AN INEQUALITY OF OSTROWSKI TYPE IN TERMS OF THE LOWER AND UPPER BOUNDS OF THE FIRST DERIVATIVE

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Abstract. In this paper an inequality of Ostrowski type in terms of the lower and upper bounds of the first derivative is found.

1. Introduction

The following result is known in the literature as Ostrowski's inequality [1].

Theorem 1. *Let $f : [a,b] \to \mathbb{R}$ be a differentiable mapping on (a,b) with the property that $|f'(t)| \leq M$ for all $t \in (a,b)$. Then*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \leq \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right](b-a)M \tag{1.1}$$

for all $x \in [a,b]$.
The constant $\frac{1}{4}$ is the best possible in the sense that it cannot be replaced by a smaller constant.

A simple proof of this fact can be done by using the identity:

$$f(x) = \frac{1}{b-a}\int_a^b f(t)\,dt + \frac{1}{b-a}\int_a^b p(x,t) f'(t)\,dt, \quad x \in [a,b], \tag{1.2}$$

where

$$p(x,t) := \begin{cases} t-a & \text{if} \quad a \leq t \leq x \\ t-b & \text{if} \quad x < t \leq b \end{cases}$$

which also holds for absolutely continuous functions $f : [a,b] \to \mathbb{R}$.

The following Ostrowski type result for absolutely continuous functions holds (see [2], [3] and [4]).

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Theorem 2. *Let $f : [a,b] \to \mathbb{R}$ be absolutely continuous on $[a,b]$. Then, for all $x \in [a,b]$, we have:*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \tag{1.3}$$
$$\leq \begin{cases} \left[\frac{1}{4} + \left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right](b-a)\|f'\|_\infty & \text{if } f' \in L_\infty[a,b]; \\ \frac{1}{(p+1)^{\frac{1}{p}}}\left[\left(\frac{x-a}{b-a}\right)^{p+1} + \left(\frac{b-x}{b-a}\right)^{p+1}\right]^{\frac{1}{p}}(b-a)^{\frac{1}{p}}\|f'\|_q & \text{if } f' \in L_q[a,b], \\ & \frac{1}{p}+\frac{1}{q}=1,\ p>1; \\ \left[\frac{1}{2} + \left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right]\|f'\|_1; \end{cases}$$

where $\|\cdot\|_r$ $(r \in [1,\infty])$ are the usual Lebesgue norms on $L_r[a,b]$, i.e.,

$$\|g\|_\infty := ess \sup_{t\in[a,b]} |g(t)|$$

and

$$\|g\|_r := \left(\int_a^b |g(t)|^r\,dt\right)^{\frac{1}{r}},\ r \in [1,\infty).$$

The constants $\frac{1}{4}$, $\frac{1}{(p+1)^{\frac{1}{p}}}$ and $\frac{1}{2}$ respectively are sharp in the sense presented in Theorem 1.

The above inequalities can also be obtained from the Fink result in [5] on choosing $n = 1$ and performing some appropriate computations.

If one drops the condition of absolute continuity and assumes that f is Hölder continuous, then one may state the result (see [6])

Theorem 3. *Let $f : [a,b] \to \mathbb{R}$ be of $r-H-$Hölder type, i.e.,*

$$|f(x) - f(y)| \leq H|x-y|^r,\ \textit{for all}\ \ x,y \in [a,b], \tag{1.4}$$

where $r \in (0,1]$ and $H > 0$ are fixed. Then for all $x \in [a,b]$ we have the inequality:

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \tag{1.5}$$
$$\leq \frac{H}{r+1}\left[\left(\frac{b-x}{b-a}\right)^{r+1} + \left(\frac{x-a}{b-a}\right)^{r+1}\right](b-a)^r.$$

The constant $\frac{1}{r+1}$ is also sharp in the above sense.

Note that if $r = 1$, i.e., f is Lipschitz continuous, then we get the following version of Ostrowski's inequality for Lipschitzian functions (with L instead of H) (see [7])

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \leq \left[\frac{1}{4} + \left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right](b-a)L. \tag{1.6}$$

Here the constant $\frac{1}{4}$ is also best.

Moreover, if one drops the condition of the continuity of the function, and assumes that it is of bounded variation, then the following result may be stated (see [8] or [10]).

Theorem 4. *Assume that* $f : [a,b] \to \mathbb{R}$ *is of bounded variation and denote by* $\bigvee_a^b$ *its total variation. Then*

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)\,dt \right| \leq \left[\frac{1}{2} + \left| \frac{x - \frac{a+b}{2}}{b-a} \right| \right] \bigvee_a^b (f) \tag{1.7}$$

for all $x \in [a,b]$.
The constant $\frac{1}{2}$ *is the best possible.*

If we assume more about f, i.e., f is monotonically increasing, then the inequality (1.7) may be improved in the following manner [9] (see also [10]).

Theorem 5. *Let* $f : [a,b] \to \mathbb{R}$ *be monotonic nondecreasing. Then for all* $x \in [a,b]$, *we have the inequality:*

$$\begin{aligned} &\left| f(x) - \frac{1}{b-a} \int_a^b f(t)\,dt \right| \\ \leq\ & \frac{1}{b-a} \left\{ [2x - (a+b)] f(x) + \int_a^b sgn(t-x) f(t)\,dt \right\} \\ \leq\ & \frac{1}{b-a} \{ (x-a)[f(x) - f(a)] + (b-x)[f(b) - f(x)] \} \\ \leq\ & \left[\frac{1}{2} + \left| \frac{x - \frac{a+b}{2}}{b-a} \right| \right] [f(b) - f(a)]. \end{aligned} \tag{1.8}$$

All the inequalities in (1.8) are sharp and the constant $\frac{1}{2}$ *is the best possible.*

In this paper we point out different Ostrowski type inequalities assuming that we know some upper and lower bounds for the derivative of the function f in the interval (a,x) and (x,b).

2. The Results

We start with the following result.

Theorem 6. *Let* $f : [a,b] \to \mathbb{R}$ *be an absolutely continuous function on* $[a,b]$ *and* $x \in [a,b]$. *Suppose that there exist the functions* m_i, $M_i : [a,b] \to \mathbb{R}$ $\left(i = \overline{1,2}\right)$ *with the properties:*

$$m_1(x) \leq f'(t) \leq M_1(x) \quad \textit{for a.e. } t \in [a,x] \tag{2.1}$$

and

$$m_2(x) \leq f'(t) \leq M_2(x) \quad \textit{for a.e. } t \in (x,b]. \tag{2.2}$$

Then we have the inequalities:

$$\begin{aligned}
(2.3) \qquad & \frac{1}{2(b-a)}\left[m_1(x)(x-a)^2 - M_2(x)(b-x)^2\right] \\
\leq\ & f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \\
\leq\ & \frac{1}{2(b-a)}\left[M_1(x)(x-a)^2 - m_2(x)(b-x)^2\right].
\end{aligned}$$

The constant $\frac{1}{2}$ *is sharp on both sides.*

Proof. As $f:[a,b]\to\mathbb{R}$ is absolutely continuous on $[a,b]$, it is differentiable a.e. on $[a,b]$ and, by applying the integration by parts formula, we may write

$$\begin{aligned}
(2.4) \qquad f(x) &= \frac{1}{b-a}\int_a^b f(t)\,dt + \frac{1}{b-a}\int_a^x (t-a)f'(t)\,dt \\
&\quad + \frac{1}{b-a}\int_x^b (t-b)f'(t)\,dt
\end{aligned}$$

for any $x\in[a,b]$.

Using the assumption (2.1) and (2.2), we have:

$$(2.5) \qquad m_1(x)(t-a) \leq (t-a)f'(t) \leq M_1(x)(t-a) \quad \text{for a.e. } t\in[a,x]$$

and

$$(2.6) \qquad M_2(x)(t-b) \leq f'(t)(t-b) \leq m_2(x)(t-b) \quad \text{for a.e. } t\in(x,b].$$

Integrating (2.5) on $[a,x]$ and (2.6) on $[x,b]$ and summing the obtained inequalities, we have

$$\begin{aligned}
& \frac{1}{2}m_1(x)(x-a)^2 - \frac{1}{2}M_2(x)(b-x)^2 \\
\leq\ & \int_a^x (t-a)f'(t)\,dt + \int_x^b (t-b)f'(t)\,dt \\
\leq\ & \frac{1}{2}M_1(x)(x-a)^2 - \frac{1}{2}m_2(x)(b-x)^2.
\end{aligned}$$

Using the representation (2.4), we deduce (2.3).

Assume that the first inequality in (2.3) holds with a constant $c>0$; that is,

$$(2.7) \qquad \frac{c}{b-a}\left[m_1(x)(x-a)^2 - M_2(x)(b-x)^2\right] \leq f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt.$$

Consider the function $f:[a,b]\to\mathbb{R}$, $f(t)=M|t-x|$, $M>0$. Then f is absolutely continuous and

$$f'(t) = \begin{cases} -M & \text{if} \quad t\in[a,x] \\ M & \text{if} \quad t\in(x,b]. \end{cases}$$

Thus, if we choose $m_1=-M$, $m_2=M$ in (2.7), we get

$$\begin{aligned}
-M\frac{c}{b-a}\left[(x-a)^2+(b-x)^2\right] &\leq 0 - \frac{M}{b-a}\int_a^b |t-x|\,dt \\
&= -\frac{M}{b-a}\left[\frac{(b-x)^2+(x-a)^2}{2}\right]
\end{aligned}$$

for all $x \in [a,b]$, implying that $c \geq \frac{1}{2}$, that is, $\frac{1}{2}$ is the best constant in the first member of (2.3).

Using a similar process, we may prove that $\frac{1}{2}$ is the best constant in the third member of (2.3) and the theorem is completely proved. ∎

Corollary 1. *If $f : [a,b] \to \mathbb{R}$ is absolutely continuous on $[a,b]$ and the derivative $f' : [a,b] \to \mathbb{R}$ is bounded above and below, that is,*

$$-\infty < m \leq f'(t) \leq M < \infty \text{ a.e. } t \in [a,b], \tag{2.8}$$

then we have the inequality

$$\begin{aligned} &\frac{1}{2(b-a)}\left[m(x-a)^2 - M(b-x)^2\right] \\ \leq\ & f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \\ \leq\ & \frac{1}{2(b-a)}\left[M(x-a)^2 - m(b-x)^2\right] \end{aligned} \tag{2.9}$$

for all $x \in [a,b]$.
The constant $\frac{1}{2}$ is the best in both inequalities.

Applying Taylor's formula

$$g(x) = g\left(\frac{a+b}{2}\right) + \left(x - \frac{a+b}{2}\right) g'\left(\frac{a+b}{2}\right) + \frac{1}{2}\left(x - \frac{a+b}{2}\right)^2 g''\left(\frac{a+b}{2}\right)$$

for $g(x) = M(x-a)^2 - m(b-x)^2$, we obtain

$$\begin{aligned} g(x) &= \frac{1}{4}(M-m)(b-a)^2 + 2\left(x - \frac{a+b}{2}\right)\left(\frac{M+m}{2}\right)(b-a) \\ &\quad + (M-m)\left(x - \frac{a+b}{2}\right)^2. \end{aligned}$$

The same formula applied for $h(x) = m(x-a)^2 - M(b-x)^2$, will reveal that

$$\begin{aligned} h(x) &= \frac{1}{4}(M-m)(b-a)^2 + 2\left(x - \frac{a+b}{2}\right)\left(\frac{M+m}{2}\right)(b-a) \\ &\quad - (M-m)\left(x - \frac{a+b}{2}\right)^2. \end{aligned}$$

Consequently, we may rewrite Corollary 1 in the following equivalent manner (see also [11]):

Corollary 2. *With the assumptions on Corollary 1, we have:*

$$\begin{aligned} &\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt - \left(x - \frac{a+b}{2}\right)\left(\frac{M+m}{2}\right) \right| \\ \leq\ & \frac{M-m}{2}(b-a)\left[\left(\frac{x - \frac{a+b}{2}}{b-a}\right)^2 + \frac{1}{4}\right] \end{aligned}$$

for all $x \in [a,b]$.

Remark 1. *If we assume that* $\|f'\|_\infty := ess \sup_{t\in[a,b]} |f'(t)|$, *then obviously we may choose in (2.9)* $m = \|f'\|_\infty$ *and* $M = \|f'\|_\infty$, *obtaining Ostrowski's inequality for absolutely continuous functions whose derivatives are essentially bounded:*

$$\begin{aligned} (2.10)\quad \left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| &\le \frac{\|f'\|_\infty}{2(b-a)}\left[(x-a)^2 + (b-x)^2\right] \\ &= \left[\frac{1}{4} + \left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right](b-a)\|f'\|_\infty, \end{aligned}$$

for all $x \in [a,b]$. *The constant* $\frac{1}{4}$ *here is best.*

Remark 2. *Ostrowski's inequality for absolutely continuous mappings in terms of* $\|f'\|_\infty$ *basically states that*

$$\begin{aligned} (2.11)\quad -\frac{\|f'\|_\infty}{2(b-a)}\left[(x-a)^2 + (b-x)^2\right] &\le f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \\ &\le \frac{\|f'\|_\infty}{2(b-a)}\left[(x-a)^2 + (b-x)^2\right] \end{aligned}$$

for all $x \in [a,b]$.

Now, if we assume that (2.1) and (2.2) hold, which can easily be obtained for particular functions, then $-\|f'\|_\infty \le m_1(x), m_2(x)$ *and* $M_1(x), M_2(x) \le \|f'\|_\infty$, *which implies:*

$$\begin{aligned} (2.12)\quad & -\frac{\|f'\|_\infty}{2(b-a)}\left[(x-a)^2 + (b-x)^2\right] \\ \le\; & \frac{1}{2(b-a)}\left[m_1(x)(x-a)^2 - M_2(x)(b-x)^2\right] \le f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \\ \le\; & \frac{1}{2(b-a)}\left[M_1(x)(x-a)^2 - m_2(x)(b-x)^2\right] \\ \le\; & \frac{\|f'\|_\infty}{2(b-a)}\left[(x-a)^2 + (b-x)^2\right]. \end{aligned}$$

Thus, the inequality (2.3) may also be regarded as a refinement of the classical Ostrowski result.

An important particular case is $x = \frac{a+b}{2}$ providing the following corollary.

Corollary 3. *Assume that the derivative* $f' : [a,b] \to \mathbb{R}$ *satisfy the conditions:*

$$(2.13)\qquad -\infty < m_1 \le f'(t) \le M_1 < \infty \text{ for a.e. } t \in \left[a, \frac{a+b}{2}\right]$$

and

$$(2.14)\qquad -\infty < m_2 \le f'(t) \le M_2 < \infty \text{ for a.e. } t \in \left(\frac{a+b}{2}, b\right].$$

Then we have the inequalities

$$\begin{aligned} (2.15)\qquad \frac{1}{8}(m_1 - M_2)(b-a) &\le f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)\,dt \\ &\le \frac{1}{8}(M_1 - m_2)(b-a). \end{aligned}$$

The constant $\frac{1}{8}$ is the best in both inequalities.

Finally, if we know some global bounds for the derivative f' on $[a,b]$, then we may state the following corollary.

Corollary 4. *Under the assumptions of Corollary 1, we have the mid-point inequality:*

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \leq \frac{1}{8}(M-m)(b-a). \tag{2.16}$$

The constant $\frac{1}{8}$ is best.

Proof. The inequality is obvious by Corollary 1 putting $x = \frac{a+b}{2}$. We observe that the function $f:[a,b]\to\mathbb{R}$, $f(x) = k\left|x - \frac{a+b}{2}\right|$, $k>0$ is absolutely continuous and $-k \leq f'(t) \leq k$ for all $t\in[a,b]$. Thus, we may choose $M=k$, $m=-k$ and as

$$\begin{aligned} \left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| &= \frac{k(b-a)}{4}, \\ \frac{1}{8}(M-m)(b-a) &= \frac{k(b-a)}{4} \end{aligned}$$

we conclude that the constant $\frac{1}{8}$ is best in (2.16). ∎

Remark 3. *Consider the mid-point inequality*

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \leq c(M-m)(b-a), \tag{2.17}$$

where $c>0$ and $f:[a,b]\to\mathbb{R}$ is an absolutely continuous function satisfying the condition (2.8).

In [12], using a technique based on the Grüss inequality, Dragomir and Wang were able to prove (2.17) with $c=\frac{1}{4}$. In [13], by using a "pre-Grüss" inequality, the authors were able to prove (2.15) with $c=\frac{1}{4\sqrt{3}}$.

Since for the mapping $f:[a,b]\to\mathbb{R}$, $f(x) = k\left|x-\frac{a+b}{2}\right|$, $k>0$, we have (see the proof of the above corollary)

$$\frac{k(b-a)}{4} \leq 2ck(b-a),$$

implying that $c\geq\frac{1}{8}$, the best constant in (2.17) is $\frac{1}{8}$.

Remark 4. *If we assume that $f:[a,b]\subset I\to\mathbb{R}$ is differentiable convex on $\mathring{I}$ and as $f'(a)\leq f'(t)\leq f'(b)$, $t\in[a,b]$, we may deduce the converse of the Hermite-Hadamard inequality*

$$0 \leq \frac{1}{b-a}\int_a^b f(t)\,dt - f\left(\frac{a+b}{2}\right) \leq \frac{1}{8}(b-a)\left[f'(b)-f'(a)\right], \tag{2.18}$$

and the constant $\frac{1}{8}$ is best.

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