SOME NEW INEQUALITIES OF OSTROWSKI TYPE

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Abstract. By the use of the Cauchy mean value theorem, some new inequalities of Ostrowski type are given.

1. Introduction

The following result is known in the literature as Ostrowski's inequality [1].

Theorem 1. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) with the property that $|f'(t)|\le M$ for all $t\in(a,b)$. Then*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\le\left[\frac{1}{4}+\frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right](b-a)M \tag{1.1}$$

for all $x\in[a,b]$.
The constant $\frac{1}{4}$ is the best possible in the sense that it cannot be replaced by a smaller constant.

A simple proof of this fact can be done by using the identity:

$$f(x)=\frac{1}{b-a}\int_a^b f(t)\,dt+\frac{1}{b-a}\int_a^b p(x,t)\,f'(t)\,dt,\quad x\in[a,b], \tag{1.2}$$

where

$$p(x,t):=\begin{cases} t-a & \text{if} \quad a\le t\le x\\ t-b & \text{if} \quad x<t\le b\end{cases}$$

which also holds for absolutely continuous functions $f:[a,b]\to\mathbb{R}$.

The following Ostrowski type result for absolutely continuous functions holds (see [2], [3] and [4]).

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Theorem 2. *Let $f : [a,b] \to \mathbb{R}$ be absolutely continuous on $[a,b]$. Then, for all $x \in [a,b]$, we have:*

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \tag{1.3}$$
$$\leq \begin{cases} \left[\frac{1}{4} + \left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right](b-a)\|f'\|_\infty & \text{if } f' \in L_\infty[a,b]; \\ \\ \frac{1}{(p+1)^{\frac{1}{p}}}\left[\left(\frac{x-a}{b-a}\right)^{p+1} + \left(\frac{b-x}{b-a}\right)^{p+1}\right]^{\frac{1}{p}}(b-a)^{\frac{1}{p}}\|f'\|_q & \text{if } f' \in L_q[a,b], \\ & \frac{1}{p}+\frac{1}{q}=1,\ p>1; \\ \left[\frac{1}{2} + \left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right]\|f'\|_1; \end{cases}$$

where $\|\cdot\|_r$ $(r \in [1,\infty])$ are the usual Lebesgue norms on $L_r[a,b]$, i.e.,

$$\|g\|_\infty := ess \sup_{t\in[a,b]} |g(t)|$$

and

$$\|g\|_r := \left(\int_a^b |g(t)|^r\,dt\right)^{\frac{1}{r}},\ r \in [1,\infty).$$

The constants $\frac{1}{4}$, $\frac{1}{(p+1)^{\frac{1}{p}}}$ and $\frac{1}{2}$ respectively are sharp in the sense presented in Theorem 1.

The above inequalities can also be obtained from the Fink result in [5] on choosing $n = 1$ and performing some appropriate computations.

If one drops the condition of absolute continuity and assumes that f is Hölder continuous, then one may state the result (see [6]):

Theorem 3. *Let $f : [a,b] \to \mathbb{R}$ be of $r-H-$Hölder type, i.e.,*

$$|f(x) - f(y)| \leq H|x-y|^r, \text{ for all } x,y \in [a,b], \tag{1.4}$$

where $r \in (0,1]$ and $H > 0$ are fixed. Then for all $x \in [a,b]$ we have the inequality:

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \tag{1.5}$$
$$\leq \frac{H}{r+1}\left[\left(\frac{b-x}{b-a}\right)^{r+1} + \left(\frac{x-a}{b-a}\right)^{r+1}\right](b-a)^r.$$

The constant $\frac{1}{r+1}$ is also sharp in the above sense.

Note that if $r = 1$, i.e., f is Lipschitz continuous, then we get the following version of Ostrowski's inequality for Lipschitzian functions (with L instead of H) (see [7])

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq \left[\frac{1}{4} + \left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right](b-a)L. \tag{1.6}$$

Here the constant $\frac{1}{4}$ is also best.

Moreover, if one drops the condition of the continuity of the function, and assumes that it is of bounded variation, then the following result may be stated (see [8]).

Theorem 4. *Assume that* $f:[a,b]\to\mathbb{R}$ *is of bounded variation and denote by* $\bigvee_a^b$ *its total variation. Then*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\le\left[\frac{1}{2}+\left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right]\bigvee_a^b(f) \tag{1.7}$$

for all $x\in[a,b]$.
The constant $\frac{1}{2}$ *is the best possible.*

If we assume more about f, i.e., f is monotonically increasing, then the inequality (1.7) may be improved in the following manner [9] (see also [10]).

Theorem 5. *Let* $f:[a,b]\to\mathbb{R}$ *be monotonic nondecreasing. Then for all* $x\in[a,b]$, *we have the inequality:*

$$\begin{aligned} &\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \\ \le\;& \frac{1}{b-a}\left\{[2x-(a+b)]\,f(x)+\int_a^b sgn(t-x)\,f(t)\,dt\right\} \\ \le\;& \frac{1}{b-a}\left\{(x-a)[f(x)-f(a)]+(b-x)[f(b)-f(x)]\right\} \\ \le\;& \left[\frac{1}{2}+\left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right][f(b)-f(a)]. \end{aligned} \tag{1.8}$$

All the inequalities in (1.8) are sharp and the constant $\frac{1}{2}$ *is the best possible.*

In this paper we point out different Ostrowski type inequalities assuming some special properties for the derivative of the function f around a given point $x\in(a,b)$.

2. The Results

The following theorem holds.

Theorem 6. *Let* $f:[a,b]\to\mathbb{R}$ *be continuous on* $[a,b]$ *and differentiable on* (a,b). *Let* $p\in(0,\infty)$ *and assume, for a given* $x\in(a,b)$, *we have that*

$$M_p(x):=\sup_{u\in(a,b)}\left\{|x-u|^{1-p}\,|f'(u)|\right\}<\infty. \tag{2.1}$$

Then we have the inequality

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\le\frac{1}{p(p+1)(b-a)}\left[(x-a)^{p+1}+(b-x)^{p+1}\right]M_p(x). \tag{2.2}$$

Proof. Let $x\in(a,b)$ and define the mapping $g_{1,x}:(a,x)\to\mathbb{R}$, $g_{1,x}(t)=(x-t)^p$.

Applying the Cauchy mean value theorem, for any $t\in(a,x)$ there exists a $\eta\in(t,x)$ such that

$$[f(t)-f(x)]\,g_{1,x}'(\eta)=[g_{1,x}(t)-g_{1,x}(x)]\,f'(\eta)$$

i.e.,

$$(-p)\left(f(t)-f(x)\right)(x-\eta)^{p-1}=(x-t)^{p}f'(\eta)$$

from where we obtain

$$|f(t)-f(x)|=\frac{(x-t)^{p}|f'(\eta)|}{p(x-\eta)^{p-1}}\leq\frac{(x-t)^{p}}{p}M_{p}(x),\ \ t\in(a,x). \tag{2.3}$$

We define the mapping $g_{2,x}:(x,b)\to\mathbb{R}$, $g_{2,x}(t)=(t-x)^{p}$. Applying the Cauchy mean value theorem, we can find a $\xi\in(x,t)$ such that

$$[f(t)-f(x)]\,p(\xi-x)^{p-1}=(t-x)^{p}f'(\xi)$$

from where we get

$$|f(t)-f(x)|=\frac{(t-x)^{p}|f'(\xi)|}{p(\xi-x)^{p-1}}\leq\frac{(t-x)^{p}}{p}M_{p}(x),\ \ t\in(x,b). \tag{2.4}$$

In conclusion, by (2.3) and (2.4) we may write

$$|f(t)-f(x)|\leq\frac{1}{p}M_{p}(x)|t-x|^{p}\quad\text{for all}\ \ t\in(a,b). \tag{2.5}$$

Integrating (2.5) over t on $[a,b]$, we get

$$\begin{aligned}&\left|f(x)-\frac{1}{b-a}\int_{a}^{b}f(t)\,dt\right|\\ \leq\ &\frac{1}{b-a}\int_{a}^{b}|f(t)-f(x)|\,dt\leq\frac{1}{p}M_{p}(x)\frac{1}{b-a}\int_{a}^{b}|t-x|^{p}\,dt\\ =\ &\frac{1}{p}M_{p}(x)\frac{1}{b-a}\left[\int_{a}^{x}(x-t)^{p}\,dt+\int_{x}^{b}(t-x)^{p}\,dt\right]\\ =\ &\frac{1}{p}M_{p}(x)\frac{(x-a)^{p+1}+(b-x)^{p+1}}{(p+1)(b-a)},\end{aligned}$$

and the inequality (2.2) is proved. ∎

Remark 1. *For $p=1$, we obtain*

$$\begin{aligned}\left|f(x)-\frac{1}{b-a}\int_{a}^{b}f(t)\,dt\right|&\leq\frac{(x-a)^{2}+(b-x)^{2}}{2(b-a)}\|f'\|_{\infty}\\ &=\left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^{2}\right]\|f'\|_{\infty}(b-a),\ x\in[a,b],\end{aligned}$$

where $\|f'\|_{\infty}:=\sup_{t\in(a,b)}|f'(t)|<\infty$, which is Ostrowski's inequality (1.1). It is obvious that for $p>1$, the accuracy order provided by (2.2) is higher than 1, as provided by the classical Ostrowski's inequality.

Remark 2. *If $p\in(0,1)$ and $f'\in L_{\infty}[a,b]$, then obviously*

$$\begin{aligned}M_{p}(x)&\leq\left(\max\{x-a,b-x\}\right)^{1-p}\|f'\|_{\infty}\\ &=\left[\frac{a+b}{2}+\left|x-\frac{a+b}{2}\right|\right]^{1-p}\|f'\|_{\infty}\end{aligned}$$

for all $x\in[a,b]$.

The following mid-point formula holds.

Corollary 1. *Let f and p be as in Theorem 6. Assume that*

$$M_p\left(\frac{a+b}{2}\right) := \sup_{u\in(a,b)}\left\{\left|\frac{a+b}{2}-u\right|^{1-p}\left|f'(u)\right|\right\} < \infty.$$

Then we have the midpoint inequality

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq \frac{(b-a)^p}{p(p+1)\,2^p}M_p\left(\frac{a+b}{2}\right). \tag{2.6}$$

Before we continue our presentation, we recall the following special means:

(a) The arithmetic mean

$$A = A(a,b) := \frac{a+b}{2}, \quad a,b \geq 0;$$

(b) The geometric mean

$$G = G(a,b) := \sqrt{ab}, \quad a,b \geq 0;$$

(c) The harmonic mean

$$H = H(a,b) := \frac{2ab}{a+b}, \quad a,b > 0;$$

(d) The logarithmic mean

$$L = L(a,b) := \begin{cases} a & \text{if} \quad a=b, \\ \frac{b-a}{\ln b-\ln a} & \text{if} \quad a\neq b, \end{cases} \quad a,b>0;$$

(e) The identric mean

$$I = I(a,b) := \begin{cases} a & \text{if} \quad a=b, \\ \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}} & \text{if} \quad a\neq b, \end{cases} \quad a,b>0;$$

(f) The $p-$logarithmic mean

$$L_p = L_p(a,b) := \begin{cases} \left[\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right]^{\frac{1}{p}} & \text{if} \quad a\neq b, \\ a & \text{if} \quad a=b, \end{cases} \quad a,b>0;$$

where $p \in \mathbb{R}\setminus\{-1,0\}$.

The following result also holds.

Theorem 7. *Let $f:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$ with $a>0$, and differentiable on (a,b). Let $p\in\mathbb{R}\setminus\{0\}$ and assume that*

$$K_p(f') := \sup_{u\in(a,b)}\left\{u^{1-p}\left|f'(u)\right|\right\} < \infty. \tag{2.7}$$

Then we have the inequality:

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq \frac{K_p(f')}{|p|(b-a)} \tag{2.8}$$

$$\times\begin{cases} 2x^{p}\left(x-A\right)+\left(b-x\right)L_{p}^{p}\left(b,x\right)-\left(x-a\right)L_{p}^{p}\left(x,a\right) & \text{if} \quad p\in\left(0,\infty\right) \\ \left(x-a\right)L_{p}^{p}\left(x,a\right)-\left(b-x\right)L_{p}^{p}\left(b,x\right)-2x^{p}\left(x-A\right) & \text{if} \quad p\in\left(-\infty,-1\right)\cup\left(-1,0\right) \\ \left(x-a\right)L^{-1}\left(a,x\right)-\left(b-x\right)L^{-1}\left(b,x\right)-\frac{2}{x}\left(x-A\right) & \text{if} \quad p=-1 \end{cases}$$

for all $x\in[a,b]$.

Proof. Consider the mapping $g:[a,b]\to\mathbb{R}$, $g(x)=x^{p}$. Applying the Cauchy mean value theorem, then for any x and $t\in[a,b]$, there exists a η between x and t such that

$$\left[f\left(t\right)-f\left(x\right)\right]g'\left(\eta\right)=\left[g\left(t\right)-g\left(x\right)\right]f'\left(\eta\right)$$

i.e.,

$$\left(f\left(t\right)-f\left(x\right)\right)p\eta^{p-1}=\left(t^{p}-x^{p}\right)f'\left(\eta\right)$$

from where we obtain:

$$\left|f\left(t\right)-f\left(x\right)\right|=\frac{\left|f'\left(\eta\right)\right|\left|t^{p}-x^{p}\right|}{\left|p\right|\eta^{p-1}}\leq\frac{K_{p}\left(f'\right)}{\left|p\right|}\left|t^{p}-x^{p}\right|.$$

In conclusion, for any $t,x\in[a,b]$, we have the inequality

$$\left|f\left(t\right)-f\left(x\right)\right|\leq\frac{K_{p}\left(f'\right)}{\left|p\right|}\left|t^{p}-x^{p}\right|. \tag{2.9}$$

Integrating (2.9) over t on $[a,b]$, we get

$$\begin{aligned}\left|f\left(x\right)-\frac{1}{b-a}\int_{a}^{b}f\left(t\right)dt\right| &\leq \frac{1}{b-a}\int_{a}^{b}\left|f\left(t\right)-f\left(x\right)\right|dt \\ &\leq \frac{K_{p}\left(f'\right)}{p}\frac{1}{b-a}\int_{a}^{b}\left|t^{p}-x^{p}\right|dt.\end{aligned}$$

For $p>0$, we have

$$\begin{aligned}\int_{a}^{b}\left|t^{p}-x^{p}\right|dt &= \int_{a}^{x}\left(x^{p}-t^{p}\right)dt+\int_{x}^{b}\left(t^{p}-x^{p}\right)dt \\ &= 2x^{p}\left(x-A\right)+\left(b-x\right)L_{p}^{p}\left(b,x\right)-\left(x-a\right)L_{p}^{p}\left(x,a\right).\end{aligned}$$

For $p\in\left(-\infty,-1\right)\cup\left(-1,0\right)$, we have

$$\begin{aligned}\int_{a}^{b}\left|x^{p}-t^{p}\right|dt &= \int_{a}^{x}\left(t^{p}-x^{p}\right)dt+\int_{x}^{b}\left(x^{p}-t^{p}\right)dt \\ &= \left(x-a\right)L_{p}^{p}\left(x,a\right)-\left(b-x\right)L_{p}^{p}\left(b,x\right)-2x^{p}\left(x-A\right)\end{aligned}$$

and, finally, for $p=-1$, we have

$$\begin{aligned}\int_{a}^{b}\left|\frac{1}{x}-\frac{1}{t}\right|dt &= \int_{a}^{x}\left(\frac{1}{t}-\frac{1}{x}\right)dt+\int_{x}^{b}\left(\frac{1}{x}-\frac{1}{t}\right)dt \\ &= \left(x-a\right)L^{-1}\left(a,x\right)-\left(b-x\right)L^{-1}\left(b,x\right)-\frac{2}{x}\left(x-A\right)\end{aligned}$$

and the theorem is proved. ∎

The following corollary is natural.

Corollary 2. *With the assumptions in Theorem 7, we have the midpoint inequality*

$$\left| f(A) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \le \frac{K_p(f')}{|p|} \times \begin{cases} \frac{1}{2}\left(L_p^p(b,A) - L_p^p(A,a)\right) & \text{if } p > 0; \\ \frac{1}{2}\left(L_p^p(A,a) - L_p^p(A,b)\right) & \text{if } p \in (-\infty,-1)\cup(-1,0); \\ \frac{1}{2}\left(L^{-1}(a,A) - L^{-1}(A,b)\right) & \text{if } p = -1. \end{cases} \tag{2.10}$$

The following theorem also holds.

Theorem 8. *Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ (with $a > 0$) and differentiable on (a,b). If*

$$P(f') := \sup_{u\in(a,b)} |uf'(u)| < \infty \tag{2.11}$$

then we have the inequality

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \le \frac{P(f')}{b-a}\left[\ln\left[\frac{[I(x,b)]^{b-x}}{[I(a,x)]^{x-a}}\right] + 2(x-A)\ln x\right] \tag{2.12}$$

for all $x \in [a,b]$.

Proof. Consider the mapping $g : [a,b] \to \mathbb{R}$, $g(t) = \ln t$. Applying the Cauchy mean value theorem for any x and $t \in [a,b]$ there exists a η between x and t such that

$$(f(t) - f(x))\,g'(\eta) = (g(t) - g(x))\,f'(\eta)$$

i.e.,

$$(f(t) - f(x))\frac{1}{\eta} = (\ln t - \ln x)\,f'(\eta)$$

from where we get

$$|f(t) - f(x)| = |\eta f'(\eta)|\,|\ln t - \ln x| \le P(f')\,|\ln t - \ln x|.$$

In conclusion, for any $t, x \in [a,b]$, we have the inequality

$$|f(t) - f(x)| \le P(f')\,|\ln t - \ln x|. \tag{2.13}$$

Integrating (2.13) over t on $[a,b]$, we get

$$
\begin{aligned}
& \left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \\
\leq & \frac{1}{b-a}\int_a^b |f(t)-f(x)|\,dt \leq P(f')\frac{1}{b-a}\int_a^b |\ln t - \ln x|\,dt \\
= & P(f')\frac{1}{b-a}\left[\int_a^x (\ln x - \ln t)\,dt + \int_x^b (\ln t - \ln x)\,dt\right] \\
= & P(f')\frac{1}{b-a}\left[(x-a)\ln x - (x-a)\ln I(a,x) + (b-x)\ln I(b,x) - (b-x)\ln x\right] \\
= & \frac{1}{b-a}\left[2(x-A)\ln x + (b-x)\ln I(x,b) - (x-a)\ln I(a,x)\right]P'(f)
\end{aligned}
$$

and the theorem is proved. ∎

The following corollary is natural.

Corollary 3. *With the assumptions of Theorem 8, we have the inequality*

$$
\left| f(A) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \leq \frac{1}{2}P(f')\ln\left[\frac{I(A,b)}{I(a,A)}\right], \tag{2.14}
$$

where $A = A(a,b) = \frac{a+b}{2}$.

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