SOME INEQUALITIES FOR THE FINITE HILBERT TRANSFORM OF A PRODUCT

S.S. DRAGOMIR

Abstract. Some inequalities for the Hilbert transform of the product of two functions are given.

1. Introduction

Let $\Omega = (-1, 1)$ where $1 \leq p < \infty$, the usual $\mathfrak{L}^p-$space with respect to the Lebesgue measure λ restricted to the open interval Ω will be denoted by $\mathfrak{L}^p(\Omega)$.

We define a linear operator T (see [24]) from the vector space $\mathfrak{L}^1(\Omega)$ into the vector space of all $\lambda-$measurable functions on Ω as follows. Let $f \in \mathfrak{L}^1(\Omega)$. The Cauchy principal value

$$\frac{1}{\pi} PV \int_{-1}^{1} \frac{f(\tau)}{\tau - t} d\tau = \lim_{\varepsilon \downarrow 0} \left[\int_{-1}^{t-\varepsilon} + \int_{t+\varepsilon}^{1} \right] \frac{f(\tau)}{\pi(\tau - t)} d\tau \tag{1.1}$$

exists for $\lambda-$almost every $t \in \Omega$.

We denote the left-hand side of (1.1) by $(Tf)(t)$ for each $t \in \Omega$ for which $(Tf)(t)$ exists. The so-defined function Tf, which we call the *finite Hilbert Transform* of f, is defined $\lambda-$almost everywhere on Ω and is $\lambda-$measurable; (see for example [1, Theorem 8.1.5]). The resulting linear operator T will be called the *finite Hilbert transform operator* or Cauchy kernel operator.

It is known that $\mathfrak{L}^1(\Omega)$ is not invariant under T, namely, $T\left(\mathfrak{L}^1(\Omega)\right) \not\subset \mathfrak{L}^1(\Omega)$ [17, Proof of Theorem 1 (b)].

The following basic results are well known and their proofs may be found in Propositions 8.1.9 and 8.2.1 of [1] respectively.

Theorem 1. *(M. Riesz) Let $1 < p < \infty$. Then $T(\mathfrak{L}^p(\Omega)) \subset \mathfrak{L}^p(\Omega)$ and the linear operator*

$$T_p : f \mapsto Tf,\ f \in \mathfrak{L}^p(\Omega)$$

on $\mathfrak{L}^p(\Omega)$ is continuous.

Theorem 2. *(Parseval) Let $1 < p < \infty$ and $q = \frac{p}{p-1}$. Then*

$$\int_{-1}^{1} (fTg + gTf)\, d\lambda = 0 \tag{1.2}$$

for every $f \in \mathfrak{L}^p(\Omega)$ and $g \in \mathfrak{L}^q(\Omega)$.

We introduce the following definition.

Date: May 29, 2001.

1991 *Mathematics Subject Classification.* Primary 26D15; Secondary 26D10.

Key words and phrases. Hilbert Transform, Analytic Inequalities.

Definition 1. *A function $f:\Omega\to\mathbb{C}$ is said to be $\alpha-$Hölder continuous* $(0<\alpha\leq 1)$ *in a subinterval Ω_0 of Ω if there exists a constant $c>0$, dependent upon Ω_0, such that*

$$|f(s)-f(t)|\leq c\,|s-t|^{\alpha},\quad s,t\in\Omega_0. \tag{1.3}$$

A function on Ω is said to be *locally $\alpha-$Hölder continuous* if it is $\alpha-$Hölder continuous in every compact subinterval of Ω. We denote by $H_{loc}^{\alpha}(\Omega)$ the space of all locally $\alpha-$Hölder continuous functions on Ω.

The class of Hölder continuous functions on Ω is independent because the finite Hilbert transform of such a function exists everywhere on Ω (see [15, Section 3.2] or [21, Lemma II.1.1]).

This is in contrast to the $\lambda-$almost everywhere existence of the finite Hilbert transform of functions in $\mathfrak{L}^1(\Omega)$.

There are continuous functions $f\in\mathfrak{L}^1(\Omega)$ such that $(Tf)(t)$ does not exist at some point $t\in\Omega$. An example is given by the function f defined by (see [24])

$$f(t)=\begin{cases}0 & \text{if}\quad -1<t\leq 0,\\[2ex] \frac{1}{\ln t-\ln 2} & \text{if}\quad 0<t<1.\end{cases}$$

It readily follows that $(Tf)(0)$ does not exist.

In paper [24] it is proved amongst others the following result.

Theorem 3. *(Okada-Elliot) The space $\mathfrak{L}^p(\Omega)\cap H_{loc}^{\alpha}(\Omega)$ is invariant under the finite Hilbert transform operator T and the restriction of T to that space is continuous whenever $1<p<\infty$. This, however, is not true when $p=1$.*

We consider the finite Hilbert transform on the open interval (a,b)

$$(Tf)(a,b;t):=\frac{1}{\pi}PV\int_a^b\frac{f(\tau)}{\tau-t}d\tau,\quad t\in(a,b).$$

The following theorem holds (see [11]).

Theorem 4. *Let $f:[a,b]\to\mathbb{R}$ be $\alpha-H-$Hölder continuous on (a,b), i.e.,*

$$|f(t)-f(s)|\leq H\,|t-s|^{\alpha}\quad\text{for all } t,s\in(a,b),\ \alpha\in(0,1],\ H>0. \tag{1.4}$$

Then we have the estimate

$$\begin{aligned}&\left|(Tf)(a,b;t)-\frac{f(t)}{\pi}\ln\left(\frac{b-t}{t-a}\right)\right| \\ &\leq\frac{H}{\alpha\pi}\left[(t-a)^{\alpha}+(b-t)^{\alpha}\right]\leq\frac{H2^{1-\alpha}}{\alpha\pi}(b-a)^{\alpha},\end{aligned} \tag{1.5}$$

for all $t\in(a,b)$.

The following result holds for monotonic functions (see [11]).

Theorem 5. *Let $f:[a,b]\to\mathbb{R}$ be a monotonic nondecreasing (nonincreasing) function on $[a,b]$. If the finite Hilbert transform $(Tf)(a,b,\cdot)$ exists in every $t\in(a,b)$, then*

$$(Tf)(a,b;t)\geq(\leq)\frac{1}{\pi}f(t)\ln\left(\frac{b-t}{t-a}\right) \tag{1.6}$$

for all $t\in(a,b)$.

Now, if we assume that the mapping $f:(a,b)\to\mathbb{R}$ is convex on (a,b), then it is locally Lipschitzian on (a,b) and then the finite Hilbert transform of f exists in every point $t\in(a,b)$.

The following result holds (see [11]).

Theorem 6. *Let $f:(a,b)\to\mathbb{R}$ be a convex mapping on (a,b). Then we have*

$$\begin{aligned}
(1.7)\qquad & \frac{1}{\pi}\left[l(t)(b-t)+\int_a^t l(s)\,ds+f(t)\ln\left(\frac{b-t}{t-a}\right)\right] \\
\le\ & (Tf)(a,b;t) \\
\le\ & \frac{1}{\pi}\left[f(t)\ln\left(\frac{b-t}{t-a}\right)+l(t)(t-a)+\int_t^b l(s)\,ds\right],
\end{aligned}$$

where $l(s)\in\left[f'_-(s),f'_+(s)\right]$, $s\in(a,b)$.

The following more practical result also holds [11]:

Corollary 1. *Let $f:(a,b)\to\mathbb{R}$ be a differentiable convex function on (a,b). Then we have the inequality*

$$\begin{aligned}
(1.8)\qquad & \frac{1}{\pi}\left[f(t)-f(a)+f'(t)(b-t)+f(t)\ln\left(\frac{b-t}{t-a}\right)\right] \\
\le\ & (Tf)(a,b;t) \\
\le\ & \frac{1}{\pi}\left[f(t)\ln\left(\frac{b-t}{t-a}\right)+f(b)-f(t)+f'(t)(t-a)\right]
\end{aligned}$$

for all $t\in(a,b)$.

In this paper we point out some inequalities for the finite Hilbert transform of the product of two functions.

For a comprehensive number of results on the numerical approximation of the Cauchy principal value integrals, see [2] – [10], [13] – [14], [16], [18] – [20], [22] – [23], [25] – [27].

2. The Results

The following lemma holds.

Lemma 1. *If f and g are locally Hölder continuous on $[a,b]$, then fg is also locally Hölder continuous on $[a,b]$ and:*

$$\begin{aligned}
(2.1)\qquad & T(fg)(a,b;t) \\
=\ & f(t)\,T(g)(a,b;t)+g(t)\,T(f)(a,b;t) \\
& -\frac{1}{\pi}f(t)\,g(t)\ln\left(\frac{b-t}{t-a}\right)+\frac{1}{\pi}PV\int_a^b\frac{(f(\tau)-f(t))(g(\tau)-g(t))}{\tau-t}d\tau
\end{aligned}$$

for any $t\in(a,b)$.

Proof. Assume that for a subinterval $[c,d]\subseteq[a,b]$, we have

$$(2.2)\qquad |f(s)-f(u)|\le L_1|s-u|^{r_1}\quad\text{for any}\quad s,u\in[c,d];$$

$$(2.3)\qquad |g(s)-g(u)|\le L_2|s-u|^{r_2}\quad\text{for any}\quad s,u\in[c,d].$$

Then

$$\begin{aligned}
|f(s)g(s)-f(u)g(u)| &= |f(s)g(s)-f(s)g(u)+f(s)g(u)-f(u)g(u)| \\
&\le |f(s)||g(s)-g(u)|+|g(u)||f(s)-f(u)| \\
&\le M_1L_1|s-u|^{r_1}+M_2L_2|s-u|^{r_2} \\
&\le |s-u|^r\left[M_1L_1|s-u|^{r_1-r}+M_2L_2|s-u|^{r_2-r}\right] \\
&\le |s-u|^r\left[M_1L_1|d-c|^{r_1-r}+M_2L_2|d-c|^{r_2-r}\right] \\
&= M|s-u|^r
\end{aligned}$$

where

$$M_1:=\sup_{s\in[c,d]}|f(s)|,\quad M_2:=\sup_{u\in[c,d]}|g(u)|,\quad r=\min(r_1,r_2),$$

and

$$M=M_1L_1|d-c|^{r_1-r}+M_2L_2|d-c|^{r_2-r},$$

proving that fg is locally Hölder continuous on $[a,b]$.

Now, for any $t,\tau\in[a,b]$, we may write that

$$(f(\tau)-f(t))(g(\tau)-g(t))=f(\tau)g(\tau)+f(t)g(t)-f(t)g(\tau)-f(\tau)g(t)$$

giving

$$\begin{aligned}
\frac{f(\tau)g(\tau)}{\tau-t} &= f(t)\cdot\frac{g(\tau)}{\tau-t}+g(t)\cdot\frac{f(\tau)}{\tau-t}-\frac{f(t)g(t)}{\tau-t} \\
&\quad+\frac{(f(\tau)-f(t))(g(\tau)-g(t))}{\tau-t}
\end{aligned}$$

for any $t,\tau\in[a,b]$, $t\neq\tau$.

Consequently,

$$\begin{aligned}
&T(fg)(a,b;t) \\
&= \frac{1}{\pi}PV\int_a^b\frac{f(\tau)g(\tau)}{\tau-t}d\tau \\
&= \frac{1}{\pi}f(t)PV\int_a^b\frac{g(\tau)}{\tau-t}d\tau+g(t)\frac{1}{\pi}PV\int_a^b\cdot\frac{f(\tau)}{\tau-t}d\tau \\
&\quad-\frac{1}{\pi}f(t)g(t)PV\int_a^b\frac{d\tau}{\tau-t}+\frac{1}{\pi}PV\int_a^b\frac{(f(\tau)-f(t))(g(\tau)-g(t))}{\tau-t}d\tau \\
&= f(t)T(g)(a,b;t)+g(t)T(f)(a,b;t) \\
&\quad-\frac{f(t)g(t)}{\pi}\ln\left(\frac{b-t}{t-a}\right)+\frac{1}{\pi}PV\int_a^b\frac{(f(\tau)-f(t))(g(\tau)-g(t))}{\tau-t}d\tau
\end{aligned}$$

for any $t\in(a,b)$, and the identity (2.1) is proved. ∎

Theorem 7. *Assume that f is of L_1-r_1-Hölder type and g is of L_2-r_2-Hölder type on $[a,b]$, where $L_1,L_2>0,\ r_1,r_2\in(0,1]$. Then we have the inequality:*

$$\begin{aligned}(2.4)\qquad &\Big|T(fg)(a,b;t)-f(t)T(g)(a,b;t)\\ &\qquad -g(t)T(f)(a,b;t)+\frac{1}{\pi}f(t)g(t)\ln\left(\frac{b-t}{t-a}\right)\Big|\\ &\leq \frac{L_1L_2}{\pi(r_1+r_2)}\left[(b-t)^{r_1+r_2}+(t-a)^{r_1+r_2}\right]\\ &\leq \frac{L_1L_2(b-a)^{r_1+r_2}}{\pi(r_1+r_2)}\end{aligned}$$

for any $t\in(a,b)$.

Proof. Taking the modulus in (2.1), we may write

$$\begin{aligned}&\Big|T(fg)(a,b;t)-f(t)T(g)(a,b;t)\\ &\qquad -g(t)T(f)(a,b;t)+\frac{1}{\pi}f(t)g(t)\ln\left(\frac{b-t}{t-a}\right)\Big|\\ \leq\ & \frac{1}{\pi}PV\int_a^b\left|\frac{(f(\tau)-f(t))(g(\tau)-g(t))}{\tau-t}\right|d\tau\leq\frac{1}{\pi}PV\int_a^b L_1L_2|\tau-t|^{r_1+r_2-1}d\tau\\ =\ & \frac{L_1L_2}{\pi}\left[\frac{(b-t)^{r_1+r_2}+(t-a)^{r_1+r_2}}{r_1+r_2}\right]\end{aligned}$$

and the first part of inequality (2.4) is proved. The second part is obvious. ∎

The best inequality we can get from (2.4) is embodied in the following corollary.

Corollary 2. *With the assumptions in Theorem 7, we have*

$$\begin{aligned}(2.5)\qquad &\Big|T(fg)\left(a,b;\frac{a+b}{2}\right)-f\left(\frac{a+b}{2}\right)T(g)\left(a,b;\frac{a+b}{2}\right)\\ &\qquad -g\left(\frac{a+b}{2}\right)T(f)\left(a,b;\frac{a+b}{2}\right)\Big|\\ &\leq \frac{L_1L_2(b-a)^{r_1+r_2}}{\pi(r_1+r_2)2^{r_1+r_2-1}}.\end{aligned}$$

The following corollary also holds.

Corollary 3. *If f and g are Lipschitzian with the constants K_1 and K_2, then we have the inequality*

$$(2.6)\qquad \left|T(fg)(a,b;t)-f(t)T(g)(a,b;t)-g(t)T(f)(a,b;t)+\frac{1}{\pi}f(t)g(t)\ln\left(\frac{b-t}{t-a}\right)\right|$$

$$(2.7)\qquad \leq\frac{K_1K_2}{\pi}\left[\frac{1}{4}(b-a)^2+\left(t-\frac{a+b}{2}\right)^2\right]\leq\frac{K_1K_2}{2\pi}(b-a)^2$$

for any $t \in (a,b)$. *In particular, for* $t = \frac{a+b}{2}$, *we have*

$$\left| T(fg)\left(a,b;\frac{a+b}{2}\right) - f\left(\frac{a+b}{2}\right) T(g)\left(a,b;\frac{a+b}{2}\right) - g\left(\frac{a+b}{2}\right) T(f)\left(a,b;\frac{a+b}{2}\right) \right| \tag{2.8}$$

$$\leq \frac{K_1 K_2}{4\pi} (b-a)^2 . \tag{2.9}$$

The following theorem also holds.

Theorem 8. *Assume that* f *and* g *are absolutely continuous on* $[a,b]$. *Then we have the inequality:*

$$\Big| T(fg)(a,b;t) - f(t) T(g)(a,b;t) \tag{2.10}$$

$$- g(t) T(f)(a,b;t) + \frac{1}{\pi} f(t) g(t) \ln\left(\frac{b-t}{t-a}\right) \Big|$$

$$\leq \frac{1}{\pi} \times \begin{cases} \left[\frac{1}{4}(b-a)^2 + \left(t - \frac{a+b}{2}\right)^2\right] \|f'\|_{[a,b],\infty} \|g'\|_{[a,b],\infty} \\ \qquad \text{if } f' \in L_\infty[a,b],\ g' \in L_\infty[a,b]; \\ \frac{\delta}{\delta+1}\left[(b-t)^{1+\frac{1}{\delta}} + (t-a)^{1+\frac{1}{\delta}}\right] \|f'\|_{[a,b],\infty} \|g'\|_{[a,b],\gamma} \\ \qquad \text{if } f' \in L_\infty[a,b],\ g' \in L_\gamma[a,b],\ \gamma > 1,\ \frac{1}{\gamma} + \frac{1}{\delta} = 1; \\ (b-a) \|f'\|_{[a,b],\infty} \|g'\|_{[a,b],1} \qquad \text{if } f' \in L_\infty[a,b],\ g' \in L_1[a,b]; \\ \frac{\beta}{\beta+1}\left[(b-t)^{1+\frac{1}{\beta}} + (t-a)^{1+\frac{1}{\beta}}\right] \|f'\|_{[a,b],\alpha} \|g'\|_{[a,b],\infty} \\ \qquad \text{if } f' \in L_\alpha[a,b],\ \alpha > 1,\ \frac{1}{\alpha} + \frac{1}{\beta} = 1, \text{ and } g' \in L_\infty[a,b]; \\ \frac{\beta\delta}{\beta+\delta}\left[(b-t)^{\frac{\delta+\beta}{\beta+\delta}} + (t-a)^{\frac{\delta+\beta}{\beta+\delta}}\right] \|f'\|_{[a,b],\alpha} \|g'\|_{[a,b],\gamma} \\ \text{if } f' \in L_\alpha[a,b],\ \alpha > 1,\ \frac{1}{\alpha} + \frac{1}{\beta} = 1, \text{ and } g' \in L_\gamma[a,b],\ \gamma > 1,\ \frac{1}{\gamma} + \frac{1}{\delta} = 1; \\ \beta\left[(b-t)^{\frac{1}{\beta}} + (t-a)^{\frac{1}{\beta}}\right] \|f'\|_{[a,b],\alpha} \|g'\|_{[a,b],1} \\ \qquad \text{if } f' \in L_\alpha[a,b],\ \alpha > 1,\ \frac{1}{\alpha} + \frac{1}{\beta} = 1, \text{ and } g' \in L_1[a,b]; \\ (b-a) \|f'\|_{[a,b],1} \|g'\|_{[a,b],\infty} \qquad \text{if } f' \in L_1[a,b],\ g' \in L_\infty[a,b]; \\ \delta\left[(b-t)^{1+\frac{1}{\delta}} + (t-a)^{1+\frac{1}{\delta}}\right] \|f'\|_{[a,b],1} \|g'\|_{[a,b],\gamma} \\ \qquad \text{if } f' \in L_1[a,b],\ g' \in L_\gamma[a,b],\ \gamma > 1,\ \frac{1}{\gamma} + \frac{1}{\delta} = 1; \end{cases} \tag{2.11}$$

Proof. Since f and g are absolutely continuous on $[a,b]$, we may write that

$$f(\tau) - f(t) = \int_t^\tau f'(u)\,du \quad \text{and} \quad g(\tau) - g(t) = \int_t^\tau g'(u)\,du$$

which implies:

$$(2.12)\qquad |f(\tau)-f(t)| \le \begin{cases} \|f'\|_{[\tau,t],\infty}|\tau-t| & \text{if } f'\in L_\infty[a,b]; \\ \\ \|f'\|_{[\tau,t],\alpha}|\tau-t|^{\frac{1}{\beta}} & \text{if } f'\in L_\alpha[a,b], \\ & \alpha>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1, \\ \|f'\|_{[\tau,t],1} \end{cases}$$

and

$$(2.13)\qquad |g(\tau)-g(t)| \le \begin{cases} \|g'\|_{[\tau,t],\infty}|\tau-t| & \text{if } g'\in L_\infty[a,b]; \\ \\ \|g'\|_{[\tau,t],\gamma}|\tau-t|^{\frac{1}{\delta}} & \text{if } g'\in L_\gamma[a,b], \\ & \gamma>1,\ \frac{1}{\gamma}+\frac{1}{\delta}=1; \\ \|g'\|_{[\tau,t],1}. \end{cases}$$

Using the identity (2.2), we get

$$(2.14)\qquad \Big| T(fg)(a,b;t) - f(t)T(g)(a,b;t) - g(t)T(f)(a,b;t) + \frac{1}{\pi}f(t)g(t)\ln\left(\frac{b-t}{t-a}\right)\Big| \le \frac{1}{\pi}PV\int_a^b \left|\frac{(f(\tau)-f(t))(g(\tau)-g(t))}{\tau-t}\right| d\tau =: I.$$

Then we have, by using (2.12) or (2.13), that

$$I \leq \frac{1}{\pi} \times \begin{cases} PV\int_a^b \|f'\|_{[\tau,t],\infty} \|g'\|_{[\tau,t],\infty} |\tau - t| \, d\tau \\ PV\int_a^b \|f'\|_{[\tau,t],\infty} \|g'\|_{[\tau,t],\alpha} |\tau - t|^{\frac{1}{\delta}} \, d\tau \\ PV\int_a^b \|f'\|_{[\tau,t],\infty} \|g'\|_{[\tau,t],1} \, d\tau \\ PV\int_a^b \|f'\|_{[\tau,t],\alpha} \|g'\|_{[\tau,t],\infty} |\tau - t|^{\frac{1}{\beta}} \, d\tau \\ PV\int_a^b \|f'\|_{[\tau,t],\alpha} \|g'\|_{[\tau,t],\gamma} |\tau - t|^{\frac{1}{\beta}+\frac{1}{\delta}-1} \, d\tau \\ PV\int_a^b \|f'\|_{[\tau,t],\alpha} \|g'\|_{[\tau,t],1} |\tau - t|^{\frac{1}{\beta}-1} \, d\tau \\ PV\int_a^b \|f'\|_{[\tau,t],1} \|g'\|_{[\tau,t],\infty} \, d\tau \\ PV\int_a^b \|f'\|_{[\tau,t],1} \|g'\|_{[\tau,t],\gamma} |\tau - t|^{\frac{1}{\delta}-1} \, d\tau \\ PV\int_a^b \|f'\|_{[\tau,t],1} \|g'\|_{[\tau,t],1} |\tau - t|^{-1} \, d\tau. \end{cases} \tag{2.15}$$

However,

$$\begin{aligned} & PV\int_a^b \|f'\|_{[\tau,t],\infty} \|g'\|_{[\tau,t],\infty} |\tau - t| \, d\tau \\ \leq \; & \|f'\|_{[a,b],\infty} \|g'\|_{[a,b],\infty} \left[\frac{(b-t)^2 + (t-a)^2}{2} \right] \\ = \; & \|f'\|_{[a,b],\infty} \|g'\|_{[a,b],\infty} \left[\frac{1}{4}(b-a)^2 + \left(t - \frac{a+b}{2} \right)^2 \right], \end{aligned}$$

$$\begin{aligned} & PV\int_a^b \|f'\|_{[\tau,t],\infty} \|g'\|_{[\tau,t],\alpha} |\tau - t|^{\frac{1}{\delta}} \, d\tau \\ \leq \; & \|f'\|_{[a,b],\infty} \|g'\|_{[a,b],\gamma} \left[\frac{(b-t)^{1+\frac{1}{\delta}} + (t-a)^{1+\frac{1}{\delta}}}{\frac{1}{\delta}+1} \right] \\ = \; & \frac{\delta}{\delta+1} \|f'\|_{[a,b],\infty} \|g'\|_{[a,b],\gamma} \left[(b-t)^{1+\frac{1}{\delta}} + (t-a)^{1+\frac{1}{\delta}} \right], \end{aligned}$$

$$PV\int_a^b \|f'\|_{[\tau,t],\infty}\|g'\|_{[\tau,t],1}\,d\tau \le \|f'\|_{[a,b],\infty}\|g'\|_{[a,b],1}(b-a),$$

$$\begin{aligned}&PV\int_a^b \|f'\|_{[\tau,t],\alpha}\|g'\|_{[\tau,t],\infty}|\tau-t|^{\frac{1}{\beta}}\,d\tau\\ \le\;& \|f'\|_{[a,b],\alpha}\|g'\|_{[a,b],\infty}\cdot\frac{\beta}{\beta+1}\left[(b-t)^{\frac{1}{\beta}+1}+(t-a)^{\frac{1}{\beta}+1}\right],\end{aligned}$$

$$\begin{aligned}&PV\int_a^b \|f'\|_{[\tau,t],\alpha}\|g'\|_{[\tau,t],\gamma}|\tau-t|^{\frac{1}{\beta}+\frac{1}{\delta}-1}\,d\tau\\ \le\;& \|f'\|_{[a,b],\alpha}\|g'\|_{[a,b],\gamma}\frac{1}{\frac{1}{\beta}+\frac{1}{\delta}}\left[(b-t)^{\frac{1}{\beta}+\frac{1}{\delta}}+(t-a)^{\frac{1}{\beta}+\frac{1}{\delta}}\right]\\ =\;& \frac{\beta\delta}{\beta+\delta}\|f'\|_{[a,b],\alpha}\|g'\|_{[a,b],\gamma}\left[(b-t)^{\frac{\delta+\beta}{\beta+\delta}}+(t-a)^{\frac{\delta+\beta}{\beta+\delta}}\right],\end{aligned}$$

$$\begin{aligned}&PV\int_a^b \|f'\|_{[\tau,t],\alpha}\|g'\|_{[\tau,t],1}|\tau-t|^{\frac{1}{\beta}-1}\,d\tau\\ \le\;& \|f'\|_{[a,b],\alpha}\|g'\|_{[a,b],1}\,\beta\left[(b-t)^{\frac{1}{\beta}}+(t-a)^{\frac{1}{\beta}}\right],\end{aligned}$$

$$PV\int_a^b \|f'\|_{[\tau,t],1}\|g'\|_{[\tau,t],\infty}\,d\tau \le (b-a)\|f'\|_{[a,b],1}\|g'\|_{[a,b],\infty}$$

and

$$\begin{aligned}&PV\int_a^b \|f'\|_{[\tau,t],1}\|g'\|_{[\tau,t],\gamma}|\tau-t|^{\frac{1}{\delta}-1}\,d\tau\\ \le\;& \|f'\|_{[a,b],1}\|g'\|_{[a,b],\gamma}\,\delta\left[(b-t)^{1+\frac{1}{\delta}}+(t-a)^{1+\frac{1}{\delta}}\right].\end{aligned}$$

For the last inequality we cannot point out a bound as above.

Using (2.14) and (2.15), we deduce the desired inequality (2.10). ∎

The following lemma also holds.

Lemma 2. *Let $f:[a,b]\to\mathbb{R}$ be locally Hölder continuous on $[a,b]$ and $g:[a,b]\to\mathbb{R}$ so that g' is absolutely continuous on $[a,b]$. Then we have the identity:*

$$
\begin{aligned}
(2.16)\qquad & T(fg)(a,b;t) \\
= {} & f(t)\,T(g)(a,b;t)+g(t)\,T(f)(a,b;t)-\frac{1}{\pi}f(t)\,g(t)\ln\left(\frac{b-t}{t-a}\right) \\
& +\frac{1}{\pi}\left[\int_a^b f(\tau)\,d\tau-(b-a)\,f(t)\right]g'(t) \\
& -\frac{1}{\pi}PV\int_a^b \frac{f(\tau)-f(t)}{\tau-t}\left(\int_t^\tau (u-\tau)\,g''(u)\,du\right)d\tau
\end{aligned}
$$

for any $t\in(a,b)$.

Proof. We use the following identity:

$$\int_\alpha^\beta \varphi(u)\,du=\varphi(\alpha)(\beta-\alpha)-\int_\alpha^\beta (u-\beta)\,\varphi'(u)\,du$$

which holds for any absolutely continuous function $\varphi:[\alpha,\beta]\to\mathbb{R}$.

Then we have

$$
\begin{aligned}
& \frac{1}{\pi}PV\int_a^b [f(\tau)-f(t)]\left[\frac{1}{\tau-t}(g(\tau)-g(t))\right]d\tau \\
= {} & \frac{1}{\pi}PV\int_a^b [f(\tau)-f(t)]\left[\frac{1}{\tau-t}\int_t^\tau g'(u)\,du\right]d\tau \\
= {} & \frac{1}{\pi}PV\int_a^b [f(\tau)-f(t)]\left[g'(t)-\frac{1}{\tau-t}\int_t^\tau (u-\tau)\,g''(u)\,du\right]d\tau \\
= {} & \frac{1}{\pi}\left[g'(t)\int_a^b f(\tau)\,d\tau-(b-a)\,f(t)\,g'(t)\right. \\
& \left.-PV\int_a^b \frac{f(\tau)-f(t)}{\tau-t}\left(\int_t^\tau (u-\tau)\,g''(u)\,du\right)d\tau\right] \\
= {} & \frac{1}{\pi}g'(t)\int_a^b f(\tau)\,d\tau-\frac{1}{\pi}(b-a)\,f(t)\,g'(t) \\
& -\frac{1}{\pi}PV\int_a^b \frac{f(\tau)-f(t)}{\tau-t}\left(\int_t^\tau (u-\tau)\,g''(u)\,du\right)d\tau.
\end{aligned}
$$

Using (2.1), we deduce (2.16). ∎

The following theorem holds.

Theorem 9. *Assume that $f:[a,b]\to\mathbb{R}$ is of $H-r-$Hölder type and $g:[a,b]\to\mathbb{R}$ is such that g' is absolutely continuous on $[a,b]$. Then we have the inequality:*

$$\begin{aligned}(2.17)\qquad &\Bigg|T(fg)(a,b;t)-f(t)T(g)(a,b;t)-g(t)T(f)(a,b;t)\\&+\frac{1}{\pi}f(t)g(t)\ln\left(\frac{b-t}{t-a}\right)-\frac{1}{\pi}\left[\int_a^b f(\tau)\,d\tau-(b-a)f(t)\right]g'(t)\Bigg|\end{aligned}$$

$$\leq\frac{H}{\pi}\begin{cases}\frac{1}{2(r+2)}\left[(b-t)^{r+2}+(t-a)^{r+2}\right]\|g''\|_{[a,b],\infty} & \text{if } g''\in L_\infty[a,b];\\ \frac{q}{(rq+q+1)(q+1)^{\frac{1}{q}}}\left[(b-t)^{r+\frac{1}{q}+1}+(t-a)^{r+\frac{1}{q}+1}\right]\|g''\|_{[a,b],p} & \text{if } g''\in L_p[a,b],\\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1;\\ \frac{1}{(r+1)}\left[(b-t)^{r+1}+(t-a)^{r+1}\right]\|g''\|_{[a,b],1}.\end{cases}$$

Proof. Using the identity (2.16), we deduce that the left side in (2.17) is upper bounded by

$$\begin{aligned}I\ &:=\frac{1}{\pi}PV\int_a^b|f(\tau)-f(t)|\frac{1}{|\tau-t|}\left|\int_t^\tau(u-\tau)g''(u)\,du\right|d\tau\\&\leq\frac{H}{\pi}PV\int_a^b|\tau-t|^{r-1}\left|\int_t^\tau(u-\tau)g''(u)\,du\right|d\tau=:J.\end{aligned}$$

We observe that

$$\left|\int_t^\tau(u-\tau)g''(u)\,du\right|\leq\|g''\|_{[t,\tau],\infty}\frac{(\tau-t)^2}{2}$$

if $g''\in L_\infty[a,b]$,

$$\left|\int_t^\tau(u-\tau)g''(u)\,du\right|\leq\|g''\|_{[t,\tau],p}\left|\int_t^\tau|t-\tau|^q\,d\tau\right|^{\frac{1}{q}}=\|g''\|_{[t,\tau],p}\frac{|t-\tau|^{\frac{q+1}{q}}}{(q+1)^{\frac{1}{q}}}$$

if $g''\in L_p[a,b]$ and, finally,

$$\left|\int_t^\tau(u-\tau)g''(u)\,du\right|\leq|t-\tau|\,\|g''\|_{[t,\tau],1}.$$

Consequently, we have

$$J \leq \frac{H}{\pi}\times\begin{cases} PV\displaystyle\int_a^b |\tau-t|^{r-1}\cdot\frac{(\tau-t)^2}{2}\cdot\|g''\|_{[t,\tau],\infty}\,d\tau \\ PV\displaystyle\int_a^b \frac{|\tau-t|^{r-1}\cdot|t-\tau|^{1+\frac{1}{q}}}{(q+1)^{\frac{1}{q}}}\|g''\|_{[t,\tau],p}\,d\tau \\ PV\displaystyle\int_a^b |\tau-t|^{r-1}\cdot|t-\tau|\,\|g''\|_{[t,\tau],1}\,d\tau \end{cases}$$

$$\leq \frac{H}{\pi}\times\begin{cases} \dfrac{1}{2}\|g''\|_{[a,b],\infty}\left[\dfrac{(b-t)^{r+2}+(t-a)^{r+2}}{r+2}\right] \\ \dfrac{1}{(q+1)^{\frac{1}{q}}}\|g''\|_{[a,b],p}\left[\dfrac{(b-t)^{r+\frac{1}{q}+1}+(t-a)^{r+\frac{1}{q}+1}}{r+\frac{1}{q}+1}\right] \\ \|g''\|_{[a,b],1}\cdot\left[\dfrac{(b-t)^{r+1}+(t-a)^{r+1}}{r+1}\right], \end{cases}$$

which proves the inequality (2.17). ∎

The following lemma also holds.

Lemma 3. *Assume that f and g are as in Lemma 2. Then we have the identity:*

$$\begin{aligned} &T(fg)(a,b;t) \\ =\ & f(t)\,T(g)(a,b;t)+g(t)\,T(f)(a,b;t)-\frac{1}{\pi}f(t)\,g(t)\ln\left(\frac{b-t}{t-a}\right) \\ &+\frac{1}{\pi}\left[\int_a^b f(\tau)\,g'(\tau)\,d\tau-[g(b)-g(a)]\,f(t)\right] \\ &-\frac{1}{\pi}PV\int_a^b\frac{f(\tau)-f(t)}{\tau-t}\left(\int_t^\tau(u-t)\,g''(u)\,du\right)d\tau \end{aligned}\tag{2.18}$$

for any $t\in(a,b)$.

Proof. In this case, we use the following identity:

$$\int_\alpha^\beta\varphi(u)\,du=\varphi(\beta)(\beta-\alpha)-\int_\alpha^\beta(u-\alpha)\,\varphi'(u)\,du$$

which holds for any absolutely continuous function $\varphi:[\alpha,\beta]\to\mathbb{R}$.

Then, as above, we have

$$\begin{aligned}
&\frac{1}{\pi}PV\int_a^b [f(\tau)-f(t)]\left[\frac{1}{\tau-t}(g(\tau)-g(t))\right]d\tau \\
&= \frac{1}{\pi}PV\int_a^b [f(\tau)-f(t)]\left[\frac{1}{\tau-t}\int_t^\tau g'(u)\,du\right]d\tau \\
&= \frac{1}{\pi}PV\int_a^b [f(\tau)-f(t)]\left[g'(t)-\frac{1}{\tau-t}\int_t^\tau (u-\tau)g''(u)\,du\right]d\tau \\
&= \frac{1}{\pi}\left[\int_a^b f(\tau)g'(\tau)\,d\tau - f(t)\,PV\int_a^b g'(\tau)\,d\tau\right. \\
&\quad \left.-PV\int_a^b \frac{f(\tau)-f(t)}{\tau-t}\left(\int_t^\tau (u-t)g''(u)\,du\right)d\tau\right] \\
&= \frac{1}{\pi}\left[\int_a^b f(\tau)g'(\tau)\,d\tau - [g(b)-g(a)]f(t)\right. \\
&\quad \left.-PV\int_a^b \frac{f(\tau)-f(t)}{\tau-t}\left(\int_t^\tau (u-t)g''(u)\,du\right)d\tau\right],
\end{aligned}$$

proving the identity (2.18). ∎

The following result also holds.

Theorem 10. *With the assumptions in Theorem 9, we have:*

$$\begin{aligned}
(2.19)\qquad &\Bigg| T(fg)(a,b;t) - f(t)T(g)(a,b;t) - g(t)T(f)(a,b;t) \\
&+\frac{1}{\pi}f(t)g(t)\ln\left(\frac{b-t}{t-a}\right) - \frac{1}{\pi}\left[\int_a^b f(\tau)g'(\tau)\,d\tau - [g(b)-g(a)]f(t)\right]\Bigg| \\
&\leq \frac{H}{\pi}\begin{cases}
\frac{1}{2(r+2)}\left[(b-t)^{r+2}+(t-a)^{r+2}\right]\|g''\|_{[a,b],\infty} & \text{if } g''\in L_\infty[a,b]; \\[2ex]
\frac{q}{(rq+q+1)(q+1)^{\frac{1}{q}}}\left[(b-t)^{r+\frac{1}{q}+1}+(t-a)^{r+\frac{1}{q}+1}\right]\|g''\|_{[a,b],p} & \text{if } g''\in L_p[a,b], \\
& p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\[2ex]
\frac{1}{(r+1)}\left[(b-t)^{r+1}+(t-a)^{r+1}\right]\|g''\|_{[a,b],1}. &
\end{cases}
\end{aligned}$$

Proof. The proof follows in a similar manner to the one in Theorem 9 by the use of Lemma 3. We omit the details. ∎

References

[1] P.L. BUTZER and R.J. NESSEL, *Fourier Analysis and Approximation Theory*, Vol. 1, Birkhaüser Verlag, Basel, 1977.

[2] M.M. CHAWLA and N. JAYARAJAN, Quadrature formulas for Cauchy principal value integrals, *Computing*, **15** (1975), 347-355.

[3] G. CRISCUOLO and G. MASTROIANNI, On the convergence of Gauss quadrature rules for the Cauchy principal value integrals, *Ricerche Mat.,* **35** (1986), 45-60.

[4] G. CRISCUOLO and G. MASTROIANNI, On the uniform convergence of Gaussian quadrature rules for Cauchy principal value integrals, *Numer. Math.,* **54** (1989), 445-461.

[5] P.J. DAVIS and P. RABINOWITZ, *Methods of Numerical Integration,* (Academic Press, Orlando, 2nd Ed., 1984).

[6] K. DIETHELM, Error estimates for a quadrature rule for Cauchy principal value integrals, *Proc. Sympos. Appl. Math.,* **48** (1994), 287-291.

[7] K. DIETHELM, Modified compound quadrature rules for strongly singular integrals, *Computing*, **52** (1994), 337-354.

[8] K. DIETHELM, Peano kernels and bounds for the error constants of Gaussian and related quadrature rules for Cauchy principal value integrals, *Numer. Math.,* to appear.

[9] K. DIETHELM, Uniform convergence to optimal order quadrature rules for Cauchy principal value integrals, *J. Comput. Appl. Math.,* **56**(3) (1994), 321-329.

[10] K. DIETHELM, Asymptotically sharp error bounds for a quadrature rule for Cauchy principal value integrals based on a piecewise linear interpolation, *Approx. Theory Appl. (N.S.),* **11**(4) (1995), 78-89.

[11] N.M. DRAGOMIR, S.S. DRAGOMIR and P. FARRELL, Some inequalities for the finite Hilbert transform, *Theory of Inequalities and Applications,* Y.J. Cho, S.S. Dragomir and S.S. Kim (Ed.), Nova Science, New York, 2001.

[12] S.S. DRAGOMIR, The application of Bullen's inequality for convex functions to the finite Hilbert transform, *in preparation.*

[13] S. EHRICH, A note on Peano constants of Gauss-Konrod quadrature schemes, *J. Comput. Appl. Math.,* **66** (1996), 177-183.

[14] D. ELLIOTT and D.F. PAGET, Gauss type quadrature rules for Cauchy principal value integrals, *Math. Comp.,* **33** (1979), 301-309.

[15] F.D. GAKHOV, *Boundary Value Problems* (English translation), Pergamon Press, Oxford, 1966.

[16] D.B. HUNTER, Some Gauss type formulae for the evaluation of Cauchy principal value integrals, *Numer. Math.,* **19** (1972), 419-424.

[17] H. KOBER, A note on Hilbert's operator, *Bull. Amer. Math. Soc.,* **48** (1942), 421-427.

[18] P. KÖHLER, Error estimates for generalized compound quadrature formulas, *IMA J. Numer. Anal.,* **13** (1993), 477-491.

[19] P. KÖHLER, Asymptotically sharp error estimates for modified compound quadrature formulae for Cauchy principal value integrals, *Computing,* **55** (1995), 225-269.

[20] P. KÖHLER, On the error of quadrature formulae for Cauchy principal value integrals based on piecewise interpolation, *Approx. Theory Appl. (N.S.),* **13**(3) (1997), 58-69.

[21] S.G. MIKHLIN and S. PRÖSSDORF, *Singular Integral Operators* (English translation), Springer Verlag, Berlin, 1986.

[22] G. MONEGATO, The numerical evaluation of one-dimensional Cauchy principal value integrals, *Computing,* **29** (1982).

[23] B. NOBLE and S. BEIGHTON, Error estimates for three methods of evaluating Cauchy principal value integrals, *J. Inst. Maths. Applics.,* **26** (1980), 341-446.

[24] S. OKADA and D. ELLIOT, Hölder continuous functions and the finite Hilbert transform, *Mathematische Nach.,* **163** (1994), 250-272.

[25] P. RABINOWITZ, Gauss-Konrod integration rules for Cauchy principal value integrals, *Math. Comp.,* **41** (1983), 63-78; Corrigendum: *Math. Comp.,* **45** (1985), 277.

[26] P. RABINOWITZ, On the convergence of Hunter's method for Cauchy principal value integrals, In *Numerical Solution of Singular Integral Equations,* A. Gerasoulis and P. Vichnevetsky, Eds. IMACS*,* 1984, 86-88.

[27] H.W. STOLLE and R. STRAUß, On the numerical integration of certain singular integrals, *Computing,* **48** (1992), 177-189.

School of Communications and Informatics, Victoria University of Technology, PO Box 14428, Melbourne City MC, 8001, Victoria, Australia.
E-mail address: sever@matilda.vu.edu.au
URL: http://rgmia.vu.edu.au/SSDragomirWeb.html