IMPROVEMENTS OF OSTROWSKI AND GENERALISED TRAPEZOID INEQUALITY IN TERMS OF THE UPPER AND LOWER BOUNDS OF THE FIRST DERIVATIVE

S.S. DRAGOMIR

Abstract. In this paper improvements of the Ostrowski and generalised Trapezoid inequalities are found in terms of the upper and lower bounds of the first derivative.

1. Introduction

The following result is well-known in the literature as Ostrowski's inequality for absolutely continuous functions whose derivatives are essentially bounded [5].

Theorem 1. *Let $f:[a,b]\to\mathbb{R}$ be absolutely continuous on $[a,b]$ whose derivative $f':[a,b]\to\mathbb{R}$ belongs to $L_\infty[a,b]$, i.e.,*

$$\|f'\|_\infty := ess\sup_{t\in[a,b]}|f'(t)|<\infty. \tag{1.1}$$

Then

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\leq\left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right](b-a)\|f'\|_\infty \tag{1.2}$$

for all $x\in[a,b]$.
The constant $\frac{1}{4}$ is the best possible in the sense that it cannot be replaced by a smaller constant.

A simple proof of this fact can be done by using the following identity valid for absolutely continuous functions:

$$f(x)=\frac{1}{b-a}\int_a^b f(t)\,dt+\frac{1}{b-a}\int_a^b p(x,t)\,f'(t)\,dt,\ \ x\in[a,b], \tag{1.3}$$

where

$$p(x,t)=\begin{cases} t-a & \text{if} \quad a\leq t\leq x \\ t-b & \text{if} \quad x<t\leq b \end{cases}.$$

An important particular case which also provides the best inequality one can get from (1.2) is $x=\frac{a+b}{2}$, obtaining the mid-point inequality:

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\leq\frac{1}{4}(b-a)\|f'\|_\infty. \tag{1.4}$$

Date: August 29, 2000.
1991 *Mathematics Subject Classification.* Primary 26D10, 26D15; Secondary 41A55, 41A99.
Key words and phrases. Ostrowski's Inequality, Quadrature formulae.

Note that in inequality (1.4) the constant $\frac{1}{4}$ is sharp.

The equality (1.4) is obtained for $f(x) = k\left|x - \frac{a+b}{2}\right|$, $k > 0$, $x \in [a,b]$. A generalised trapezoid type inequality that is similar to the Ostrowski inequality is the following (see [2]).

Theorem 2. *Let f be as in Theorem 1. Then we have:*

$$\left|\frac{(b-x)f(b)+(x-a)f(a)}{b-a} - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq \left[\frac{1}{4} + \left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right](b-a)\|f'\|_\infty \tag{1.5}$$

for all $x \in [a,b]$.
The constant $\frac{1}{4}$ is best possible in the above sense.

A simple proof of this fact can be obtained by employing the following identity valid for absolutely continuous functions:

$$\frac{(b-x)f(b)+(x-a)f(a)}{b-a} = \frac{1}{b-a}\int_a^b f(t)\,dt + \frac{1}{b-a}\int_a^b (t-x)f'(t)\,dt \tag{1.6}$$

for all $x \in [a,b]$.

A particularly important case is for $x = \frac{a+b}{2}$, obtaining the trapezoid inequality

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq \frac{1}{4}(b-a)\|f'\|_\infty. \tag{1.7}$$

Note that in (1.7) the constant $\frac{1}{4}$ is the best possible.

The equality in (1.7) is obtained for $f(x) = k\left|x - \frac{a+b}{2}\right|$, $k > 0$, $x \in [a,b]$.

2. Some Inequalities

We suppose that the absolutely continuous function $g : [a,b] \to \mathbb{R}$ satisfies the standing condition

$$-\infty < m \leq g'(t) \leq M < \infty \quad \text{for a.e. } t \in [a,b], \tag{2.1}$$

and ask the question of finding an Ostrowski like inequality in terms of the difference $M - m$.

The following result holds.

Theorem 3. *Assume that the absolutely continuous function $g : [a,b] \to \mathbb{R}$ satisfies the condition (2.1). Then*

$$\left|g(x) - \frac{1}{b-a}\int_a^b g(t)\,dt - \left(x - \frac{a+b}{2}\right)\left(\frac{m+M}{2}\right)\right| \leq \frac{1}{2}\left[\frac{1}{4} + \left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right](M-m)(b-a), \tag{2.2}$$

for all $x \in [a,b]$.
The inequality (2.2) is sharp.

Proof. Consider the auxiliary function $f : [a, b] \to \mathbb{R}$, $f(x) = g(x) - \left(x - \frac{a+b}{2}\right)\left(\frac{m+M}{2}\right)$ which is absolutely continuous and as $f'(x) = g'(x) - \frac{m+M}{2}$, we get by (2.1) that

$$|f'(x)| \leq \frac{M-m}{2} \quad \text{for a.e. } x \in [a, b]$$

which shows that $f' \in L_\infty[a, b]$ and $\|f'\|_\infty \leq \frac{M-m}{2}$.

If we apply the Ostrowski inequality for the mapping f, we may write

$$\begin{aligned} &\left| g(x) - \left(x - \frac{a+b}{2}\right)\left(\frac{m+M}{2}\right) \right. \\ &\left. - \frac{1}{b-a}\int_a^b \left[g(x) - \left(x - \frac{a+b}{2}\right)\left(\frac{m+M}{2}\right)\right] dx \right| \\ \leq \; &\left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a}\right)^2\right](b-a)\frac{M-m}{2}, \end{aligned}$$

which is clearly equivalent to (2.2).

Since for $m = -\|g'\|_\infty$, $M = \|g'\|_\infty$ where $g' \in L_\infty[a, b]$, we recapture the Ostrowski inequality which is a sharp inequality, we may deduce that (2.2) is also sharp. ∎

The following corollary is interesting.

Corollary 1. *Assume that $g : [a, b] \to \mathbb{R}$ is an absolutely continuous function satisfying the condition (2.1). Then we have the mid-point inequality:*

$$\left| g\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b g(t)\,dt \right| \leq \frac{1}{8}(M-m)(b-a). \tag{2.3}$$

The constant $\frac{1}{8}$ is best in the sense that it cannot be replaced by a smaller constant.

Proof. The inequality follows by (2.2) on choosing $x = \frac{a+b}{2}$. To prove the sharpness of the constant $\frac{1}{8}$, assume that

$$\left| g\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b g(t)\,dt \right| \leq c(M-m)(b-a) \tag{2.4}$$

with $c > 0$.

If in (2.4) we choose $g(t) = k\left|t - \frac{a+b}{2}\right|$, $t \in [a, b]$, $k > 0$ which is absolutely continuous and $M = k$, $m = -k$, then we get

$$k \cdot \frac{(b-a)}{4} \leq 2ck(b-a)$$

which implies that $c \geq \frac{1}{8}$. ∎

Remark 1. *In* [3]*, using a technique based on the Grüss inequality, Dragomir and Wang were able to prove (2.4) with the constant $c = \frac{1}{4}$. By using a "pre-Grüss" inequality, the authors of* [6] *were able to prove (2.4) with a better constant $c = \frac{1}{4\sqrt{3}}$. Now, we know that the best possible constant in (2.4) is $c = \frac{1}{8}$; and then, the problem of estimating the error in the mid-point formula in terms of the difference $(M-m)$ is completely solved.*

Remark 2. *The following inequality is well known in the literature as the Hermite-Hadamard inequality (see for example [4]):*

$$g\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_{a}^{b}g\left(t\right)dt\leq\frac{g\left(a\right)+g\left(b\right)}{2}, \tag{2.5}$$

provided that $g:[a,b]\to\mathbb{R}$ is convex on $[a,b]$.

If we assume that $g:I\subseteq\mathbb{R}\to\mathbb{R}$ is differentiable convex on $\mathring{I}$ ($\mathring{I}$ is the interior of I) and $a,b\in\mathring{I}$, then, by (2.3), we have the following reverse inequality

$$0\leq\frac{1}{b-a}\int_{a}^{b}g\left(t\right)dt-g\left(\frac{a+b}{2}\right)\leq\frac{1}{8}\left[g'\left(b\right)-g'\left(a\right)\right]\left(b-a\right). \tag{2.6}$$

Now, we are able to point out the following version for the generalised trapezoid formula.

Theorem 4. *Assume that the function $g:[a,b]\to\mathbb{R}$ fulfills the hypothesis of Theorem 3. Then*

$$\begin{aligned}&\left|\frac{\left(b-x\right)g\left(b\right)+\left(x-a\right)g\left(a\right)}{b-a}\right.\\&\left.-\frac{1}{b-a}\int_{a}^{b}g\left(t\right)dt+\left(x-\frac{a+b}{2}\right)\left(\frac{m+M}{2}\right)\right|\\\leq&\ \frac{1}{2}\left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^{2}\right]\left(M-m\right)\left(b-a\right),\end{aligned} \tag{2.7}$$

for all $x\in[a,b]$.

The inequality (2.7) is sharp.

Proof. Consider the auxiliary function $f:[a,b]\to\mathbb{R}$, $f\left(x\right)=g\left(x\right)-\left(x-\frac{a+b}{2}\right)\left(\frac{m+M}{2}\right)$. Then, as in the proof of Theorem 3, we may state that $\left\Vert f'\right\Vert_{\infty}\leq\frac{M-m}{2}$ and applying the inequality (1.5) we may write:

$$\begin{aligned}&\left|\frac{1}{b-a}\left[\left(b-x\right)\left(g\left(b\right)-\left(b-\frac{a+b}{2}\right)\left(\frac{m+M}{2}\right)\right)\right.\right.\\&\left.+\left(x-a\right)\left(g\left(a\right)-\left(a-\frac{a+b}{2}\right)\left(\frac{m+M}{2}\right)\right)\right]\\&\left.-\frac{1}{b-a}\int_{a}^{b}\left[g\left(x\right)-\left(x-\frac{a+b}{2}\right)\left(\frac{m+M}{2}\right)\right]dx\right|\\\leq&\ \left[\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^{2}+\frac{1}{4}\right]\frac{M-m}{2}\cdot\left(b-a\right)\end{aligned}$$

for all $x\in[a,b]$, which is clearly equivalent to (2.7).

The sharpness of (2.7) follows by the sharpness of (1.5) and we omit the details. ∎

The following corollary is interesting.

Corollary 2. *Assume that $g : [a,b] \to \mathbb{R}$ is an absolutely continuous function satisfying the condition (2.1). Then we have the trapezoid inequality:*

$$\left| \frac{g(a)+g(b)}{2} - \frac{1}{b-a}\int_a^b g(t)\,dt \right| \leq \frac{1}{8}(M-m)(b-a). \tag{2.8}$$

The constant $\frac{1}{8}$ is best.

Proof. The inequality follows by (2.7) choosing $x = \frac{a+b}{2}$. To prove the sharpness of the constant $\frac{1}{8}$, assume that

$$\left| \frac{g(a)+g(b)}{2} - \frac{1}{b-a}\int_a^b g(t)\,dt \right| \leq c(M-m)(b-a) \tag{2.9}$$

with $c > 0$.

If in (2.9) we choose $g(t) = k\left|t - \frac{a+b}{2}\right|$, $t \in [a,b]$, $k > 0$ which is absolutely continuous and $M = k$, $m = -k$, then we get

$$k \cdot \frac{(b-a)}{4} \leq 2ck(b-a) \tag{2.10}$$

implying that $c \geq \frac{1}{8}$. ∎

Remark 3. *In* [1]*, the authors proved, among others, the inequality (2.8), however, the problem of the best constant was not considered.*

Remark 4. *With the assumptions of Remark 3, we may state the following counterpart inequality for the (HH)-inequality:*

$$0 \leq \frac{g(a)+g(b)}{2} - \frac{1}{b-a}\int_a^b g(t)\,dt \leq \frac{1}{8}\left[g'(b) - g'(a)\right](b-a). \tag{2.11}$$

Note that, since by Bullen's inequality for convex functions [4, p. 2], we have

$$0 \leq \frac{1}{b-a}\int_a^b g(t)\,dt - g\left(\frac{a+b}{2}\right) \leq \frac{g(a)+g(b)}{2} - \frac{1}{b-a}\int_a^b g(t)\,dt,$$

then, by (2.11), we may obtain (2.6) as well.

3. Some Quadrature Formulae

Consider the division

$$I_n : a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$$

of the interval $[a,b]$ and denote $h_i := x_{i+1} - x_i$ $\left(i = \overline{0, n-1}\right)$ and $\nu(h) := \max\left\{h_i | i = \overline{0, n-1}\right\}$.

If $g : [a,b] \to \mathbb{R}$ is absolutely continuous, then let

$$\begin{aligned} m_i &: = ess \inf_{t \in [x_i, x_{i+1}]} g'(t), \quad M_i := ess \sup_{t \in [x_i, x_{i+1}]} g'(t), \\ m &: = ess \inf_{t \in [a,b]} g'(t) \quad \text{and} \quad M := ess \sup_{t \in [a,b]} g'(t). \end{aligned}$$

Assume that $-\infty < m < M < \infty$. Then obviously

$$m \leq m_i \leq M_i \leq M \quad \text{for all} \quad i \in \{0, \ldots, n-1\}.$$

For a sequence of intermediate points $\boldsymbol{\xi} = \left(\xi_0, \xi_1, \dots \xi_{n-1}\right)$ with $\xi_i \in [x_i, x_{i+1}]$, $i = \overline{0, n-1}$, define the perturbed Riemann sum

$$A\left(g, I_n, \boldsymbol{\xi}\right) := \sum_{i=0}^{n-1} g\left(\xi_i\right) h_i - \sum_{i=0}^{n-1} h_i \left(\xi_i - \frac{x_i + x_{i+1}}{2}\right)\left(\frac{m_i + M_i}{2}\right). \tag{3.1}$$

Then we may state the following quadrature result.

Theorem 5. *Let $g : [a, b] \to \mathbb{R}$ be an absolutely continuous function whose derivative is essentially bounded on $[a, b]$. Then*

$$\int_a^b g\left(t\right) dt = A\left(g, I_n, \boldsymbol{\xi}\right) + R\left(g, I_n, \boldsymbol{\xi}\right), \tag{3.2}$$

where the remainder $R\left(g, I_n, \boldsymbol{\xi}\right)$ satisfies the estimate

$$\begin{aligned} \left|R\left(g, I_n, \boldsymbol{\xi}\right)\right| &\leq \frac{1}{2}\sum_{i=0}^{n-1}\left\{\left[\left(\xi_i - \frac{x_i + x_{i+1}}{2}\right)^2 + \frac{1}{4}h_i^2\right]\left(M_i - m_i\right)\right\} \\ &\leq \frac{1}{2}\left(M - m\right)\sum_{i=0}^{n-1}\left[\left(\xi_i - \frac{x_i + x_{i+1}}{2}\right)^2 + \frac{1}{4}h_i^2\right] \\ &\leq \frac{1}{4}\left(M - m\right)\sum_{i=0}^{n-1} h_i^2 \leq \frac{1}{4}\left(M - m\right)\left(b - a\right)\nu\left(h\right). \end{aligned} \tag{3.3}$$

The inequalities are sharp.

The proof is obvious by applying the inequality (2.2) on the intervals $[x_i, x_{i+1}]$ for the intermediate points ξ_i $\left(i = \overline{0, n-1}\right)$ and simple algebraic manipulations.

Corollary 3. *Assume that g satisfies the assumptions of Theorem 5. If $M\left(g, I_n\right)$ denotes the mid-point rule, i.e.,*

$$M\left(g, I_n\right) := \sum_{i=0}^{n-1} g\left(\frac{x_i + x_{i+1}}{2}\right) h_i \tag{3.4}$$

then we have

$$\int_a^b g\left(t\right) dt = M\left(g, I_n\right) + R\left(g, I_n\right), \tag{3.5}$$

where the remainder $R\left(g, I_n\right)$ satisfies the estimate

$$\begin{aligned} \left|R\left(g, I_n\right)\right| &\leq \frac{1}{8}\sum_{i=0}^{n-1}\left(M_i - m_i\right) h_i^2 \leq \frac{1}{8}\left(M - m\right)\sum_{i=0}^{n-1} h_i^2 \\ &\leq \frac{1}{8}\left(M - m\right)\left(b - a\right)\nu\left(h\right). \end{aligned} \tag{3.6}$$

The constant $\frac{1}{8}$ is sharp in all inequalities.

Now, for a sequence of intermediate points $\boldsymbol{\xi} = \left(\xi_0, \xi_1, \dots \xi_{n-1}\right)$ we can also consider the perturbed generalised trapezoid rule:

$$\begin{aligned} B\left(g, I_n, \boldsymbol{\xi}\right) &:= \sum_{i=0}^{n-1}\left[\left(x_{i+1} - \xi_i\right) g\left(x_{i+1}\right) + \left(\xi_i - x_i\right) g\left(x_i\right)\right] \\ &\quad + \sum_{i=0}^{n-1} h_i \left(\xi_i - \frac{x_i + x_{i+1}}{2}\right)\left(\frac{m_i + M_i}{2}\right). \end{aligned} \tag{3.7}$$

Then we may state the following quadrature result.

Theorem 6. *Let $g:[a,b]\to\mathbb{R}$ be an absolutely continuous function whose derivative is essentially bounded on $[a,b]$. Then*

$$\int_a^b g(t)\,dt = B(g,I_n,\boldsymbol{\xi}) + W(g,I_n,\boldsymbol{\xi}), \tag{3.8}$$

where the remainder $W(g,I_n,\boldsymbol{\xi})$ satisfies the estimate

$$\begin{aligned} |W(g,I_n,\boldsymbol{\xi})| &\le \frac{1}{2}\sum_{i=0}^{n-1}\left\{\left[\left(\xi_i-\frac{x_i+x_{i+1}}{2}\right)^2+\frac{1}{4}h_i^2\right](M_i-m_i)\right\} \\ &\le \frac{1}{2}(M-m)\sum_{i=0}^{n-1}\left[\left(\xi_i-\frac{x_i+x_{i+1}}{2}\right)^2+\frac{1}{4}h_i^2\right] \\ &\le \frac{1}{4}(M-m)\sum_{i=0}^{n-1}h_i^2 \le \frac{1}{4}(M-m)(b-a)\nu(h). \end{aligned} \tag{3.9}$$

The inequalities are sharp.

The proof follows by Theorem 4 and we omit the details.

Corollary 4. *Assume that g satisfies the assumptions of Theorem 5. If $T(g,I_n)$ denotes the trapezoid rule, that is,*

$$T(g,I_n) := \sum_{i=0}^{n-1} h_i\left[\frac{g(x_i)+g(x_{i+1})}{2}\right] \tag{3.10}$$

then we have

$$\int_a^b g(t)\,dt = T(g,I_n) + W(g,I_n), \tag{3.11}$$

where the remainder $W(g,I_n)$ satisfies the estimate

$$\begin{aligned} |W(g,I_n)| &\le \frac{1}{8}\sum_{i=0}^{n-1}(M_i-m_i)h_i^2 \le \frac{1}{8}(M-m)\sum_{i=0}^{n-1}h_i^2 \\ &\le \frac{1}{8}(M-m)(b-a)\nu(h). \end{aligned} \tag{3.12}$$

The constant $\frac{1}{8}$ is sharp in all inequalities.

4. Applications for Special Means

Recall the following means:

The arithmetic mean

$$A = A(a,b) := \frac{a+b}{2}, \quad a,b\ge 0;$$

The geometric mean

$$G = G(a,b) := \sqrt{ab}, \quad a,b\ge 0;$$

The harmonic mean

$$H = H(a,b) := \frac{2ab}{a+b}, \quad a,b>0;$$

The logarithmic mean

$$L = L(a,b) := \begin{cases} a & \text{if} \quad a = b, \\ \frac{b-a}{\ln b - \ln a} & \text{if} \quad a \neq b, \end{cases} \quad a,b > 0;$$

The identric mean

$$I = I(a,b) := \begin{cases} a & \text{if} \quad a = b, \\ \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}} & \text{if} \quad a \neq b, \end{cases} \quad a,b > 0;$$

The $p-$logarithmic mean

$$L_p = L_p(a,b) := \begin{cases} \left[\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right]^{\frac{1}{p}} & \text{if} \quad a \neq b, \\ a & \text{if} \quad a = b, \end{cases} \quad a,b > 0;$$

where $p \in R\backslash\{-1,0\}$.

If $L_0 := I$ and $L_{-1} := L$, then the function $p \mapsto L_p$ is monotonically increasing over $p \in \mathbb{R}$ and in particular

$$H \leq G \leq L \leq I \leq A.$$

In this section we point out some applications to special means of the inequality

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \leq \frac{1}{8}(M-m)(b-a), \tag{4.1}$$

where $m \leq f(t) \leq M$ for $t \in [a,b]$.

(1) Consider the function $f : [a,b] \subset (0,\infty) \to \mathbb{R}$, $f(x) = x^p$, $p \in \mathbb{R}\diagdown\{-1,0\}$. Then $f'(x) = px^{p-1}$ which is strictly increasing if $p \in (-\infty,0)\cup[1,\infty)\diagdown\{-1\}$ and strictly decreasing for $p \in (0,1)$. So

$$pa^{p-1} \leq f'(x) \leq pb^{p-1}, \quad x \in [a,b] \text{ if } p \in (-\infty,0)\cup[1,\infty)\diagdown\{-1\} \tag{4.2}$$

and

$$pb^{p-1} \leq f'(x) \leq pa^{p-1}, \quad x \in [a,b] \text{ if } p \in (0,1). \tag{4.3}$$

Consequently, using (4.1) – (4.3) we have the inequality,

$$\left|A^p - L_p^p\right| \leq \begin{cases} \frac{p}{8}\left(b^{p-1}-a^{p-1}\right)(b-a) & \text{if} \quad p \in (-\infty,0)\cup[1,\infty)\diagdown\{-1\} \\ \frac{p}{8}\left(a^{p-1}-b^{p-1}\right)(b-a) & \text{if} \quad p \in (0,1). \end{cases}$$

(2) Consider the function $f : [a,b] \subset (0,\infty) \to \mathbb{R}$, $f(x) = \frac{1}{x}$. Then $f'(x) = -\frac{1}{x^2}$ which is strictly increasing on $[a,b]$. So

$$-\frac{1}{a^2} \leq f'(x) \leq -\frac{1}{b^2}, \; x \in [a,b]. \tag{4.4}$$

Consequently, using (4.1) and (4.4) we may state

$$\left|\frac{1}{A} - \frac{1}{L}\right| \leq \frac{1}{8}\left(\frac{1}{a^2} - \frac{1}{b^2}\right)(b-a),$$

which is equivalent to

$$0 \leq A - L \leq \frac{1}{4} \cdot \frac{A^2 L}{G^4} (b-a)^2 . \tag{4.5}$$

(3) Consider the function $f(x) = \ln x$, $x \in [a,b] \subset (0,\infty)$. Then $f'(x) = \frac{1}{x}$ which is strictly decreasing on $[a,b]$. Thus,

$$\frac{1}{b} \leq f'(x) \leq \frac{1}{a}, \quad x \in [a,b] . \tag{4.6}$$

Consequently, using (4.1) and (4.60 we may state that

$$|\ln A - \ln I| \leq \frac{1}{8}\left(\frac{1}{a} - \frac{1}{b}\right)(b-a),$$

which is equivalent to

$$1 \leq \frac{A}{I} \leq \exp\left[\frac{1}{8} \cdot \frac{(b-a)^2}{G^2}\right]. \tag{4.7}$$

References

[1] R.P. AGARWAL and S.S. DRAGOMIR, An applications of Hayashi's inequality for differentiable functions, *Comp. Math. Appl.,* **32** (6) (1996), 85-99.

[2] P. CERONE and S.S. DRAGOMIR, Trapezoidal type rules from an inequalities point of view, in *Analytic-Computational Methods in Applied Mathematics*, G.A. Anastassiou (Ed), CRC Press, New York, 2000, 65-134.

[3] S.S. DRAGOMIR and S. WANG, An inequality of Ostrowski-Grüss' type and its applications to the estimation of error bounds for some special means and for some numerical quadrature rules, *Computers Math. Applic.,* **11** (1997), 15-20.

[4] S.S. DRAGOMIR and C.E.M. PEARCE, *Selected Topics on Hermite-Hadamard Inequalities and Applications,* RGMIA Monographs, Victoria Univ., 2000, `http://rgmia.vu.edu.au/monographs.html`

[5] A. OSTROWSKI, Uber die Absolutabweichung einer differentienbaren Funktionen von ihren Integralmittelwert, *Comment. Math. Hel,* **10** (1938), 226-227.

[6] M. MATIĆ, J. PEČARIĆ and N. UJEVIĆ, Improvement and further generalisation of some inequalities of Ostrowski-Grüss type, *Comp. Math. Appl.,* to appear.

School of Communications and Informatics, Victoria University of Technology, PO Box 14428, Melbourne City MC, Victoria 8001, Australia.

E-mail address: `sever@matilda.vu.edu.au`

URL: `http://rgmia.vu.edu.au/SSDragomirWeb.html`