A GRÜSS TYPE INEQUALITY FOR ISOTONIC LINEAR FUNCTIONALS AND APPLICATIONS

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Abstract. An inequality for a normalised isotonic linear functional of Grüss type and particular cases for integrals and norms are established. Applications in obtaining a counterpart for the Cauchy-Buniakowski-Schwartz inequality for functionals and Jessen's inequality for convex functions are also given.

1. Introduction

Let L be a linear class of real-valued functions $g : E \to \mathbb{R}$ having the properties

(L1) $f, g \in L$ imply $(\alpha f + \beta g) \in L$ for all $\alpha, \beta \in \mathbb{R}$;

(L2) $\mathbf{1} \in L$, i.e., if $f(t) = 1$, $t \in E$, then $f \in L$.

An isotonic linear functional $A : L \to \mathbb{R}$ is a functional satisfying

(A1) $A(\alpha f + \beta g) = \alpha A(f) + \beta A(g)$ for all $f, g \in L$ and $\alpha, \beta \in \mathbb{R}$;

(A2) If $f \in L$ and $f \geq 0$, then $A(f) \geq 0$.

The mapping A is said to be normalised if

(A3) $A(\mathbf{1}) = 1$.

Usual examples of isotonic linear functionals that are normalised are the following ones

$$A(f) = \frac{1}{\mu(X)} \int_X f(x)\, d\mu(x) \quad \text{if} \quad \mu(X) < \infty$$

or

$$A_w(f) := \frac{1}{\int_X w(x)\, d\mu(x)} \int_X w(x) f(x)\, d\mu(x),$$

where $w(x) \geq 0$, $\int_X w(x)\, d\mu(x) > 0$, X is a measurable space and μ a positive measure on X.

In particular, for $\bar{x} = (x_1, \dots, x_n)$, $\bar{w} := (w_1, \dots, w_n) \in \mathbb{R}^n$ with $w_i \geq 0$, $W_n := \sum_{i=1}^n w_i > 0$, we have

$$A(\bar{x}) := \frac{1}{n} \sum_{i=1}^n x_i$$

and

$$A_{\bar{w}}(\bar{x}) := \frac{1}{W_n} \sum_{i=1}^n w_i x_i,$$

are normalised isotonic functionals on $\mathbb{R}^n$.

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In 1988, D. Andrica and C. Badea [1], proved the following generalisation of the Grüss inequality for isotonic linear functionals.

Theorem 1. *If $f, g \in L$ so that $fg \in L$ and $m \leq f \leq M$, $n \leq g \leq N$ where m, M, n, N are given real numbers, then for any normalised isotonic linear functional $A : L \to \mathbb{R}$ one has the inequality*

$$|A(fg) - A(f)A(g)| \leq \frac{1}{4}(M-m)(N-n). \tag{1.1}$$

The constant $\frac{1}{4}$ in (1.1) is best possible in the sense that it cannot be replaced by a smaller constant.

In this paper we point out a refinement of the Grüss inequality (1.1) for isotonic linear functionals. Applications for the Cauchy-Buniakowski-Schwartz and Jessen's inequality are also provided.

2. A Grüss Type Inequality

The following result holds.

Theorem 2. *Let $f, g \in L$ be such that $fg \in L$ and assume that there exists the real numbers n and N so that*

$$n \leq g \leq N. \tag{2.1}$$

Then for any normalised isotonic linear functional $A : L \to \mathbb{R}$ for which $|f - A(f) \cdot \mathbf{1}| \in L$ one has the inequality

$$|A(fg) - A(f)A(g)| \leq \frac{1}{2}(N-n)A(|f - A(f) \cdot \mathbf{1}|). \tag{2.2}$$

The constant $\frac{1}{2}$ in (2.2) is best possible in the sense that it cannot be replaced by a smaller constant.

Proof. Using the linearity property of A, we have

$$\begin{aligned} & A\left[(f - A(f) \cdot \mathbf{1})\left(g - \frac{n+N}{2} \cdot \mathbf{1}\right)\right] \\ = {} & A[(f - A(f) \cdot \mathbf{1})g] - \frac{n+N}{2}A[f - A(f) \cdot \mathbf{1}] \\ = {} & A(fg) - A(f)A(g) - \frac{n+N}{2}[A(f) - A(f) \cdot A(\mathbf{1})] \\ = {} & A(fg) - A(f)A(g) \end{aligned} \tag{2.3}$$

since, by the normality property of A, $A(\mathbf{1}) = 1$.

From (2.1) we may easily deduce that

$$\left|g - \frac{n+N}{2} \cdot \mathbf{1}\right| \leq \frac{M-n}{2} \cdot \mathbf{1}. \tag{2.4}$$

It is known that if $h \in L$ so that $|h| \in L$, then, by the monotonicity and linearity of A, one has

$$|A(h)| \leq A(|h|). \tag{2.5}$$

Using this property, the monotonicity property of A and condition (2.4), we deduce

$$\begin{aligned} &\left| A\left[(f - A(f)\cdot \mathbf{1}) \left(g - \frac{n+N}{2}\cdot \mathbf{1}\right)\right]\right| \\ \le\ & A\left(\left| (f - A(f)\cdot \mathbf{1}) \left(g - \frac{n+N}{2}\cdot \mathbf{1}\right)\right| \right) \\ \le\ & \frac{N-n}{2} A\left(|f - A(f)\cdot \mathbf{1}| \right). \end{aligned} \tag{2.6}$$

Utilising (2.3) and (2.6) we deduce the desired result (2.2).

To prove the sharpness of the constant $\frac{1}{2}$, we assume that (2.2) holds with a constant $c > 0$ for $A = \frac{1}{b-a}\int_a^b$, $L = L[a,b]$ (the Lebesgue space of integrable functions on $[a,b]$) and g satisfying the condition (2.1) on the interval $[a,b]$, i.e., one has the inequality

$$\begin{aligned} &\left| \frac{1}{b-a}\int_a^b f(x)\, g(x)\, dx - \frac{1}{b-a}\int_a^b f(x)\, dx \cdot \frac{1}{b-a}\int_a^b g(x)\, dx \right| \\ &\qquad \le c(N-n)\cdot \frac{1}{b-a}\int_a^b \left| f(x) - \frac{1}{b-a}\int_a^b f(y)\, dy \right| dx. \end{aligned} \tag{2.7}$$

If we choose $g = f$ and $f : [a,b] \to \mathbb{R}$,

$$f(x) = \begin{cases} -1 & \text{if } \ x \in \left[a, \frac{a+b}{2}\right] \\ 1 & \text{if } \ x \in \left(\frac{a+b}{2}, b\right] \end{cases}$$

then

$$\begin{aligned} \frac{1}{b-a}\int_a^b f^2(x)\, dx - \left(\frac{1}{b-a}\int_a^b f(x)\, dx \right)^2 &= 1, \\ \frac{1}{b-a}\int_a^b \left| f(x) - \frac{1}{b-a}\int_a^b f(y)\, dy \right| dx &= 1, \\ m = -1,\ M &= 1. \end{aligned}$$

and by (2.7) we deduce $c \ge \frac{1}{2}$. ∎

The following corollaries are natural consequences of the above result.

Corollary 1. *Let $f \in L$ be such that $f^2 \in L$ and there exists the real numbers m, M so that*

$$m \le f \le M. \tag{2.8}$$

Then for any $A : L \to \mathbb{R}$ a normalised isotonic linear functional so that $|f - A(f)\cdot \mathbf{1}| \in L$ one has the inequality

$$0 \le A\left(f^2\right) - [A(f)]^2 \le \frac{1}{2}(M-m)\, A\left(|f - A(f)\cdot \mathbf{1}| \right). \tag{2.9}$$

The constant $\frac{1}{2}$ is sharp.

Corollary 2. *Let $f,g \in L$ so that $fg \in L$ and f satisfy (2.8) while g satisfies (2.1). Then for any normalised isotonic linear functional $A : L \to \mathbb{R}$ so that $|f - A(f) \cdot \mathbf{1}|, |g - A(g) \cdot \mathbf{1}| \in L$ one has the inequality:*

$$
\begin{aligned}
(2.10) \quad & |A(fg) - A(f)A(g)| \\
& \leq \frac{1}{2}[(M-m)(N-n)]^{\frac{1}{2}}[A(|f - A(f)\cdot \mathbf{1}|)A(|g - A(g)\cdot \mathbf{1}|)]^{\frac{1}{2}}.
\end{aligned}
$$

The constant $\frac{1}{2}$ is sharp.

Remark 1. *Using Hölder's inequality for isotonic linear functionals, we may state the following inequalities as well*

$$
\begin{aligned}
& |A(fg) - A(f)A(g)| \\
\leq\ & \frac{1}{2}(N-n)A(|f - A(f)\cdot \mathbf{1}|) \ \ \text{if} \ \ |f - A(f)\cdot \mathbf{1}| \in L, \\
\leq\ & \frac{1}{2}(N-n)[A(|f - A(f)\cdot \mathbf{1}|^p)]^{\frac{1}{p}} \ \ \text{if} \ \ |f - A(f)\cdot \mathbf{1}|^p \in L, \ p > 1 \\
\leq\ & \sup_{t \in E}|f(t) - A(f)|;
\end{aligned}
$$

provided $f,g \in L$ and $fg \in L$ while g satisfies the condition (2.1).

If f and g fulfill the conditions (2.8) and (2.1), then we have the following *refinement of the Grüss inequality* (1.1)

$$
\begin{aligned}
(2.11) \qquad |A(fg) - A(f)A(g)| & \leq \frac{1}{2}(N-n)A(|f - A(f)\cdot \mathbf{1}|) \\
& \leq \frac{1}{2}(N-n)\left[A\left(f^2\right) - [A(f)]^2\right]^{\frac{1}{2}} \\
& \leq \frac{1}{4}(M-m)(N-n).
\end{aligned}
$$

The constants $\frac{1}{2}$, $\frac{1}{2}$ and $\frac{1}{4}$ are sharp in (2.11).

The following weighted version of Theorem 2 also holds.

Theorem 3. *Let $f,g,h \in L$ be such that $h \geq 0$, fh, gh, $fgh \in L$ and there exists the real constants n,N so that (2.1) holds. Then for any $B : L \to \mathbb{R}$ an isotonic linear functional so that $B(h) > 0$, $h\left|f - \frac{1}{B(h)} \cdot \mathbf{1}\right| \in L$ one has the inequality:*

$$
\begin{aligned}
(2.12) \quad & \left|\frac{B(fgh)}{B(h)} - \frac{B(fh)}{B(h)} \cdot \frac{B(gh)}{B(h)}\right| \\
& \leq \frac{1}{2}(N-n)\frac{1}{B(h)}B\left[h\left|f - \frac{1}{B(h)}B(hf)\cdot \mathbf{1}\right|\right].
\end{aligned}
$$

The constant $\frac{1}{2}$ is best possible.

Proof. Apply Theorem 1 for the functional $A_h : L \to \mathbb{R}$,

$$
A_h(f) := \frac{1}{B(h)}B(hf),
$$

that is a normalised isotonic linear functional on L. ∎

Similar corollaries may be stated from the weighted inequality (2.12), but we omit the details.

3. Applications for Integral and Discrete Inequalities

Let $(\Omega, \mathcal{A}, \mu)$ be a measurable space consisting of a set Ω, a $\sigma-$algebra of parts of Ω and a countably additive and positive measure μ on $\mathcal{A}$ with values in $\mathbb{R}\cup\{\infty\}$.

For a $\mu-$measurable function $w : \Omega \to \mathbb{R}$ with $w(x) \geq 0$ for $\mu-$a.e. $x \in \Omega$, assume $\int_\Omega w(x)\, d\mu(x) > 0$. Consider the Lebesgue space $L_w(\Omega, \mu) := \{f : \Omega \to \mathbb{R},\ f$ is measurable on $\int_\Omega w(x)|f(x)|\, d\mu(x) < \infty\}$.

If $f, g : \Omega \to \mathbb{R}$ are $\mu-$measurable functions and $f, g, fg \in L_w(\Omega, \mu)$, then we may consider the Čebyšev functional

$$
\begin{aligned}
T_w(f,g) := \frac{1}{\int_\Omega w(x)\, d\mu(x)} \int_\Omega w(x) f(x) g(x)\, d\mu(x) \\
- \frac{1}{\int_\Omega w(x)\, d\mu(x)} \int_\Omega w(x) f(x)\, d\mu(x) \\
\times \frac{1}{\int_\Omega w(x)\, d\mu(x)} \int_\Omega w(x) g(x)\, d\mu(x).
\end{aligned}
$$

We may also consider the functional

$$
\begin{aligned}
D_w(f) := \frac{1}{\int_\Omega w(x)\, d\mu(x)} \\
\times \int_\Omega w(x) \left| f(x) - \frac{1}{\int_\Omega w(y)\, d\mu(y)} \int_\Omega w(y) f(y)\, d\mu(y) \right| d\mu(x).
\end{aligned}
$$

Applying Theorem 2 for the normalised isotonic linear functional

$$
A(f) := \frac{1}{\int_\Omega w(x)\, d\mu(x)} \int_\Omega w(x) f(x)\, d\mu(x),
$$

$A : L_w(\Omega, \mu) \to \mathbb{R}$, we may recapture the following result due to Cerone and Dragomir [2]. Note that the proof of this result in [2] is different to the one in Theorem 2.

Theorem 4. *Let $w, f, g : \Omega \to \mathbb{R}$ be $\mu-$measurable functions with $w \geq 0$ $\mu-$a.e. on Ω and $\int_\Omega w(x)\, d\mu(x) > 0$. If $f, g, fg \in L_w(\Omega, \mu)$ and there exists the constants n, N so that*

$$
-\infty < n \leq g(x) \leq N < \infty \quad \text{for } \mu\text{-a.e. } x \in \Omega, \tag{3.1}
$$

then we have the inequality

$$
|T_w(f,g)| \leq \frac{1}{2}(N - n) D_w(f). \tag{3.2}
$$

The constant $\frac{1}{2}$ is sharp in the sense that it cannot be replaced by a smaller constant.

Remark 2. *If $\Omega = [a,b]$ and $w(x) = 1$ in Theorem 4, then we recapture the result obtained in* [3]

$$
\begin{aligned}
\left| \frac{1}{b-a} \int_a^b f(x) g(x)\, dx - \frac{1}{b-a} \int_a^b f(x)\, dx \cdot \frac{1}{b-a} \int_a^b g(x)\, dx \right| \\
\leq \frac{1}{2}(N-n) \cdot \frac{1}{b-a} \int_a^b \left| f(x) - \frac{1}{b-a} \int_a^b f(y)\, dy \right| dx
\end{aligned} \tag{3.3}
$$

provided $n \leq g(x) \leq N$ for a.e. $x \in [a,b]$.

Note that the proof in Theorem 2 is different to the one in [3], using only the linearity and monotonicity properties of the functional A. We should also remark that in [3] the authors did not show the sharpness of the constant $\frac{1}{2}$.

Now, if we consider the normalised isotonic linear functional

$$A_{\bar{w}}(\bar{x}) := \frac{1}{W_n}\sum_{i=1}^{n} w_i x_i, \tag{3.4}$$

$A_{\bar{w}} : \mathbb{R}^n \to \mathbb{R}$, where $w_i \geq 0$ $\left(i = \overline{1,n}\right)$ and $W_n := \sum_{i=1}^{n} w_i > 0$, the by Theorem 2 we may obtain the following discrete inequality obtained by Cerone and Dragomir in [2].

Theorem 5. *Let $\bar{a} = (a_1, \dots, a_n)$, $\bar{b} = (b_1, \dots, b_n) \in \mathbb{R}$ be such that there exists the constants $b, B \in \mathbb{R}$ so that*

$$b \leq b_i \leq B \text{ for each } i \in \{1, \dots, n\}. \tag{3.5}$$

Then one has the inequality

$$\left|\frac{1}{W_n}\sum_{i=1}^{n} w_i a_i b_i - \frac{1}{W_n}\sum_{i=1}^{n} w_i a_i \cdot \frac{1}{W_n}\sum_{i=1}^{n} w_i b_i\right| \leq \frac{1}{2}(B-b)\frac{1}{W_n}\sum_{i=1}^{n} w_i \left|a_i - \frac{1}{W_n}\sum_{j=1}^{n} w_j a_j\right|. \tag{3.6}$$

The constant $\frac{1}{2}$ is sharp in (3.6).

4. A Counterpart of the (CBS)-Inequality

The following inequality is known in the literature as the Cauchy-Buniakowski-Schwartz's inequality for isotonic linear functionals or the (CBS)-inequality, for short,

$$[A(fg)]^2 \leq A\left(f^2\right) A\left(g^2\right), \tag{4.1}$$

provided $f, g : E \to \mathbb{R}$ are with the property that $fg, f^2, g^2 \in L$ and $A : L \to \mathbb{R}$ is any isotonic linear functional.

Making use of the Grüss inequality (2.12), we may prove the following counterpart of the (CBS)-inequality for isotonic linear functionals.

Theorem 6. *Let $k, l : E \to \mathbb{R}$ be such that $k^2, l^2, kl \in L$ and there exists the real constants $\gamma, \Gamma \in \mathbb{R}$ so that*

$$\gamma \leq \frac{k}{l} \leq \Gamma. \tag{4.2}$$

Then for any isotonic linear functional $A : L \to \mathbb{R}$ so that $|l|\left|A\left(l^2\right)k - A(kl)\,l\right| \in L$, one has the inequality:

$$\begin{aligned} 0 &\leq A\left(k^2\right) A\left(l^2\right) - [A(kl)]^2 \\ &\leq \frac{1}{2}(\Gamma - \gamma) A\left[|l|\left|A\left(l^2\right)k - A(kl)\,l\right|\right]. \end{aligned}$$

The constant $\frac{1}{2}$ is sharp.

Proof. We choose in (2.12) $f = g = \frac{k}{l}$, $h = l^2$ and $B = A$ to get

$$\begin{aligned} 0 &\leq \frac{A\left(k^2\right)}{A\left(l^2\right)} - \frac{\left[A\left(kl\right)\right]^2}{\left[A\left(l^2\right)\right]^2} \\ &\leq \frac{1}{2}\left(\Gamma-\gamma\right)\frac{1}{A\left(l^2\right)}A\left[l^2\left|\frac{k}{l}-\frac{1}{A\left(l^2\right)}A\left(kl\right)\right|\right], \end{aligned}$$

provided $A\left(l^2\right) \neq 0$, which is equivalent to

$$\begin{aligned} 0 &\leq A\left(k^2\right)A\left(l^2\right) - \left[A\left(kl\right)\right]^2 \\ &\leq \frac{1}{2}\left(\Gamma-\gamma\right)A\left(l^2\right)A\left[\left|kl-\frac{l^2}{A\left(l^2\right)}A\left(kl\right)\right|\right], \end{aligned}$$

which is clearly equivalent to (4.3). ∎

The following integral inequality holds.

Corollary 3. *Let $w, f, g : \Omega \to \mathbb{R}$ be a $\mu-$measurable function with $w \geq 0$ μ-a.e. on Ω. If $f, g \in L^2_w\left(\Omega,\mu\right) := \left\{f : \Omega \to \mathbb{R},\ \int_\Omega w\left(y\right)f^2\left(y\right)d\mu\left(y\right) < \infty\right\}$ and there exists γ, Γ so that*

$$-\infty < \gamma \leq \frac{f}{g} \leq \Gamma < \infty \quad \text{for } \mu\text{-a.e. } x \in \Omega, \tag{4.3}$$

then one has the inequality:

$$\begin{aligned} 0 &\leq \int_\Omega w\left(x\right)f^2\left(x\right)d\mu\left(x\right)\int_\Omega w\left(x\right)g^2\left(x\right)d\mu\left(x\right) \\ &\quad - \left[\int_\Omega w\left(x\right)f\left(x\right)g\left(x\right)d\mu\left(x\right)\right]^2 \\ &\leq \frac{1}{2}\left(\Gamma-\gamma\right)\int_\Omega w\left(x\right)\left|g\left(x\right)\right|\left|\left(\int_\Omega w\left(y\right)g^2\left(y\right)d\mu\left(y\right)\right)f\left(x\right)\right. \\ &\quad \left. - g\left(x\right)\int_\Omega w\left(y\right)f\left(y\right)g\left(y\right)d\mu\left(y\right)\right|d\mu\left(x\right) \\ &= \frac{1}{2}\left(\Gamma-\gamma\right)\int_\Omega w\left(x\right)\left|g\left(x\right)\right|\left|\int_\Omega w\left(y\right)g\left(y\right)\begin{vmatrix} f\left(x\right) & g\left(x\right) \\ f\left(y\right) & g\left(y\right)\end{vmatrix}d\mu\left(y\right)\right|d\mu\left(x\right). \end{aligned} \tag{4.4}$$

The constant $\frac{1}{2}$ is sharp.

Remark 3. *In particular, if $f, g \in L^2\left(\Omega,\mu\right)$ and the condition (4.3) holds, then*

$$\begin{aligned} 0 &\leq \int_\Omega f^2\left(x\right)d\mu\left(x\right)\int_\Omega g^2\left(x\right)d\mu\left(x\right) - \left[\int_\Omega f\left(x\right)g\left(x\right)d\mu\left(x\right)\right]^2 \\ &\leq \frac{1}{2}\left(\Gamma-\gamma\right)\int_\Omega \left|g\left(x\right)\right|\left|\int_\Omega g\left(y\right)\begin{vmatrix} f\left(x\right) & g\left(x\right) \\ f\left(y\right) & g\left(y\right)\end{vmatrix}d\mu\left(y\right)\right|d\mu\left(x\right). \end{aligned} \tag{4.5}$$

The constant $\frac{1}{2}$ is sharp.

The following discrete inequality also holds.

Corollary 4. *Let $\bar{a} = \left(a_1,\dots,a_n\right)$, $\bar{b} = \left(b_1,\dots,b_n\right)$ and $\bar{w} = \left(w_1,\dots,w_n\right)$ be the sequences of real numbers so that $w_i \geq 0 \ \ (i = 1,\dots,n)$, $W_n := \sum_{i=1}^n w_i > 0$ and*

$$\gamma \leq \frac{a_i}{b_i} \leq \Gamma \quad \text{for each } i \in \left\{1,\dots,n\right\}. \tag{4.6}$$

Then one has the inequality

$$\begin{aligned}(4.7)\qquad 0 &\leq \sum_{i=1}^{n} w_i a_i^2 \sum_{i=1}^{n} w_i b_i^2 - \left(\sum_{i=1}^{n} w_i a_i b_i\right)^2 \\ &\leq \frac{1}{2}(\Gamma-\gamma)\sum_{i=1}^{n} w_i b_i \left|\sum_{j=1}^{n} w_j b_j \left|\begin{array}{cc} a_i & b_i \\ a_j & b_j \end{array}\right|\right|.\end{aligned}$$

The constant $\frac{1}{2}$ *is sharp.*

Remark 4. *If* $\bar{a},\bar{b}$ *satisfy (4.6), then one has the inequality*

$$\begin{aligned}(4.8)\qquad 0 &\leq \sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2 - \left(\sum_{i=1}^{n} a_i b_i\right)^2 \\ &\leq \frac{1}{2}(\Gamma-\gamma)\sum_{i=1}^{n} b_i \left|\sum_{j=1}^{n} b_j \left|\begin{array}{cc} a_i & b_i \\ a_j & b_j \end{array}\right|\right|.\end{aligned}$$

The constant $\frac{1}{2}$ *is sharp.*

5. A Converse for Jessen's Inequality

In [4], the author has proved the following converse of Jessen's inequality for normalized isotonic linear functionals.

Theorem 7. *Let* $\Phi : (\alpha,\beta) \subseteq \mathbb{R} \to \mathbb{R}$ *be a differentiable convex function on* (α,β), $f : E \to (\alpha,\beta)$ *so that* $\Phi\circ f,\ f,\ \Phi'\circ f,\ (\Phi'\circ f)\cdot f \in L$. *If* $A : L \to \mathbb{R}$ *is an isotonic linear and normalised functional, then*

$$\begin{aligned}(5.1)\qquad 0 &\leq A(\Phi\circ f) - \Phi(A(f)) \\ &\leq A[(\Phi'\circ f)\cdot f] - A(f)A(\Phi'\circ f) \\ &\leq \frac{1}{4}[\Phi'(\beta)-\Phi'(\alpha)](\beta-\alpha) \quad \textit{(if } \alpha,\beta \textit{ are finite).}\end{aligned}$$

We can state the following result improving the inequality (5.1).

Theorem 8. *Let* $\Phi : [\alpha,\beta] \to \mathbb{R}$ *with* $-\infty < \alpha < \beta < \infty$, *and* f, A *are as in Theorem 7, then one has the inequality*

$$\begin{aligned}(5.2)\qquad 0 &\leq A(\Phi\circ f) - \Phi(A(f)) \\ &\leq A[(\Phi'\circ f)\cdot f] - A(f)A(\Phi'\circ f) \\ &\leq \frac{1}{2}[\Phi'(\beta)-\Phi'(\alpha)]A(|f - A(f)\cdot\mathbf{1}|),\end{aligned}$$

provided $|f - A(f)\cdot\mathbf{1}| \in L$.

Proof. Taking into account that $\alpha \leq f \leq \beta$ and Φ' is monotonic on $[\alpha,\beta]$, we have $\Phi'(\alpha) \leq \Phi'\circ f \leq \Phi'(\beta)$. Applying Theorem 2, we deduce

$$\begin{aligned}&A[(\Phi'\circ f)\cdot f] - A(f)A(\Phi'\circ f) \\ \leq\ &\frac{1}{2}[\Phi'(\beta)-\Phi'(\alpha)]A(|f - A(f)\cdot\mathbf{1}|),\end{aligned}$$

and the theorem is proved. ∎

The following corollary addressing the integral case also holds.

Corollary 5. *Let $\Phi : [\alpha, \beta] \subset \mathbb{R} \to \mathbb{R}$ be a differentiable convex function on (α, β) and $f : \Omega \to [\alpha, \beta]$ so that $\Phi \circ f,\ f,\ \Phi' \circ f,\ (\Phi' \circ f) \cdot f \in L_w(\Omega, \mu)$, where $w \geq 0$ μ-a.e. on Ω with $\int_\Omega w(x)\, d\mu(x) > 0$. Then we have the inequality:*

$$
\begin{aligned}
(5.3)\qquad 0 &\leq \frac{1}{\int_\Omega w(x)\, d\mu(x)} \int_\Omega w(x)\, \Phi(f(x))\, d\mu(x) \\
&\quad - \Phi\left(\frac{1}{\int_\Omega w(x)\, d\mu(x)} \int_\Omega w(x)\, f(x)\, d\mu(x)\right) \\
&\leq \frac{1}{\int_\Omega w(x)\, d\mu(x)} \int_\Omega w(x)\, \Phi'(f(x))\, f(x)\, d\mu(x) \\
&\quad - \frac{1}{\int_\Omega w(x)\, d\mu(x)} \int_\Omega w(x)\, \Phi'(f(x))\, d\mu(x) \\
&\quad \times \frac{1}{\int_\Omega w(x)\, d\mu(x)} \int_\Omega w(x)\, f(x)\, d\mu(x) \\
&\leq \frac{1}{2}\left[\Phi'(\beta) - \Phi'(\alpha)\right] \frac{1}{\int_\Omega w(x)\, d\mu(x)} \\
&\quad \times \int_\Omega w(x) \left| f(x) - \frac{1}{\int_\Omega w(y)\, d\mu(y)} \int_\Omega w(y)\, f(y)\, d\mu(y) \right| d\mu(x).
\end{aligned}
$$

Remark 5. *If $\mu(\Omega) < \infty$ and $\Phi \circ f,\ f,\ \Phi' \circ f,\ (\Phi' \circ f) \cdot f \in L(\Omega, \mu)$, then we have the inequality:*

$$
\begin{aligned}
(5.4)\qquad 0 &\leq \frac{1}{\mu(\Omega)} \int_\Omega \Phi(f(x))\, d\mu(x) - \Phi\left(\frac{1}{\mu(\Omega)} \int_\Omega f(x)\, d\mu(x)\right) \\
&\leq \frac{1}{\mu(\Omega)} \int_\Omega \Phi'(f(x))\, f(x)\, d\mu(x) \\
&\quad - \frac{1}{\mu(\Omega)} \int_\Omega \Phi'(f(x))\, d\mu(x) \cdot \frac{1}{\mu(\Omega)} \int_\Omega f(x)\, d\mu(x) \\
&\leq \frac{1}{2}\left[\Phi'(\beta) - \Phi'(\alpha)\right] \frac{1}{\mu(\Omega)} \int_\Omega \left| f(x) - \frac{1}{\mu(\Omega)} \int_\Omega f(y)\, d\mu(y) \right| d\mu(x).
\end{aligned}
$$

The case of functions of a real variable is embodied in the following inequality that provides a counterpart for the Jensen's integral inequality

$$
\begin{aligned}
(5.5)\qquad 0 &\leq \frac{1}{b-a} \int_a^b \Phi(f(x))\, dx - \Phi\left(\frac{1}{b-a} \int_a^b f(x)\, dx\right) \\
&\leq \frac{1}{b-a} \int_a^b \Phi'(f(x))\, f(x)\, dx \\
&\quad - \frac{1}{b-a} \int_a^b \Phi'(f(x))\, dx \cdot \frac{1}{b-a} \int_a^b f(x)\, dx \\
&\leq \frac{1}{2}\left[\Phi'(\beta) - \Phi'(\alpha)\right] \frac{1}{b-a} \int_a^b \left| f(x) - \frac{1}{b-a} \int_a^b f(y)\, dy \right| dx.
\end{aligned}
$$

The following discrete inequality is valid as well.

Corollary 6. *Let $\Phi : [\alpha, \beta] \to \mathbb{R}$ be a differentiable convex function on (α, β). If $x_i \in [\alpha, \beta]$ and $w_i \geq 0$ $(i = 1, \ldots, n)$ with $W_n > 0$, then one has the counterpart of*

Jensen's discrete inequality:

$$0 \leq \frac{1}{W_n} \sum_{i=1}^{n} w_i \Phi(x_i) - \Phi\left(\frac{1}{W_n} \sum_{i=1}^{n} w_i x_i\right) \tag{5.6}$$
$$\leq \frac{1}{W_n} \sum_{i=1}^{n} w_i \Phi'(x_i) x_i - \frac{1}{W_n} \sum_{i=1}^{n} w_i \Phi'(x_i) \frac{1}{W_n} \sum_{i=1}^{n} w_i x_i$$
$$\leq \frac{1}{2} [\Phi'(\beta) - \Phi'(\alpha)] \frac{1}{W_n} \sum_{i=1}^{n} w_i \left| x_i - \frac{1}{W_n} \sum_{j=1}^{n} w_j x_j \right| .$$

Remark 6. *In particular, we get the discrete inequality:*

$$0 \leq \frac{1}{n} \sum_{i=1}^{n} \Phi(x_i) - \Phi\left(\frac{1}{n} \sum_{i=1}^{n} x_i\right) \tag{5.7}$$
$$\leq \frac{1}{n} \sum_{i=1}^{n} \Phi'(x_i) x_i - \frac{1}{n} \sum_{i=1}^{n} \Phi'(x_i) \frac{1}{n} \sum_{i=1}^{n} x_i$$
$$\leq \frac{1}{2} [\Phi'(\beta) - \Phi'(\alpha)] \frac{1}{n} \sum_{i=1}^{n} \left| x_i - \frac{1}{n} \sum_{j=1}^{n} x_j \right| .$$

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