1}{p}}, \\ \sum_{k=0}^{n-1}h_k\int_{x_k}^{x_{k+1}}|g'(t)|\,dt. \end{cases}$$

As

$$\sup_{t\in[x_k,x_{k+1}]}|g'(t)| \le \sup_{t\in[a,b]}|g'(t)| = \|g'\|_\infty,$$

the first inequality in (10) is obtained.
Using Hölder's discrete inequality, we can state that

$$\sum_{k=0}^{n-1}h_k^{1+\frac{1}{q}}\left(\int_{x_k}^{x_{k+1}}|g'(t)|^p dt\right)^{\frac{1}{p}} \le \left[\sum_{k=0}^{n-1}\left(h_k^{1+\frac{1}{q}}\right)^q\right]^{\frac{1}{q}} \times\left[\sum_{k=0}^{n-1}\left[\left(\int_{x_k}^{x_{k+1}}|g'(t)|^p dt\right)^{\frac{1}{p}}\right]^p\right]^{\frac{1}{p}} = \left(\sum_{k=0}^{n-1}h_k^{q+1}\right)^{\frac{1}{q}}\|g'\|_p$$

which proves the second inequality in (10).

For the last inequality, we observe that

$$\sum_{k=0}^{n-1} h_k \int_{x_k}^{x_{k+1}} |g'(t)|\, dt \le \nu(h) \sum_{k=0}^{n-1} \int_{x_k}^{x_{k+1}} |g'(t)|\, dt$$
$$= \nu(h) \int_a^b |g'(t)|\, dt$$
$$= \nu(h) \|g'\|_1$$

and the theorem is completely proved. ■

In practical applications, it is more convenient to consider the equidistant partitioning of the interval $[a, b]$. Thus, let

$$I_n : x_j = a + j \cdot \frac{b-a}{n}, j = 0, ..., n$$

be an equidistant partition of (a, b), and define the sum

$$\mathcal{E}(g, I_n, x) := \sum_{k=0}^{n-1} E\left(-2\pi i x\left(a + k \cdot \frac{b-a}{n}\right), -2\pi i\left(a + (k+1)\frac{b-a}{n}\right)\right) \times \int_{a+k\cdot\frac{b-a}{n}}^{a+(k+1)\frac{b-a}{n}} g(t)\, dt. \quad (11)$$

The following corollary of Theorem 3 holds:

Corollary 4: We have

$$\mathcal{F}(g)(x) = \mathcal{E}_n(g, x) + R_n(g, x), \quad (12)$$

where $\mathcal{E}_n(g, \cdot)$ approximates the Fourier transform at every point $x \in [a, b], x \neq 0$. The error of approximation $R_n(g, \cdot)$ satisfies the bound

$$|R_n(g, x)| \le \begin{cases} \frac{(b-a)^2}{3n} \|g'\|_\infty \text{ if } g' \in L_\infty[a, b], \\ \frac{(b-a)^{1+\frac{1}{q}} 2^{\frac{1}{q}}}{[(q+1)(q+2)]^{\frac{1}{q}} n} \cdot \|g'\|_p \\ \text{if } g' \in L_p[a, b], \frac{1}{p} + \frac{1}{q} = 1, p > 1; \\ \frac{b-a}{n} \|g'\|_1 \end{cases} . \quad (13)$$

IV. SOME NUMERICAL EXPERIMENTS

In the following we numerically illustrate the approximation for the Fourier transform provided by

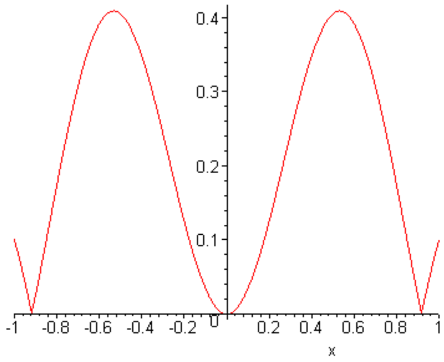

Figure 1: Er_n for $n = 1$.

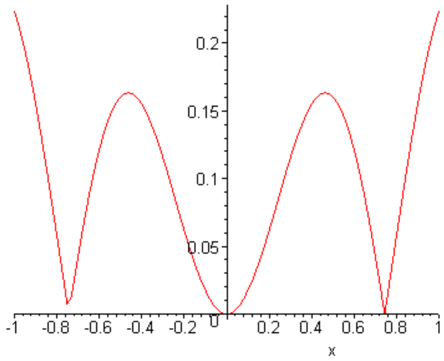

Figure 2: Er_n for $n = 3$.

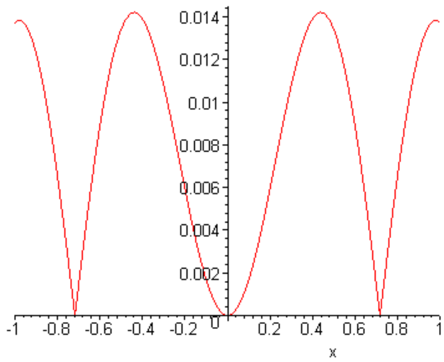

Figure 3: Er_n for $n = 10$.

$$\mathcal{E}(g, I_n, x) := \sum_{k=0}^{n-1} E\left(2\pi i x\left(a + k \cdot \frac{b-a}{n}\right), -2\pi\, i\left(a + (k+1)\frac{b-a}{n}\right)\right) \times \int_{a+k\cdot\frac{b-a}{n}}^{a+(k+1)\frac{b-a}{n}} g(t)\, dt. \quad (14)$$

If we consider, for instance, the quadratic function $g(t) = t^2, t \in [-1, 1]$, then the plots of the error $Er_n(x) := |R_n(g, x)|, x \in [-1, 1]$ for $n = 1, n = 3$ and $n = 10$, respectively, are depicted in Figure 1, 2 and 3.

V. CONCLUSION

An approximation of the Fourier Transform for absolutely continuous functions in terms of the exponential mean and the integral of the function, and accurate estimates for the remainder are pointed out. Some numerical experiments for the composite rule that confirm the theoretical error obtained are also performed.

References

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