THE GENERALISED INTEGRATION BY PARTS FORMULA FOR APPELL SEQUENCES AND RELATED RESULTS

S.S. DRAGOMIR

Abstract. A generalised integration by parts formula for sequences of absolutely continuous functions that satisfy the $w-$Appell condition and different estimates for the remainder are provided. Applications for particular instances of such sequences are pointed out as well.

1. Introduction

In [6], Matić et. al introduced the concept of *harmonic sequences of polynomials* by assuming that the polynomial $\{P_n\}_{n\in\mathbb{N}}$ satisfies the condition

$$P_0 = 1,\ \ P_n'(t) = P_{n-1}(t) \quad \text{for all } t \in \mathbb{R} \text{ and } n \in \mathbb{N}. \tag{1.1}$$

With this assumption, they proved the following generalised Taylor's formula:

Theorem 1. *Let $I \subset \mathbb{R}$ be a closed interval and $a \in I$. If $f : I \to \mathbb{R}$ is any function such that, for some $n \in \mathbb{N}$, $f^{(n)}$ is absolutely continuous, then for any $x \in I$*

$$f(x) = f(a) + \sum_{k=1}^{n} (-1)^{k+1} \left[P_k(x) f^{(k)}(x) - P_k(a) f^{(k)}(a)\right] + R_n(f; a, x), \tag{1.2}$$

where

$$R_n(f; a, x) = (-1)^n \int_a^x P_n(t) f^{(n+1)}(t)\, dt \tag{1.3}$$

and $\{P_n\}_{n\in\mathbb{N}}$ is a harmonic sequence of polynomials.

As examples of such polynomials, they mentioned the following

$$P_n(t) := \frac{1}{n!}(t-x)^n,\ \ t \in \mathbb{R};$$

or

$$P_n(t) := \frac{1}{n!}\left(t - \frac{a+x}{2}\right)^n,\ \ t \in \mathbb{R};$$

or

$$P_n(t) := \frac{(x-a)^n}{n!} B_n\left(\frac{t-a}{x-a}\right),\ n \geq 1,\ P_0(t) = 1,$$

where $B_n(\cdot)$ are the *Bernoulli's polynomials,* or

$$P_n(t) := \frac{(x-a)^n}{n!} E_n\left(\frac{t-a}{x-a}\right),\ n \geq 1,\ P_0(t) = 1,$$

where $E_n(\cdot)$ are the *Euler's polynomials.*

Date: January 25, 2001.

1991 *Mathematics Subject Classification.* Primary 26D15; Secondary 26D10.

Key words and phrases. Integration by parts formula, Appell polynomials, Korkine identity, Analytic inequalities.

Amongst others, they proved the following general estimation result for the remainder $R_n(f;a,x)$.

Corollary 1. *Under the assumptions of Theorem 1 and if $x \geq a$, then*

$$|R_n(f;a,x)| \leq \begin{cases} \max\limits_{t\in[a,x]} |P_n(t)| \int_a^x \left|f^{(n+1)}(s)\right| ds; \\ \\ \max\limits_{t\in[a,x]} \left|f^{(n+1)}(t)\right| \int_a^x |P_n(s)|\, ds; \\ \\ \left(\int_a^x |P_n(s)|^q\, ds\right)^{\frac{1}{q}} \left(\int_a^x \left|f^{(n+1)}(s)\right|^p ds\right)^{\frac{1}{p}}, \\ \quad \text{where } \frac{1}{p}+\frac{1}{q}=1,\ p>1. \end{cases} \tag{1.4}$$

Now, if one would choose in (1.2) $f(x) = \int_a^b g(t)\, dt$ and put $x = b$, we could then state the following generalised integration by parts formula

$$\int_a^b g(t)\, dt = \sum_{k=1}^{n} (-1)^{k+1} \left[P_k(b)\, g^{(k-1)}(b) - P_k(a)\, g^{(k-1)}(a)\right] + S_n(f;a,b), \tag{1.5}$$

where

$$S_n(f;a,b) = (-1)^n \int_a^b P_n(t)\, g(t)\, dt. \tag{1.6}$$

Using the classical notation for the Lebesgue norms,

$$\|h\|_\infty := ess \sup_{t\in[a,b]} |h(t)|,$$

$$\|h\|_p := \left(\int_a^b |h(t)|^p\, dt\right)^{\frac{1}{p}},\ p \geq 1$$

the remainder (1.6) may be bounded in the following manner

$$|S_n(g;a,b)| \leq \begin{cases} \|P_n\|_\infty \left\|g^{(n)}\right\|_1, \\ \\ \|P_n\|_1 \left\|g^{(n)}\right\|_\infty, \\ \\ \|P_n\|_q \left\|g^{(n)}\right\|_p, \text{ where } \frac{1}{p}+\frac{1}{q}=1,\ p>1. \end{cases} \tag{1.7}$$

For other results based on the integration by parts formula (1.5), see [1] – [5] and [7] – [9].

2. The Generalised Integration by Parts Formula

We shall start with the following definition.

Definition 1. *Let $w : [a,b] \to \mathbb{R}$ be an absolutely continuous function on $[a,b]$. The sequence of absolutely continuous functions $\{w_k\}_{k=\overline{0,n}}$ $(n \geq 1)$ defined on $[a,b]$ are said to be of $w-$Appell type if*

$$w_0 = w \quad a.e.\ on\ [a,b]; \tag{2.1}$$

$$w_k' = w_{k-1} \quad a.e.\ on\ [a,b], \quad for\ all\ k = 1,\dots,n. \tag{2.2}$$

Remark 1. *It is obvious that any sequence of harmonic polynomials is a sequence of $w-$Appell type with $w=1$.*

Remark 2. *Having given an absolutely continuous function $w:[a,b]\to\mathbb{R}$ we may construct a sequence of $w-$Appell type in the following canonical fashion:*

$$\begin{aligned} w_1(t) &= \int_a^t w(s)\,ds + c_1,\ t\in[a,b],\ c_1\in\mathbb{R}; \\ w_2(t) &= \int_a^t w_1(s)\,ds + c_2,\ t\in[a,b],\ c_2\in\mathbb{R}; \\ &\cdots \\ w_n(t) &= \int_a^t w_{n-1}(s)\,ds + c_n,\ t\in[a,b],\ c_n\in\mathbb{R}. \end{aligned}$$

The following generalised integration by parts formula associated with the sequence $\{w_k\}_{k=\overline{0,n}}$ naturally holds.

Lemma 1. *Let $w:[a,b]\to\mathbb{R}$ be an absolutely continuous function on $[a,b]$ and $\{w_k\}_{k=\overline{0,n}}$ a sequence of $w-$Appell type. If $g:[a,b]\to\mathbb{R}$ is such that $g^{(n-1)}$is absolutely continuous on $[a,b]$ and $w_n g^{(n)}\in L_1[a,b]$, then we have the equality*

$$\int_a^b w(t)\,g(t)\,dt = A_n(w,g;a,b) + R_n(w,g;a,b), \tag{2.3}$$

where

$$A_n(w,g;a,b) = \sum_{k=1}^n (-1)^{k-1}\left[w_k(b)\,g^{(k-1)}(b) - w_k(a)\,g^{(k-1)}(a)\right] \tag{2.4}$$

and

$$R_n(w,g;a,b) = (-1)^n \int_a^b w_n(t)\,g^{(n)}(t)\,dt. \tag{2.5}$$

Proof. We prove (2.3) by mathematical induction.

For $n=1$, we have

$$\int_a^b w(t)\,g(t)\,dt = w_1(b)\,g(b) - w_1(a)\,g(a) - \int_a^b w_1(t)\,g^{(1)}(t)\,dt$$

which follows on applying the integration by parts formula taking into account that $w_1'=w$ a.e. on $[a,b]$.

Assume that (2.3) holds for $m\in\{1,\dots,n-1\}$, i.e.,

$$\begin{aligned} \int_a^b w(t)\,g(t)\,dt &= \sum_{k=1}^m (-1)^{k-1}\left[w_k(b)\,g^{(k-1)}(b) - w_k(a)\,g^{(k-1)}(a)\right] \\ &\quad + (-1)^m \int_a^b w_m(t)\,g^{(m)}(t)\,dt. \end{aligned} \tag{2.6}$$

As w_m and $g^{(m)}$ are absolutely continuous and $w'_{m+1} = w_m$ a.e. on $[a,b]$, then, by the integration by parts formula we may write

$$\int_a^b w_m(t)\, g^{(m)}(t)\, dt = \int_a^b w'_{m+1}(t)\, g^{(m)}(t)\, dt = w_{m+1}(b)\, g^{(m)}(b) - w_{m+1}(a)\, g^{(m)}(a) - \int_a^b w_{m+1}(t)\, g^{(m+1)}(t)\, dt. \tag{2.7}$$

Using (2.6), we get

$$\begin{aligned}\int_a^b w(t)\, g(t)\, dt &= \sum_{k=1}^{m} (-1)^{k-1} \left[w_k(b)\, g^{(k-1)}(b) - w_k(a)\, g^{(k-1)}(a) \right] \\ &\quad + (-1)^m \left[w_{m+1}(b)\, g^{(m)}(b) - w_{m+1}(a)\, g^{(m)}(a) \right. \\ &\quad \left. - \int_a^b w_{m+1}(t)\, g^{(m+1)}(t)\, dt \right] \\ &= \sum_{k=1}^{m+1} (-1)^{k-1} \left[w_k(b)\, g^{(k-1)}(b) - w_k(a)\, g^{(k-1)}(a) \right] \\ &\quad + (-1)^{m+1} \int_a^b w_{m+1}(t)\, g^{(m+1)}(t)\, dt\end{aligned}$$

showing that (2.3) holds for $n = m+1$ as well.

The lemma is thus proved. ∎

Remark 3. *If $w_n = P_n$, $\{P_n\}_{n\in\mathbb{N}}$ is a sequence of harmonic polynomials, then by (2.3) we recapture the identity (1.6).*

The following result concerning estimates of the remainder $R_n(w,g;a,b)$ holds.

Theorem 2. *With the assumptions of Lemma 1, we have the estimates:*

$$|R_n(g;a,b)| \le \begin{cases} \|w_n\|_1 \left\|g^{(n)}\right\|_\infty & \text{if} \quad w_n \in L_1[a,b],\ g^{(n)} \in L_\infty[a,b]; \\ \|w_n\|_q \left\|g^{(n)}\right\|_p, & \text{if} \quad w_n \in L_q[a,b],\ g^{(n)} \in L_p[a,b]; \\ & \text{and} \quad \frac{1}{p}+\frac{1}{q}=1,\ p>1; \\ \|w_n\|_\infty \left\|g^{(n)}\right\|_1 & \text{if} \quad w_n \in L_\infty[a,b]. \end{cases} \tag{2.8}$$

The proof follows by the representation (2.5) and Hölder's inequalities for $\|\cdot\|_p$ −norms ($p \in [1,\infty]$).

In what follows, we point out a number of useful examples.

1. Define $w_0^{(1)}(t) := e^{\alpha t}$, $\alpha \in \mathbb{R}\setminus\{0\}$ and consider the sequence $w_k^{(1)}(t) := \frac{1}{\alpha^k} e^{\alpha t}$, $k = 0,1,\dots$. Then

$$\frac{dw_k^{(1)}(t)}{dt} = \frac{1}{\alpha^{k-1}} e^{\alpha t} = w_{k-1}^{(1)}(t),\ \text{for } k = 1,2,\dots$$

showing that $\left\{w_k^{(1)}\right\}_{k\in\mathbb{N}}$ is an $w_0^{(1)}$−Appell type sequence.

If we use Lemma 1, we may state the identity:

$$\int_a^b e^{\alpha t} g(t)\,dt = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{\alpha^k}\left[e^{\alpha b} g^{(k-1)}(b) - e^{\alpha a} g^{(k-1)}(a)\right] + \frac{(-1)^n}{\alpha^n}\int_a^b e^{\alpha t} g^{(n)}(t)\,dt. \tag{2.9}$$

As

$$\left\|w_n^{(1)}\right\|_1 = \int_a^b \left|\frac{e^{\alpha t}}{\alpha^n}\right| dt = \frac{1}{|\alpha|^n}\int_a^b e^{\alpha t} dt = \frac{1}{|\alpha|^n}\cdot\left|\frac{e^{\alpha b}-e^{\alpha a}}{\alpha}\right|,$$

$$\left\|w_n^{(1)}\right\|_q = \left(\int_a^b \left|\frac{e^{\alpha t}}{\alpha^n}\right|^q dt\right)^{\frac{1}{q}} = \frac{1}{|\alpha|^n}\left(\int_a^b e^{\alpha q t} dt\right)^{\frac{1}{q}} = \frac{1}{|\alpha|^n}\left|\frac{e^{\alpha q b}-e^{\alpha q a}}{\alpha q}\right|^{\frac{1}{q}},$$

$$\begin{aligned}\left\|w_n^{(1)}\right\|_\infty &= \frac{1}{|\alpha|^n}\sup_{t\in[a,b]}\left|e^{\alpha t}\right| = \frac{1}{|\alpha|^n}\max\left\{e^{\alpha a}, e^{\alpha b}\right\}\\ &= \frac{1}{|\alpha|^n}\cdot\frac{e^{\alpha a}+e^{\alpha b}+\left|e^{\alpha a}-e^{\alpha b}\right|}{2}.\end{aligned}$$

Then, by Theorem 2 we may state the inequality:

$$\left|\int_a^b e^{\alpha t} g(t)\,dt - \sum_{k=1}^{n}\frac{(-1)^{k-1}}{\alpha^k}\left[e^{\alpha b} g^{(k-1)}(b) - e^{\alpha a} g^{(k-1)}(a)\right]\right| \tag{2.10}$$

$$\leq \begin{cases} \dfrac{\left|e^{\alpha b}-e^{\alpha a}\right|}{|\alpha|^{n+1}}\left\|g^{(n)}\right\|_\infty & \text{if } \quad g^{(n)}\in L_\infty[a,b];\\[2ex] \dfrac{1}{|\alpha|^n}\left|\dfrac{e^{\alpha q b}-e^{\alpha q a}}{\alpha q}\right|^{\frac{1}{q}}\left\|g^{(n)}\right\|_p, & \text{if } \quad g^{(n)}\in L_p[a,b];\\ & \text{and } \quad \frac{1}{p}+\frac{1}{q}=1,\ p>1;\\[2ex] \dfrac{1}{|\alpha|^n}\cdot\dfrac{e^{\alpha a}+e^{\alpha b}+\left|e^{\alpha a}-e^{\alpha b}\right|}{2}\left\|g^{(n)}\right\|_1, & \end{cases}$$

where $\alpha\in\mathbb{R}\setminus\{0\}$.

2. Define $w_0^{(2)}(t) = t^\alpha$, $\alpha > -1$ and consider

$$w_k^{(2)}(t) := \frac{t^{\alpha+k}}{(\alpha+1)(\alpha+2)\cdots(\alpha+k)},\quad k = 0,1,\ldots.$$

Then

$$\frac{dw_k^{(1)}(t)}{dt} = \frac{t^{\alpha+k-1}}{(\alpha+1)(\alpha+2)\cdots(\alpha+k-1)} = w_{k-1}^{(2)}(t),\ \text{for } k=1,2,\ldots,$$

which shows that $\left\{w_k^{(2)}\right\}_{k=0,1,\ldots}$ is a $w_0^{(2)}-$Appell type sequence.

If we use Lemma 1, we may state the identity:

$$\int_a^b t^\alpha g(t)\,dt = \sum_{k=1}^{n}(-1)^{k-1}\frac{\left[b^{\alpha+k}g^{(k-1)}(b) - a^{\alpha+k}g^{(k-1)}(a)\right]}{(\alpha+1)(\alpha+2)\cdots(\alpha+k)} + \frac{(-1)^n}{(\alpha+1)\cdots(\alpha+n)}\int_a^b t^{\alpha+n}g^{(n)}(t)\,dt,\ [a,b]\in(0,\infty). \tag{2.11}$$

As, for $[a,b]\subset(0,\infty)$,

$$\begin{aligned}\left\|w_n^{(2)}\right\|_1 &= \int_a^b\left|\frac{t^{\alpha+n}}{(\alpha+1)\cdots(\alpha+n)}\right|dt=\frac{b^{\alpha+n+1}-a^{\alpha+n+1}}{(\alpha+1)\cdots(\alpha+n)(\alpha+n+1)}\\ &= \frac{(b-a)}{(\alpha+1)\cdots(\alpha+n)}L_{\alpha+n}^{\alpha+n}(a,b),\end{aligned}$$

where

$$L_m(a,b):=\begin{cases} b & \text{if} \quad a=b\\ \left[\dfrac{b^{m+1}-a^{m+1}}{(m+1)(b-a)}\right]^{\frac{1}{m}} & \text{if} \quad a\neq b\end{cases}$$

is the $m-$logarithmic mean, $m\in\mathbb{R}\setminus\{-1,0\}$; and for $q>1$

$$\begin{aligned}\left\|w_n^{(2)}\right\|_q &= \left(\int_a^b\left|\frac{t^{\alpha+n}}{(\alpha+1)\cdots(\alpha+n)}\right|^q dt\right)^{\frac{1}{q}}\\ &= \frac{1}{(\alpha+1)\cdots(\alpha+n)}\left[\frac{b^{(\alpha+n)q+1}-a^{(\alpha+n)q+1}}{((\alpha+n)q+1)(b-a)}\cdot(b-a)\right]^{\frac{1}{q}\cdot\frac{(\alpha+n)q}{(\alpha+n)q}}\\ &= \frac{(b-a)^{\frac{1}{q}}}{(\alpha+1)\cdots(\alpha+n)}L_{(\alpha+n)q}^{\alpha+n}(a,b),\end{aligned}$$

and, finally

$$\left\|w_n^{(2)}\right\|_\infty=\sup_{t\in[a,b]}\left|\frac{t^{\alpha+n}}{(\alpha+1)\cdots(\alpha+n)}\right|=\frac{1}{(\alpha+1)\cdots(\alpha+n)}b^{\alpha+n},\quad n\geq 1$$

then by Theorem 2, we may state the inequality

$$\begin{aligned}&\left|\int_a^b t^\alpha g(t)\,dt\right.\\ &\left.-\sum_{k=1}^{n}\frac{(-1)^{k-1}}{(\alpha+1)(\alpha+2)\cdots(\alpha+k)}\left[b^{\alpha+k}g^{(k-1)}(b)-a^{\alpha+k}g^{(k-1)}(a)\right]\right|\\ &\leq\begin{cases}\dfrac{(b-a)L_{\alpha+n}^{\alpha+n}(a,b)}{(\alpha+1)\cdots(\alpha+n)}\left\|g^{(n)}\right\|_\infty & \text{if} \quad g^{(n)}\in L_\infty[a,b];\\ \dfrac{(b-a)^{\frac{1}{q}}L_{(\alpha+n)q}^{\alpha+n}(a,b)}{(\alpha+1)\cdots(\alpha+n)}\left\|g^{(n)}\right\|_p, & \text{if} \quad g^{(n)}\in L_p[a,b];\\ & \text{and} \quad \frac{1}{p}+\frac{1}{q}=1,\ p>1;\\ \dfrac{b^{\alpha+n}}{(\alpha+1)\cdots(\alpha+n)}\left\|g^{(n)}\right\|_1, & \text{if} \quad g^{(n)}\in L_1[a,b],\end{cases}\end{aligned}\tag{2.12}$$

for all $\alpha>-1$.

3. Define $w_0^{(3)}(t)=\sin(\alpha t)$, $\alpha\in\mathbb{R}\setminus\{0\}$ and consider the sequence

$$w_k^{(3)}(t)=\frac{(-1)^k}{\alpha^k}\sin\left(\alpha t+k\cdot\frac{\pi}{2}\right).$$

Then

$$\begin{aligned}\frac{dw_k^{(3)}(t)}{dt} &= \frac{(-1)^k}{\alpha^k}\alpha\cos\left[\alpha t + k\cdot\frac{\pi}{2}\right] = \frac{(-1)^k}{\alpha^{k-1}}\sin\left[\frac{\pi}{2} - \left(\alpha t + k\frac{\pi}{2}\right)\right] \\ &= \frac{(-1)^{k-1}}{\alpha^{k-1}}\sin\left(\alpha t + k\frac{\pi}{2} - \frac{\pi}{2}\right) = \frac{(-1)^{k-1}}{\alpha^{k-1}}\sin\left(\alpha t + (k-1)\frac{\pi}{2}\right) \\ &= w_{k-1}^{(3)}(t), \quad t\in\mathbb{R},\end{aligned}$$

which shows that $\left\{w_k^{(3)}\right\}_{k=0,1,\ldots}$ is an $w_0^{(3)}-$Appell type sequence.

If we use Lemma 1, we may state the identity

$$\begin{aligned}&\int_a^b \sin(\alpha t)\cdot g(t)\,dt \\ &= \sum_{k=1}^n \frac{1}{\alpha^k}\left[\sin\left(\alpha a + k\frac{\pi}{2}\right)\cdot g^{(k-1)}(a) - \sin\left(\alpha b + k\frac{\pi}{2}\right)\cdot g^{(k-1)}(b)\right] \\ &\qquad + \frac{1}{\alpha^n}\int_a^b \sin\left(\alpha t + k\frac{\pi}{2}\right) g^{(n)}(t)\,dt.\end{aligned} \tag{2.13}$$

We compute

$$\begin{aligned}\left\|w_n^{(3)}\right\|_2 &:= \left[\frac{1}{\alpha^{2n}}\int_a^b \sin^2\left(\alpha t + n\frac{\pi}{2}\right)dt\right]^{\frac{1}{2}} \\ &= \frac{1}{|\alpha|^n}\left[\int_a^b\left[\frac{1-\cos(2\alpha t + n\cdot\pi)}{2}\right]dt\right]^{\frac{1}{2}} \\ &= \frac{1}{|\alpha|^n}\left[\frac{1}{2}(b-a) - \frac{1}{4\alpha}\left[\sin(2\alpha b + n\pi) - \sin(2\alpha a + n\pi)\right]\right]^{\frac{1}{2}} \\ &= \frac{(b-a)^{\frac{1}{2}}}{\sqrt{2}\,|\alpha|^n}\left[1 - \frac{\sin(2\alpha b + n\pi) - \sin(2\alpha a + n\pi)}{2\alpha(b-a)}\right]^{\frac{1}{2}}.\end{aligned}$$

Consequently, using Theorem 2 for Hilbertian norms, we may state that

$$\begin{aligned}&\Bigg|\int_a^b \sin(\alpha t)\, g(t)\,dt \\ &\quad - \sum_{k=1}^n \frac{1}{\alpha^k}\left[\sin\left(\alpha a + k\frac{\pi}{2}\right)\cdot g^{(k-1)}(a) - \sin\left(\alpha b + k\frac{\pi}{2}\right)\cdot g^{(k-1)}(b)\right]\Bigg| \\ &\leq \frac{(b-a)^{\frac{1}{2}}}{\sqrt{2}\,|\alpha|^n}\left[1 - \frac{\sin(2b\alpha + n\pi) - \sin(2a\alpha + n\pi)}{2\alpha(b-a)}\right]^{\frac{1}{2}}\left\|g^{(n)}\right\|_2,\end{aligned} \tag{2.14}$$

provided $g^{(n)} \in L_2[a,b]$.

4. Define $w_0^{(4)}(t) = \cos(\alpha t)$, $\alpha \in \mathbb{R} \setminus \{0\}$ and consider the sequence $w_k^{(4)}(t) = \frac{(-1)^k}{\alpha^k} \cos\left(\alpha t + k \cdot \frac{\pi}{2}\right)$. Then

$$\begin{aligned} \frac{dw_k^{(4)}(t)}{dt} &= \frac{(-1)^k}{\alpha^k}\left[-\alpha \sin\left[\alpha t + k \cdot \frac{\pi}{2}\right]\right] = \frac{(-1)^{k-1}}{\alpha^{k-1}} \sin\left(\alpha t + k\frac{\pi}{2}\right) \\ &= \frac{(-1)^{k-1}}{\alpha^{k-1}} \cos\left[\frac{\pi}{2} - \left(\alpha t + k\frac{\pi}{2}\right)\right] = \frac{(-1)^{k-1}}{\alpha^{k-1}} \cos\left[\alpha t + (k-1)\frac{\pi}{2}\right] \\ &= w_{k-1}^{(4)}(t), \ t \in \mathbb{R}, \end{aligned}$$

which shows that $\left\{w_k^{(4)}\right\}_{k=0,1,\dots}$ is an $w_0^{(4)}-$Appell type sequence.

If we use Lemma 1, we may state the identity

$$\begin{aligned} &\int_a^b \cos(\alpha t)\, g(t)\, dt \\ &= \sum_{k=1}^{n} \frac{1}{\alpha^k}\left[\cos\left(\alpha a + k\frac{\pi}{2}\right) \cdot g^{(k-1)}(a) - \cos\left(\alpha b + k\frac{\pi}{2}\right) \cdot g^{(k-1)}(b)\right] \\ &\quad + \frac{1}{\alpha^n}\int_a^b \cos\left(\alpha t + k\frac{\pi}{2}\right) g^{(n)}(t)\, dt. \end{aligned} \tag{2.15}$$

We compute

$$\begin{aligned} \left\|w_n^{(4)}\right\|_2 \ &:= \left[\frac{1}{\alpha^{2n}}\int_a^b \cos^2\left(\alpha t + n\frac{\pi}{2}\right) dt\right]^{\frac{1}{2}} \\ &= \frac{1}{|\alpha|^n}\left[\int_a^b \left[\frac{1 + \cos(2\alpha t + n \cdot \pi)}{2}\right] dt\right]^{\frac{1}{2}} \\ &= \frac{1}{|\alpha|^n}\left[\frac{1}{2}(b-a) + \frac{1}{4\alpha}\left[\sin(2\alpha b + n\pi) - \sin(2\alpha a + n\pi)\right]\right]^{\frac{1}{2}} \\ &= \frac{(b-a)^{\frac{1}{2}}}{\sqrt{2}\,|\alpha|^n}\left[1 + \frac{\sin(2\alpha b + n\pi) - \sin(2\alpha a + n\pi)}{2\alpha(b-a)}\right]^{\frac{1}{2}}. \end{aligned}$$

Consequently, using Theorem 2 for Hilbertian norms, we may state that

$$\begin{aligned} &\left|\int_a^b \cos(\alpha t)\, g(t)\, dt \right. \\ &\left. - \sum_{k=1}^{n} \frac{1}{\alpha^k}\left[\cos\left(\alpha a + k\frac{\pi}{2}\right) \cdot g^{(k-1)}(a) - \cos\left(\alpha b + k\frac{\pi}{2}\right) \cdot g^{(k-1)}(b)\right]\right| \\ &\qquad \leq \frac{(b-a)^{\frac{1}{2}}}{\sqrt{2}\,|\alpha|^n}\left[1 + \frac{\sin(2b\alpha + n\pi) - \sin(2a\alpha + n\pi)}{2\alpha(b-a)}\right]^{\frac{1}{2}} \left\|g^{(n)}\right\|_2, \end{aligned} \tag{2.16}$$

provided $g^{(n)} \in L_2[a,b]$.

3. A Perturbed Version via Korkine's Identity

The following identity which can be easily proved by direct computation is known in the literature as Korkine's identity:

$$(3.1)\quad \frac{1}{b-a}\int_a^b u(t)\,v(t)\,dt-\frac{1}{b-a}\int_a^b u(t)\,dt\cdot\frac{1}{b-a}\int_a^b v(t)\,dt = \frac{1}{2(b-a)^2}\int_a^b\int_a^b (u(t)-u(s))(v(t)-v(s))\,dtds,$$

provided that $u,v:[a,b]\to\mathbb{R}$ are measurable and all the involved integrals exist.

The following representation of the weighted integral $\int_a^b w(t)\,g(t)\,dt$ holds.

Lemma 2. *Let $w:[a,b]\to\mathbb{R}$ be an absolutely continuous function on $[a,b]$ and $\{w_k\}_{k=\overline{0,n+1}}$ be a sequence of $w-$Appell type. If $g:[a,b]\to\mathbb{R}$ is such that $g^{(n-1)}$ is absolutely continuous on $[a,b]$ and $w_n g^{(n)}\in L_1[a,b]$, then we have the equality*

$$(3.2)\quad \int_a^b w(t)\,g(t)\,dt = B_n(w,g;a,b)+S_n(w,g;a,b),$$

where

$$(3.3)\quad B_n(w,g;a,b)=A_n(w,g;a,b)+(-1)^n\left[w_{n+1}(b)-w_{n+1}(a)\right]\left[g^{(n-1)};a,b\right]$$

and the remainder $S_n(w,g;a,b)$ can be represented by

$$(3.4)\quad S_n(w,g;a,b)=\frac{(-1)^n}{2(b-a)}\int_a^b\int_a^b (w_n(t)-w_n(s))\left(g^{(n)}(t)-g^{(n)}(s)\right)dtds$$

where $\left[g^{(n-1)};a,b\right]$ is the devided difference, i.e., we recall that

$$\left[g^{(n-1)};a,b\right]:=\frac{g^{(n-1)}(b)-g^{(n-1)}(a)}{b-a}.$$

Proof. Using Korkine's identity, we may write

$$\begin{aligned}\int_a^b w_n(t)\,g^{(n)}(t)\,dt &= \frac{1}{b-a}\int_a^b w_n(t)\,dt\cdot\int_a^b g^{(n)}(t)\,dt\\ &\quad+\frac{1}{2(b-a)}\int_a^b\int_a^b (w_n(t)-w_n(s))\left(g^{(n)}(t)-g^{(n)}(s)\right)dtds\\ &= \left[w_{n+1}(b)-w_{n+1}(a)\right]\cdot\frac{g^{(n-1)}(b)-g^{(n-1)}(a)}{b-a}\\ &\quad+\frac{1}{2(b-a)}\int_a^b\int_a^b (w_n(t)-w_n(s))\left(g^{(n)}(t)-g^{(n)}(s)\right)dtds,\end{aligned}$$

giving the following representation for the remainder $R_n(w,g;a,b)$:

$$\begin{aligned}R_n(w,g;a,b) &= (-1)^n\left[w_{n+1}(b)-w_{n+1}(a)\right]\cdot\frac{g^{(k-1)}(b)-g^{(k-1)}(a)}{b-a}\\ &\quad+\frac{(-1)^n}{2(b-a)}\int_a^b\int_a^b (w_n(t)-w_n(s))\left(g^{(n)}(t)-g^{(n)}(s)\right)dtds.\end{aligned}$$

Using the identity (2.3) we deduce (3.2). ∎

For an absolutely continuous function $h:[\alpha,\beta]\to\mathbb{R}$, we denote

$$\|h\|_{[\alpha,\beta],1}:=\left|\int_{\alpha}^{\beta}|h(\tau)|\,d\tau\right|.$$

Using this notation, we may state the following result involving the estimation of the remainder $S(w,g;a,b)$ in the perturbed formula (3.2).

Theorem 3. *With the assumption of Lemma 2, we have*

$$\begin{aligned}(3.5)\qquad |S(w,g;a,b)| &\leq \frac{1}{2(b-a)}\int_a^b\int_a^b \|w_{n+1}\|_{[s,t],1}\left\|g^{(n+1)}\right\|_{[s,t],1}dtds\\ &=: M_n\left(w_{n+1},g^{(n+1)}\right).\end{aligned}$$

Proof. Using the representation (3.4), we may write

$$\begin{aligned}|S(w,g;a,b)| &\leq \frac{1}{2(b-a)}\int_a^b\int_a^b |w_n(t)-w_n(s)|\left|g^{(n)}(t)-g^{(n)}(s)\right|dtds\\ &= \frac{1}{2(b-a)}\int_a^b\int_a^b \left|\int_s^t w_{n+1}(\tau)\,d\tau\right|\left|\int_s^t g^{(n+1)}(\sigma)\,d\sigma\right|dtds\\ &\leq \frac{1}{2(b-a)}\int_a^b\int_a^b \|w_{n+1}\|_{[s,t],1}\left\|g^{(n+1)}\right\|_{[s,t],1}dtds\end{aligned}$$

and the estimate (3.5) is obtained. ∎

In practical applications some bounds of $M_n\left(w_{n+1},g^{(n+1)}\right)$ could be more useful. Now, it obvious that

$$\begin{aligned}&M_n\left(w_{n+1},g^{(n+1)}\right)\\ &\leq \frac{1}{2(b-a)} ess\sup_{(t,s)\in[a,b]}\left\{\|w_{n+1}\|_{[s,t],1}\right\}\cdot\int_a^b\int_a^b\left\|g^{(n+1)}\right\|_{[s,t],1}dtds\\ &= \frac{1}{2(b-a)}\left\|\|w_{n+1}\|_{[\cdot,\cdot],1}\right\|_{\infty}\left\|\left\|g^{(n+1)}\right\|_{[\cdot,\cdot],1}\right\|_{1}.\end{aligned}$$

Also, by Hölder's integral inequality for double integrals, we may write that

$$\begin{aligned}M_n\left(w_{n+1},g^{(n+1)}\right) &\leq \frac{1}{2(b-a)}\left(\int_a^b\int_a^b\|w_{n+1}\|_{[s,t],1}^q dtds\right)^{\frac{1}{q}}\\ &\quad\times\left(\int_a^b\int_a^b\left\|g^{(n+1)}\right\|_{[s,t],1}^p dtds\right)^{\frac{1}{p}}\\ &= \frac{1}{2(b-a)}\left\|\|w_{n+1}\|_{[\cdot,\cdot],1}\right\|_{q}\left\|\left\|g^{(n+1)}\right\|_{[\cdot,\cdot],1}\right\|_{p},\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1$, $p>1$.

In a similar manner,

$$M_n\left(w_{n+1},g^{(n+1)}\right)\leq\frac{1}{2(b-a)}\left\|\|w_{n+1}\|_{[\cdot,\cdot],1}\right\|_{1}\left\|\left\|g^{(n+1)}\right\|_{[\cdot,\cdot],1}\right\|_{\infty}.$$

Consequently, we may state the following corollary.

Corollary 2. *With the assumptions of Lemma 2, we have the following bounds for the remainder $S_n(w,g;a,b)$:*

(3.6)
$$|S_n(w,g;a,b)| \leq \begin{cases} \frac{1}{2(b-a)} \left\| \|w_{n+1}\|_{[\cdot,\cdot],1} \right\|_{\infty} \left\| \|g^{(n+1)}\|_{[\cdot,\cdot],1} \right\|_{1}; \\ \frac{1}{2(b-a)} \left\| \|w_{n+1}\|_{[\cdot,\cdot],1} \right\|_{q} \left\| \|g^{(n+1)}\|_{[\cdot,\cdot],1} \right\|_{p}, \ p>1, \ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{1}{2(b-a)} \left\| \|w_{n+1}\|_{[\cdot,\cdot],1} \right\|_{1} \left\| \|g^{(n+1)}\|_{[\cdot,\cdot],1} \right\|_{\infty}, \end{cases}$$

where $\|\cdot\|_s$ $(s\in[1,\infty])$ are the usual Lebesgue norms on $L_s([a,b]\times[a,b])^2$.

Now, if we use the natural notations

$$\|h\|_{[t,s],\beta} := \left| \int_t^s |h(\tau)|^{\beta} d\tau \right|^{\frac{1}{\beta}}, \ \beta>1$$

and

$$\|h\|_{[t,s],\infty} := ess\sup\{|h(\tau)|, \ \tau\in[t,s] \ ([s,t])\},$$

then by Hölder's integral inequality, we have

$$\|w_{n+1}\|_{[s,t],1} \leq |s-t|^{\frac{1}{\alpha}} \|w_{n+1}\|_{[s,t],\beta}, \ \beta>1, \ \frac{1}{\beta}+\frac{1}{\alpha}=1 \ \text{ or } \ \alpha=1, \ \beta=\infty \tag{3.7}$$

and

$$\left\|g^{(n+1)}\right\|_{[s,t],1} \leq |s-t|^{\frac{1}{\gamma}} \left\|g^{(n+1)}\right\|_{[s,t],\delta}, \ \gamma>1, \ \frac{1}{\gamma}+\frac{1}{\delta}=1 \ \text{ or } \ \gamma=1, \ \delta=\infty. \tag{3.8}$$

Multiplying (3.7) and (3.8) and integrating over $(t,s)\in[a,b]^2$, we deduce

$$M_n\left(w_{n+1},g^{(n+1)}\right) \leq \frac{1}{2(b-a)} \int_a^b \int_a^b |s-t|^{\frac{1}{\alpha}+\frac{1}{\gamma}} \|w_{n+1}\|_{[s,t],\beta} \left\|g^{(n+1)}\right\|_{[s,t],\delta} dtds, \tag{3.9}$$

where $\alpha,\beta>1$, $\frac{1}{\beta}+\frac{1}{\alpha}=1$ or $\alpha=1$, $\beta=\infty$; γ,$\delta>1$, $\frac{1}{\gamma}+\frac{1}{\delta}=1$ or $\gamma=1$ and $\delta=\infty$, which is an inequality interesting in itself.

The terms $M_n\left(w_{n+1},g^{(n+1)}\right)$ can be bounded in a simpler form as follows.

Corollary 3. *With the assumptions of Lemma 2 and if α, β, γ, δ are as above and $r_i>1$, $\left(i=\overline{1,3}\right)$, $\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}=1$, then we have the inequality:*

$$M_n\left(w_{n+1},g^{(n+1)}\right) \leq K_{r_1} \left\| \|w_{n+1}\|_{[\cdot,\cdot],\beta} \right\|_{r_2} \left\| \left\|g^{(n+1)}\right\|_{[\cdot,\cdot],\delta} \right\|_{r_3}, \tag{3.10}$$

where

$$K_{r_1} = \frac{2^{\frac{1}{r_1}-1}(b-a)^{\frac{1}{\alpha}+\frac{1}{\delta}+\frac{2}{r_1}-1}}{\left[\left(\frac{1}{\alpha}+\frac{1}{\delta}\right)r_1+1\right]^{\frac{1}{r_1}} \left[\left(\frac{1}{\alpha}+\frac{1}{\delta}\right)r_1+2\right]^{\frac{1}{r_1}}}.$$

Proof. Using Hölder's integral inequality for three terms with $r_i > 1$, $\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} = 1$, we may write:

$$\begin{aligned} (3.11) \quad & \int_a^b \int_a^b |s-t|^{\frac{1}{\alpha}+\frac{1}{\gamma}} \|w_{n+1}\|_{[s,t],\beta} \left\|g^{(n+1)}\right\|_{[s,t],\delta} dtds \\ \leq & \left(\int_a^b \int_a^b |s-t|^{\left(\frac{1}{\alpha}+\frac{1}{\gamma}\right)r_1} dtds\right)^{\frac{1}{r_1}} \times \left(\int_a^b \int_a^b \|w_{n+1}\|_{[s,t],\beta}^{r_2} dtds\right)^{\frac{1}{r_2}} \\ & \times \left(\int_a^b \int_a^b \left\|g^{(n+1)}\right\|_{[s,t],\delta}^{r_3} dtds\right)^{\frac{1}{r_3}} \\ = & \left(\int_a^b \int_a^b |s-t|^{\left(\frac{1}{\alpha}+\frac{1}{\gamma}\right)r_1} dtds\right)^{\frac{1}{r_1}} \left\|\|w_{n+1}\|_{[\cdot,\cdot],\beta}\right\|_{r_2} \left\|\left\|g^{(n+1)}\right\|_{[\cdot,\cdot],\delta}\right\|_{r_3}. \end{aligned}$$

We remark that, for a positive p, we have

$$\begin{aligned} \int_a^b \int_a^b |x-y|^p \, dxdy &= \int_a^b \left(\int_a^b |y-x|^p \, dy\right) dx \\ &= \int_a^b \left(\int_a^x (x-y)^p \, dy + \int_x^b (y-x)^p \, dy\right) dx \\ &= \int_a^b \left[\frac{(x-a)^{p+1} + (b-x)^{p+1}}{p+1}\right] dx = \frac{2(b-a)^{p+2}}{(p+1)(p+2)} \end{aligned}$$

and then

$$\begin{aligned} \left(\int_a^b \int_a^b |s-t|^{\left(\frac{1}{\alpha}+\frac{1}{\gamma}\right)r_1} dtds\right)^{\frac{1}{r_1}} &= \left[\frac{2(b-a)^{\left(\frac{1}{\alpha}+\frac{1}{\gamma}\right)r_1+2}}{\left[\left(\frac{1}{\alpha}+\frac{1}{\gamma}\right)r_1+1\right]\left[\left(\frac{1}{\alpha}+\frac{1}{\gamma}\right)r_1+2\right]}\right]^{\frac{1}{r_1}} \\ &= \frac{2^{\frac{1}{r_1}} (b-a)^{\frac{1}{\alpha}+\frac{1}{\delta}+\frac{2}{r_1}}}{\left[\left(\frac{1}{\alpha}+\frac{1}{\delta}\right)r_1+1\right]^{\frac{1}{r_1}} \left[\left(\frac{1}{\alpha}+\frac{1}{\delta}\right)r_1+2\right]^{\frac{1}{r_1}}}. \end{aligned}$$

Using (3.9) and (3.11) we obtain (3.10). ∎

Remark 4. *If one would use the examples **1-4** considered in Section 2, some particular inequalities may be stated. We omit the details.*

4. Some Bounds Via a Grüss Type Inequality

In [4], Cheng and Sun have proved the following integral inequality of Grüss type for univariate real functions:

Lemma 3. *Let $h, g : [a,b] \to \mathbb{R}$ be two integrable functions such that*

$$(4.1) \qquad \varphi \leq g(x) \leq \Phi \text{ for a.e. } x \in [a,b].$$

Then

$$\begin{aligned} (4.2) \quad & \left|\frac{1}{b-a}\int_a^b h(t) g(t) dt - \frac{1}{b-a}\int_a^b h(t) dt \cdot \frac{1}{b-a}\int_a^b g(t) dt\right| \\ \leq & \frac{1}{2}(\Phi - \varphi) \frac{1}{b-a}\int_a^b \left|h(t) - \frac{1}{b-a}\int_a^b h(s) ds\right| dt. \end{aligned}$$

For a generalization of this result in the abstract settings of Lebesgue integrals and weighted means, see [3].

The following result also holds.

Theorem 4. *Let* $w : [a,b] \to \mathbb{R}$ *be an absolutely continuous function on* $[a,b]$ *and* $\{w_k\}_{k=\overline{0,n+1}}$ *be a sequence of* $w-$*Appell type. If* $g : [a,b] \to \mathbb{R}$ *is such that* $g^{(n-1)}$ *is absolutely continuous on* $[a,b]$, $w_n g^{(n)} \in L_1[a,b]$, *and there exists the constants* γ_n, Γ_n *so that*

$$-\infty < \gamma_n \leq g^{(n)} \leq \Gamma_n < \infty \text{ a.e. on } [a,b],$$

then we have the equality

$$\int_a^b w(t)\, g(t)\, dt = B_n(w,g;a,b) + S_n(w,g;a,b),$$

where

$$B_n(w,g;a,b) = A_n(w,g;a,b) + (-1)^n [w_{n+1}(b) - w_{n+1}(a)] \left[g^{(n-1)};a,b\right], \tag{4.3}$$

$A_n(w,g;a,b)$ *is given in equation (2.4) and the remainder* $S_n(w,g;a,b)$ *satisfies the estimate*

$$|S_n(w,g;a,b)| \leq \frac{1}{2}(\Gamma_n - \gamma_n) \int_a^b |w_n(t) - [w_{n+1};a,b]|\, dt \tag{4.4}$$

where $[w_{n+1};a,b]$ *is the devided difference.*

Proof. From (3.4) we have

$$\begin{aligned} & S_n(w,g;a,b) \\ = \; & (b-a)(-1)^n \left| \frac{1}{b-a}\int_a^b w_n(t)\, g^{(n)}(t)\, dt \right. \\ & \left. - \frac{1}{b-a}\int_a^b w_n(t)\, dt \cdot \frac{1}{b-a}\int_a^b g^{(n)}(t)\, dt \right|. \end{aligned}$$

Thus, by the use of Lemma 3 we have

$$\begin{aligned} & |S_n(w,g;a,b)| \\ \leq \; & \frac{1}{2}(\Gamma_n - \gamma_n) \int_a^b \left| w_n(t) - \frac{1}{b-a}\int_a^b w_n(s)\, ds \right| dt \end{aligned}$$

and the theorem is proved. ∎

Remark 5. *If one would use the examples* ***1-4*** *considered in Section 2, some particular inequalities may be stated. We omit the details.*

References

[1] P. CERONE, Three point rules in numerical integration, *Nonlinear Analysis,* **47** (2001), 2341-2352.

[2] P. CERONE and S.S. DRAGOMIR, New bounds for a perturbed generalised Taylor's formula, *East Asian Math. J.,* **17**(2) (2001), 197-215.

[3] P. CERONE and S.S. DRAGOMIR, A refinement of the Grüss inequality and applications, *submitted.*

[4] X.-L. CHENG and J. SUN, A note on the perturbed trapezoid inequality, *J. Ineq. Pure & Appl. Math.*, **3**(2002), No. 2, Article 29. [ON LINE: http://jipam.vu.edu.au].

[5] S.S. DRAGOMIR, Better bounds in some Ostrowski-Grüss type inequality, *Tamkang J. Math.,* **32**(3) (2001), 211-216.
[6] M. MATIĆ, J.E. PEČARIĆ and N. UJEVIĆ, On new estimation of the remainder in generalised Taylor's formula, *Math. Ineq. & Appl.,* **2**(3) (1999), 343-361.
[7] C.E.M. PEARCE and J.E. PEČARIĆ, On Anastassiou's generalisation of the Ostrowski inequality and related results, *J. Comput. Anal. and Applic.,* **2**(3) (2000), 215-235.
[8] C.E.M. PEARCE, J.E. PEČARIĆ, N. UJEVIĆ and S. VAROŠANEC, Generalisations of some inequalities of Ostrowski-Grüss type, *Math. Ineq. & Appl.,* **3**(1) (2000), 25-34.
[9] J.E. PEČARIĆ and S. VAROŠANEC, Harmonic polynomials and generalisation of Ostrowski inequality with applications in numerical integration, *Nonlinear Analysis,* **47** (2001), 2365-2374.

School of Communications and Informatics, Victoria University of Technology, PO Box 14428, Melbourne City MC, Victoria 8001, Australia.
E-mail address: `sever@matilda.vu.edu.au`
URL: `http://rgmia.vu.edu.au/SSDragomirWeb.html`