THE MEDIAN PRINCIPLE FOR INEQUALITIES AND APPLICATIONS

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Abstract. The "Median Principle" for different integral inequalities of Grüss and Ostrowski type is applied.

1. Introduction

There are many mathematical inequalities whose right hand side may be expressed in terms of the sup-norm of a certain derivative for the involved functions.

For instance, in Numerical Analysis, the integral of a function $f:[a,b]\to\mathbb{R}$ may be represented by

$$\int_a^b f(t)\,dt = A_n(I_n,f) + R_n\left(I_n, f^{(r)}\right), \tag{1.1}$$

where $A_n(I_n,f)$ is the *quadrature rule* defined on a given division

$$I_n : a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b,$$

of the interval $[a,b]$ and $R_n\left(I_n, f^{(r)}\right)$ is the *remainder*, usually expressed in the integral form

$$R_n\left(I_n, f^{(r)}\right) = \int_a^b K_n(I_n,t)\, f^{(r)}(t)\,dt, \tag{1.2}$$

where $K_n(I_n,\cdot):[a,b]\to\mathbb{R}$ is an appropriate *Peano kernel* and $f^{(r)}$ is the $r-$th derivative of f assumed to be essentially bounded on $[a,b]$.

If the integral

$$\int_a^b |K_n(I_n,t)|\,dt$$

can be exactly computed or bounded above by different techniques, then we have the *error estimate*

$$\left|R_n\left(I_n, f^{(r)}\right)\right| \le \left\|f^{(r)}\right\|_{[a,b],\infty} \int_a^b |K_n(I_n,t)|\,dt, \tag{1.3}$$

where

$$\left\|f^{(r)}\right\|_{[a,b],\infty} := ess\sup_{t\in[a,b]}\left|f^{(r)}(t)\right|,$$

that provides a large class of examples of inequalities mentioned above.

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In Analytic Inequalities Theory, the results such as Ostrowski's inequality

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq \left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right](b-a)\|f'\|_{[a,b],\infty}, \quad x\in[a,b], \tag{1.4}$$

provided f is absolutely continuous with $f'\in L_\infty[a,b]$, or Čebyšev's inequality

$$\left|\frac{1}{b-a}\int_a^b f(t)\,g(t)\,dt-\frac{1}{b-a}\int_a^b f(t)\,dt\cdot\frac{1}{b-a}\int_a^b g(t)\,dt\right| \leq \frac{1}{12}(b-a)^2\|f'\|_{[a,b],\infty}\|g'\|_{[a,b],\infty}, \tag{1.5}$$

provided f and g are absolutely continuous with $f',g'\in L_\infty[a,b]$, are other natural examples.

Since, in order to estimate $\left\|f^{(r)}\right\|_{[a,b],\infty}$, in practice it is usually necessary to find the quantities

$$M_r := \sup_{t\in[a,b]} f^{(r)}(t) \quad\text{and}\quad m_r := \inf_{t\in[a,b]} f^{(r)}(t)$$

(as $\left\|f^{(r)}\right\|_{[a,b],\infty} = \max\{|M_r|,|m_r|\}$), the knowledge of $\left\|f^{(r)}\right\|_{[a,b],\infty}$ may be as difficult as the knowledge of M_r and m_r.

Consequently, it is a natural problem in trying to establish inequalities where instead of $\left\|f^{(r)}\right\|_{[a,b],\infty}$, one would have the positive quantity M_r-m_r. We also must note that for functions whose derivatives $f^{(r)}$ have a "modest variation", the quantity M_r-m_r may usually be a lot smaller than $\left\|f^{(r)}\right\|_{[a,b],\infty}$.

We note that there are many examples of inequalities where the bounds are expressed in terms of M_r-m_r, from which we would like to mention only the celebrated result due to Grüss

$$\left|\frac{1}{b-a}\int_a^b f(t)\,g(t)\,dt-\frac{1}{b-a}\int_a^b f(t)\,dt\cdot\frac{1}{b-a}\int_a^b g(t)\,dt\right| \leq \frac{1}{4}(M-m)(N-n), \tag{1.6}$$

provided $f,g\in L[a,b]$ with

$$m\leq f\leq M,\quad n\leq g\leq N \quad \text{a.e. on } [a,b]. \tag{1.7}$$

It is the main purpose of this paper to point out a general strategy for transforming an inequality whose right hand side is expressed in terms of $\left\|f^{(r)}\right\|_{[a,b],\infty}$ into an inequality for which the same side will be expressed in terms of the quantity $M_r-m_r>0$. We call this method the "Median Principle". A formal statement of this method is provided in the next section. Applications for some well-known inequalities are given as well.

2. The Median Principle

Consider the class of polynomials

$$\mathcal{P}_n^0 := \left\{ P_n | P_n(x) = x^n + a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_{n-1} x + a_n,\ a_i \in \mathbb{R} \right\}.$$

The following result, that will be called the "*Median Principle*", holds.

Theorem 1. *Let $f : [a,b] \subset \mathbb{R} \to \mathbb{R}$ be a function so that $f^{(n-1)}$ is absolutely continuous and $f^{(n)} \in L_\infty [a,b]$. Assume that the following inequality holds*

$$L\left(f, f^{(1)}, \dots, f^{(n-1)}, f^{(n)}; a, b\right) \le R\left(\left\|f^{(n)}\right\|_{[a,b],\infty}; a, b\right), \tag{2.1}$$

where $L(\cdot, \cdot, \cdots; a, b) : \mathbb{R}^{(n+1)} \to \mathbb{R}$ is a general function, $R : [0, \infty) \to \mathbb{R}$ and R is monotonic nondecreasing on $[0, \infty)$.

If $g : [a,b] \to \mathbb{R}$ is such that $g^{(n-1)}$ is absolutely continuous and

$$-\infty < \gamma \le g^{(n-1)}(x) \le \Gamma < \infty \text{ for a.e. } x \in [a,b], \tag{2.2}$$

then one has the inequality

$$\begin{aligned} \sup_{P_n \in \mathcal{P}_n^0} L\Bigg(g - \frac{\gamma+\Gamma}{2} P_n, g^{(1)} - \frac{\gamma+\Gamma}{2} P_n^{(1)}, \\ \dots, g^{(n-1)} - \frac{\gamma+\Gamma}{2} P_n^{(n-1)}, g^{(n)} - \frac{\gamma+\Gamma}{2}; a, b\Bigg) \\ \le R\left(\frac{\Gamma-\gamma}{2}; a, b\right). \end{aligned} \tag{2.3}$$

Proof. Let $P_n \in \mathcal{P}_n^{(0)}$ and define $f : [a,b] \to \mathbb{R}$, $f(x) = g(x) - \frac{\gamma+\Gamma}{2} P_n^{(n)}(x)$, where g satisfies (2.2).

Obviously, $f^{(n)} \in L_\infty [a,b]$ and

$$f^{(n)}(x) = g^{(n)}(x) - \frac{\gamma+\Gamma}{2} P_n^{(n)}(x) = g^{(n)}(x) - \frac{\gamma+\Gamma}{2}.$$

Also,

$$\left|f^{(n)}(x)\right| = \left|g^{(n)}(x) - \frac{\gamma+\Gamma}{2}\right| \le \frac{\Gamma-\gamma}{2}$$

giving

$$\left\|f^{(n)}\right\|_{[a,b],\infty} \le \frac{\Gamma-\gamma}{2}.$$

Using the inequality (2.1) and the monotonicity of R, we deduce the desired inequality (2.3). ∎

Remark 1. *Similar results may be obtained if the function L or/and R depend on other function h, l, etc.*

The example provided in the next sections will show how the "Median Principle" works in practice.

3. Inequalities of the $0^{\text{th}}-$Degree

An inequality of the form

$$L(h;a,b) \leq R\left(\|h\|_{[a,b],\infty};a,b\right), \tag{3.1}$$

i.e., no derivatives of the function f are involved, is said to be of the 0^{th} degree.

For example, the following inequality:

$$\left|\int_a^b h(x)\,l(x)\,dx\right| \leq \|h\|_{[a,b],\infty}\int_a^b |l(x)|\,dx, \tag{3.2}$$

provided $h \in L_\infty[a,b]$ and $l \in L_1[a,b]$, and

$$\left|\int_a^b h(x)\,du(x)\right| \leq \|h\|_{[a,b],\infty}\bigvee_a^b(u), \tag{3.3}$$

provided $h \in C[a,b]$ (the class of continuous functions) and $u \in BV[a,b]$ (the class of functions of bounded variation), are inequalities of $0^{th}-$degree. As a generalisation of (3.3), if $h,l \in C[a,b]$ and $u \in BV[a,b]$, then also

$$\left|\int_a^b h(x)\,l(x)\,du(x)\right| \leq \|h\|_{[a,b],\infty}\|l\|_{[a,b],\infty}\bigvee_a^b(u). \tag{3.4}$$

Here and in (3.3), $\bigvee_a^b(u)$ denotes the total variation of u in $[a,b]$.

The following result holds.

Theorem 2. *Let $f,l:[a,b]\to\mathbb{R}$ be such that there exists the constants $m,M\in\mathbb{R}$ with the property*

$$-\infty < m \leq f(x) \leq M < \infty \text{ for a.e. } x\in[a,b], \tag{3.5}$$

and $l \in L_1[a,b]$, such that

$$\int_a^b l(x)\,dx = 0. \tag{3.6}$$

Then we have the inequality

$$\left|\int_a^b f(x)\,l(x)\,dx\right| \leq \frac{1}{2}(M-m)\int_a^b |l(x)|\,dx. \tag{3.7}$$

The constant $\frac{1}{2}$ is sharp.

Proof. Using the "*Median Principle*" for the inequality (3.2) we have

$$\left|\int_a^b \left(f(x)-\frac{m+M}{2}\right)l(x)\,dx\right| \leq \frac{1}{2}(M-m)\int_a^b |l(x)|\,dx \tag{3.8}$$

and since $\int_a^b l(x)\,dx = 0$, we deduce (3.7).

Now, assume that (3.7) holds with a constant $C>0$, i.e.,

$$\left|\int_a^b f(x)\,l(x)\,dx\right| \leq C(M-m)\int_a^b |l(x)|\,dx. \tag{3.9}$$

If we choose $f = l$ and $f : [a,b] \to \mathbb{R}$ where

$$f(x) = \begin{cases} -1, & x \in \left[a, \frac{a+b}{2}\right] \\ \\ 1, & x \in \left(\frac{a+b}{2}, b\right]. \end{cases}$$

Then

$$\left| \int_a^b f(x)\, l(x)\, dx \right| = b - a,$$
$$\int_a^b |l(x)|\, dx = b - a,$$
$$m = -1,\ M = 1$$

and thus, by (3.9) we deduce $C \geq \frac{1}{2}$, and the theorem is then proved. ∎

Corollary 1. *Let $f, g : [a,b] \to \mathbb{R}$ be such that f satisfies (3.5) and $g \in L_1[a,b]$. Then we have the inequalities*

$$\left| \frac{1}{b-a} \int_a^b f(x)\, g(x)\, dx - \frac{1}{b-a} \int_a^b f(x)\, dx \cdot \frac{1}{b-a} \int_a^b g(x)\, dx \right| \leq \frac{1}{2}(M-m) \frac{1}{b-a} \int_a^b \left| g(x) - \frac{1}{b-a} \int_a^b g(y)\, dy \right| dx. \tag{3.10}$$

The constant $\frac{1}{2}$ is sharp in (3.10).

Proof. Follows by (3.7) on choosing $l(x) = g(x) - \frac{1}{b-a} \int_a^b g(y)\, dy$. ∎

Remark 2. *The inequality (3.6) was proved in a different, more complicated, way in* [4]. *Generalisations for abstract Lebesgue integrals, the weighted and discrete cases were obtained in* [1].

The following result also holds.

Theorem 3. *Let $f : [a,b] \to \mathbb{R}$ be a continuous function with the property that*

$$-\infty < m \leq f(x) \leq M < \infty \text{ for a.e. } x \in [a,b], \tag{3.11}$$

and $u \in BV[a,b]$ with the property that

$$u(a) = u(b). \tag{3.12}$$

Then we have the inequality:

$$\left| \int_a^b f(x)\, du(x) \right| \leq \frac{1}{2}(M-m) \bigvee_a^b (u). \tag{3.13}$$

The constant $\frac{1}{2}$ is sharp.

Proof. Using the "Median Principle" for the inequality (3.3) we have

$$\left| \int_a^b \left(f(x) - \frac{m+M}{2} \right) du(x) \right| \leq \frac{1}{2}(M-m) \bigvee_a^b (u). \tag{3.14}$$

Since $\int_a^b du(x) = u(b) - u(a) = 0$, from (3.14) we deduce (3.13).

Now, assume that the inequality (3.13) holds with a constant $D>0$. That is,

$$\left|\int_a^b f(x)\,du(x)\right| \leq D(M-m)\bigvee_a^b(u). \tag{3.15}$$

Consider $a=0$, $b=1$, $f:[0,1]\to\mathbb{R}$, $f(x)=x$, and $u:[0,1]\to\mathbb{R}$ given by

$$u(x)=\begin{cases} 0 & \text{if } \ x=0 \text{ or } x=1. \\ \\ 1 & \text{if } \ x\in(0,1). \end{cases}$$

We have

$$\int_0^1 f(x)\,du(x) = u(x)f(x)\big|_0^1 - \int_0^1 u(x)\,df(x) = -\int_0^1 u(x)\,dx = -1,$$
$$M=1,\ m=0$$

and

$$\bigvee_a^b(u)=2.$$

Then, by (3.15) we deduce $2D\geq 1$ giving $D\geq\frac{1}{2}$, and the theorem is thus proved. ∎

Another result generalizing the above ones also holds.

Theorem 4. *Let $f,l:[a,b]\to\mathbb{R}$ be continuous and f is such that the condition (3.11) holds. If $u\in BV([a,b])$ is such that*

$$\int_a^b l(x)\,du(x)=0, \tag{3.16}$$

then we have the inequality:

$$\left|\int_a^b f(x)\,l(x)\,du(x)\right| \leq \frac{1}{2}(M-m)\|l\|_{[a,b],\infty}\bigvee_a^b(u). \tag{3.17}$$

The constant $\frac{1}{2}$ in (3.17) is sharp.

Proof. Follows by the "Median Principle" applied for the inequality (3.4). The sharpness of the constant follows by Theorem 3 on choosing $l=1$. ∎

As a corollary of the above result, we may state the following Grüss type inequality.

Corollary 2. *Let $f,g\in C[a,b]$ and f is such that (3.11) holds. If $u\in BV[a,b]$ and $u(b)\neq u(a)$, then one has the inequality*

$$\begin{aligned} &\left|\frac{1}{u(b)-u(a)}\int_a^b f(x)\,g(x)\,du(x)\right. \\ &\qquad\left.-\frac{1}{u(b)-u(a)}\int_a^b f(x)\,du(x)\cdot\frac{1}{u(b)-u(a)}\int_a^b g(x)\,du(x)\right| \\ &\leq\frac{1}{2}(M-m)\frac{1}{|u(b)-u(a)|}\left\|g-\frac{1}{u(b)-u(a)}\int_a^b g(y)\,du(y)\right\|_{[a,b],\infty}\bigvee_a^b(u). \end{aligned} \tag{3.18}$$

The constant $\frac{1}{2}$ is sharp in (3.18).

Proof. We choose in Theorem 4, $l : [a,b] \to \mathbb{R}$,

$$l(x) = g(x) - \frac{1}{u(b) - u(a)} \int_a^b g(y)\, du(y), \quad x \in [a,b].$$

Then, obviously

$$\int_a^b l(x)\, du(x) = 0,$$

and by (3.17) we deduce (3.18).

To prove the sharpness of the constant $\frac{1}{2}$ in (3.18), we assume that it holds with a constant $C > 0$, i.e.,

$$\begin{aligned} &\left| \frac{1}{u(b) - u(a)} \int_a^b f(x)\, g(x)\, du(x) \right. \\ &\quad \left. - \frac{1}{u(b) - u(a)} \int_a^b f(x)\, du(x) \cdot \frac{1}{u(b) - u(a)} \int_a^b g(x)\, du(x) \right| \\ &\leq C(M - m) \frac{1}{|u(b) - u(a)|} \left\| g - \frac{1}{u(b) - u(a)} \int_a^b g(y)\, du(y) \right\|_{[a,b],\infty} \bigvee_a^b (u). \end{aligned} \tag{3.19}$$

Let us choose $f = g$, $f : [a,b] \to \mathbb{R}$, with $f(t) = t$ and $u : [a,b] \to \mathbb{R}$ given by

$$u(t) = \begin{cases} -1, & \text{if } t = a \\ 0 & \text{if } t \in (a,b), \\ 1, & \text{if } t = b. \end{cases}$$

Then

$$\begin{aligned} \frac{1}{u(b) - u(a)} \int_a^b f(x)\, g(x)\, du(x) &= \frac{1}{2} \int_a^b u^2 du(x) \\ &= \frac{1}{2} \left[x^2 u(x) \Big|_a^b - 2 \int_a^b x u(x)\, dx \right] \\ &= \frac{b^2 + a^2}{2}, \end{aligned}$$

$$\begin{aligned} \int_a^b f(x)\, du(x) &= \int_a^b g(x)\, du(x) = \frac{1}{2} \int_a^b x du(x) \\ &= x u(x) \Big|_a^b - \int_a^b u(x)\, dx = b + a, \end{aligned}$$

$$\left\| g - \frac{1}{u(b) - u(a)} \int_a^b g(s)\, ds \right\|_\infty = \sup_{x \in [a,b]} \left| x - \frac{a+b}{2} \right| = \frac{b-a}{2},$$

$$\bigvee_a^b (u) = 2, \; M = b, \; m = a$$

and thus, by (3.19), we get

$$\left|\frac{a^2+b^2}{2}-\frac{(a+b)^2}{4}\right| \leq C(b-a)\frac{1}{2}\cdot\frac{(b-a)}{2}\cdot 2,$$

giving $C \geq \frac{1}{2}$, and the corollary is proved. ∎

For other results of this type see [6].

4. Inequalities of the 1^{st}−Degree

An inequality that contains at most the first derivative of the involved functions will be called an inequality of the 1^{st}−degree.

For example, Ostrowski's inequality

$$\left|h(x)-\frac{1}{b-a}\int_a^b h(t)\,dt\right| \leq \left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right]\|h'\|_{[a,b],\infty}(b-a), \quad x\in[a,b]; \tag{4.1}$$

provided h is absolutely continuous and $h' \in L_\infty[a,b]$, is such an inequality,

Also, the generalised-trapezoid inequality:

$$\left|\frac{(x-a)h(a)+(b-x)h(b)}{b-a}-\frac{1}{b-a}\int_a^b h(t)\,dt\right| \leq \left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right]\|h'\|_{[a,b],\infty}(b-a), \quad x\in[a,b]; \tag{4.2}$$

provided h is absolutely continuous, $h' \in L_\infty[a,b]$, is another example of such an inequality.

In both the inequalities above, the constant $\frac{1}{4}$ is sharp in the sense that it cannot be replaced by a smaller constant.

If one would like examples of such inequalities for two functions, the following Ostrowski's inequality obtained in [7] is the most suitable

$$\left|\frac{1}{b-a}\int_a^b h(x)\,l(x)\,dx-\frac{1}{b-a}\int_a^b h(x)\,dx\cdot\frac{1}{b-a}\int_a^b l(x)\,dx\right| \leq \frac{1}{8}(b-a)(M-m)\|h'\|_{[a,b],\infty}, \tag{4.3}$$

provided $-\infty < m \leq h(x) \leq M < \infty$ for a.e. $x \in [a,b]$, and l is absolutely continuous and such that $l' \in L_\infty[a,b]$. The constant $\frac{1}{8}$ is sharp.

Another example of such an inequality is the Čebyšev one

$$\left|\frac{1}{b-a}\int_a^b h(x)\,l(x)\,dx-\frac{1}{b-a}\int_a^b h(x)\,dx\cdot\frac{1}{b-a}\int_a^b l(x)\,dx\right| \leq \frac{1}{12}(b-a)^2\|h'\|_{[a,b],\infty}\|l'\|_{[a,b],\infty}, \tag{4.4}$$

provided h,l are absolutely continuous and $h',l'\in L_{\infty}[a,b]$. The constant $\frac{1}{12}$ here is sharp.

The following perturbed version of Ostrowski's inequality holds.

Theorem 5. *Let $f:[a,b]\to\mathbb{R}$ be an absolutely continuous function on $[a,b]$ such that*

$$-\infty<\gamma\leq f'(x)\leq\Gamma<\infty \text{ for a.e. } x\in[a,b]. \tag{4.5}$$

Then one has the inequality:

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt-\frac{\gamma+\Gamma}{2}\left(x-\frac{a+b}{2}\right)\right| \leq\frac{1}{2}\left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right](\Gamma-\gamma)(b-a) \tag{4.6}$$

for any $x\in[a,b]$.

The constant $\frac{1}{2}$ is sharp.

Proof. Consider the function $h(x)=f(x)-\frac{\gamma+\Gamma}{2}x$, $x\in[a,b]$. Applying the "Median Principle" for Ostrowski's inequality, we get

$$\left|h(x)-\frac{\gamma+\Gamma}{2}x-\frac{1}{b-a}\int_a^b\left(h(t)-\frac{\gamma+\Gamma}{2}t\right)dt\right| \leq\frac{1}{2}(\Gamma-\gamma)\left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right](b-a),$$

which is clearly equivalent to (4.6).

The sharpness of the constant follows by the sharpness of Ostrowski's inequality on choosing $\gamma=-\|f'\|_{[a,b],\infty}$, $\Gamma=\|f'\|_{[a,b],\infty}$. We omit the details. ∎

Remark 3. *For a different proof of this fact, see* [5].

Now, we may give a perturbed version of the generalised trapezoid inequality (4.2) as well (see also [5]).

Theorem 6. *Let f be as in Theorem 5. Then one has the inequality*

$$\left|\frac{(x-a)f(a)+(b-x)f(b)}{b-a}-\frac{\gamma+\Gamma}{2}\left(x-\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b h(t)\,dt\right| \leq\frac{1}{2}(\Gamma-\gamma)\left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right](b-a),$$

for any $x\in[a,b]$.

The constant $\frac{1}{2}$ is sharp.

The proof follows by the inequality (4.2) and we omit the details.

Now, we are able to point out the following perturbation of the second Ostrowski's inequality (4.3).

Theorem 7. *Let $f:[a,b]\to\mathbb{R}$ be an absolutely continuous function on $[a,b]$ such that the derivative $f':[a,b]\to\mathbb{R}$ satisfies the condition*

$$-\infty<\gamma\leq f'(x)\leq\Gamma<\infty \text{ for a.e. } x\in[a,b]. \tag{4.7}$$

If $g:[a,b]\to\mathbb{R}$ is such that

$$-\infty<m\leq g(x)\leq M<\infty \text{ for a.e. } x\in[a,b], \tag{4.8}$$

then we have the inequality:

$$\begin{aligned}\Bigg|\frac{1}{b-a}\int_a^b f(x)g(x)\,dx-\frac{1}{b-a}\int_a^b f(x)\,dx\cdot\frac{1}{b-a}\int_a^b g(x)\,dx \\ -\frac{\gamma+\Gamma}{2}\cdot\frac{1}{b-a}\int_a^b\left(x-\frac{a+b}{2}\right)g(x)\,dx\Bigg| \\ \leq\frac{1}{16}(b-a)(M-m)(\Gamma-\gamma)\end{aligned} \tag{4.9}$$

The constant $\frac{1}{16}$ is best possible.

Proof. Consider $h(x)=f(x)-\frac{\gamma+\Gamma}{2}x$. Applying the "Median Principle" for the Ostrowski's inequality (4.3), we have:

$$\begin{aligned}\Bigg|\frac{1}{b-a}\int_a^b\left(f(x)-\frac{\gamma+\Gamma}{2}x\right)g(x)\,dx \\ -\frac{1}{b-a}\int_a^b\left(f(x)-\frac{\gamma+\Gamma}{2}x\right)dx\cdot\frac{1}{b-a}\int_a^b g(x)\,dx\Bigg| \\ \leq\frac{1}{16}(b-a)(M-m)(\Gamma-\gamma)\end{aligned} \tag{4.10}$$

that after some elementary computations is equivalent to (4.9).

The sharpness of the constant $\frac{1}{16}$ follows by the fact that the constant $\frac{1}{8}$ is sharp in (4.3) on taking $\gamma=-\|f'\|_{[a,b],\infty}$, $\Gamma=\|f'\|_{[a,b],\infty}$. We omit the details. ∎

5. Inequalities of the n^{TH}–Degree

In [2], the authors proved the following identity

$$\begin{aligned}\int_a^b f(t)\,dt=\sum_{k=0}^{n-1}\left[\frac{(b-x)^{k+1}+(-1)^k(x-a)^{k+1}}{(k+1)!}\right]f^{(k)}(x) \\ +(-1)^n\int_a^b K_n(x,t)f^{(n)}(t)\,dt,\end{aligned} \tag{5.1}$$

provided $f^{(n-1)}$ is absolutely continuous on $[a,b]$ and the kernel $K_n:[a,b]^2\to\mathbb{R}$ is given by

$$K_n(x,t):=\begin{cases}\dfrac{(t-a)^n}{n!} & \text{if } a\leq t\leq x\leq b,\\[2ex] \dfrac{(t-b)^n}{n!} & \text{if } a\leq x<t\leq b.\end{cases} \tag{5.2}$$

Using the representation (5.1), they proved the following inequality

$$\left|\int_a^b f(t)\,dt - \sum_{k=0}^{n-1}\left[\frac{(b-x)^{k+1}+(-1)^k(x-a)^{k+1}}{(k+1)!}\right]f^{(k)}(x)\right| \leq \frac{1}{(n+1)!}\left\|f^{(n)}\right\|_\infty\left[(x-a)^{n+1}+(b-x)^{n+1}\right] \tag{5.3}$$

for any $x \in [a,b]$, and in particular, for $x = \frac{a+b}{2}$

$$\left|\int_a^b f(t)\,dt - \sum_{k=0}^{n-1}\left[\frac{1+(-1)^k}{(k+1)!}\right]\cdot\frac{(b-a)^{k+1}}{2^{k+1}}f^{(k)}\left(\frac{a+b}{2}\right)\right| \leq \frac{1}{2^n(n+1)!}\left\|f^{(n)}\right\|_\infty(b-a)^{n+1}. \tag{5.4}$$

The following result holds.

Theorem 8. *Let $f : [a,b] \subset \mathbb{R} \to \mathbb{R}$ be a function such that the derivative $f^{(n-1)}$ is absolutely continuous on $[a,b]$ and there exists the constants $\gamma_n, \Gamma_n \in \mathbb{R}$ so that*

$$-\infty < \gamma_n \leq f^{(n)}(t) \leq \Gamma_n < \infty \text{ for a.e. } t \in [a,b]. \tag{5.5}$$

Then we have the inequality

$$\left|\int_a^b f(t)\,dt - \sum_{k=0}^{n-1}\left[\frac{(b-x)^{k+1}+(-1)^k(x-a)^{k+1}}{(k+1)!}\right]f^{(k)}(x) - (-1)^n\frac{\Gamma_n+\gamma_n}{2}\left[\frac{(x-a)^{n+1}+(-1)^{n+1}(b-x)^{n+1}}{(n+1)!}\right]\right| \leq \frac{\Gamma_n-\gamma_n}{2(n+1)!}\left[(x-a)^{n+1}+(b-x)^{n+1}\right]. \tag{5.6}$$

Proof. Observe, by (5.1), we have that

$$\begin{aligned}\int_a^b K_n(x,t)\,dt &= \frac{1}{n!}\left[\int_a^x (t-a)^n\,dt + \int_x^b (t-b)^n\,dt\right] \\ &= \frac{(x-a)^{n+1}-(x-b)^{n+1}}{(n+1)!} \\ &= \frac{(x-a)^{n+1}+(-1)^{n+1}(b-x)^{n+1}}{(n+1)!}.\end{aligned} \tag{5.7}$$

Taking the modulus in (5.1) and using the fact that

$$\left|f^{(n+1)}(t) - \frac{\Gamma_n+\gamma_n}{2}\right| \leq \frac{\Gamma_n-\gamma_n}{2} \quad \text{for a.e. } t \in [a,b]$$

and

$$\int_a^b |K_n(x,t)|\,dt = \frac{1}{(n+1)!}\left[(x-a)^{n+1}+(b-x)^{n+1}\right],$$

we easily deduce (5.6). ∎

Corollary 3. *With the assumptions in Theorem 8, one has the inequality:*

$$\left|\int_a^b f(t)\,dt-\sum_{k=0}^{n-1}\left[\frac{1+(-1)^k}{(k+1)!}\right]\cdot\frac{(b-a)^{k+1}}{2^{k+1}}f^{(k)}\left(\frac{a+b}{2}\right)\right.$$
$$\left.-(-1)^n\frac{\Gamma_n+\gamma_n}{2}\left[\frac{1+(-1)^{n+1}}{(n+1)!}\right]\frac{(b-a)^{n+1}}{2^{n+1}}\right|\leq\frac{\Gamma_n-\gamma_n}{2^{n+1}(n+1)!}(b-a)^{n+1}. \tag{5.8}$$

In [3], the authors also obtained the following identity

$$\int_a^b f(t)\,dt=\sum_{k=0}^{n-1}\frac{1}{(k+1)!}\left[(x-a)^{k+1}f^{(k)}(a)+(-1)^k(b-x)^{k+1}f^{(k)}(b)\right]+\frac{1}{n!}\int_a^b(x-t)^n f^{(n)}(t)\,dt, \tag{5.9}$$

provided $f^{(n-1)}$ is absolutely continuous on $[a,b]$.

By the use of this identity, they obtained the inequality

$$\left|\int_a^b f(t)\,dt-\sum_{k=0}^{n-1}\frac{1}{(k+1)!}\left[(x-a)^{k+1}f^{(k)}(a)+(-1)^k(b-x)^{k+1}f^{(k)}(b)\right]\right|\leq\frac{1}{(n+1)!}\left\|f^{(n)}\right\|_\infty\left[(x-a)^{n+1}+(b-x)^{n+1}\right], \tag{5.10}$$

for any $x\in[a,b]$.

In particular, for $x=\frac{a+b}{2}$ we get the inequality:

$$\left|\int_a^b f(t)\,dt-\sum_{k=0}^{n-1}\frac{1}{(k+1)!}\left(\frac{b-a}{2}\right)^{k+1}\left[f^{(k)}(a)+(-1)^k f^{(k)}(b)\right]\right|\leq\frac{1}{2^n(n+1)!}\left\|f^{(n)}\right\|_\infty(b-a)^{n+1}. \tag{5.11}$$

Finally, we may state the following result.

Theorem 9. *With the assumptions in Theorem 8, we have the inequality*

$$\left|\int_a^b f(t)\,dt-\sum_{k=0}^{n-1}\left[\frac{(b-x)^{k+1}+(-1)^k(x-a)^{k+1}}{(k+1)!}\right]f^{(k)}(x)\right.$$
$$\left.-\frac{1}{(n+1)!}\cdot\frac{\Gamma_n+\gamma_n}{2}\left[(x-a)^{n+1}+(-1)^n(b-x)^{n+1}\right]\right|\leq\frac{\Gamma_n-\gamma_n}{2(n+1)!}\left[(x-a)^{n+1}+(b-x)^{n+1}\right], \tag{5.12}$$

for any $x\in[a,b]$.

In particular, for $x=\frac{a+b}{2}$, we have the corollary

Corollary 4. *With the above assumptions, we have*

$$\left|\int_a^b f(t)\,dt - \sum_{k=0}^{n-1} \frac{1}{(k+1)!}\left(\frac{b-a}{2}\right)^{k+1}\left[f^{(k)}(a) + (-1)^k f^{(k)}(b)\right] - \frac{\Gamma_n + \gamma_n}{2}\left[\frac{1+(-1)^n}{(n+1)!}\right]\left(\frac{b-a}{2}\right)^{n+1}\right| \leq \frac{1}{2^{n+1}(n+1)!}(\Gamma_n - \gamma_n)(b-a)^{n+1}. \tag{5.13}$$

The interested reader may find many other examples which can be treated in a similar fashion. We omit the details.

References

[1] P. Cerone and S.S. Dragomir, A Refinement of the Grüss Inequality and Applications, RGMIA Res. Rep. Coll.,**5**(2002), No. 2, Article 14. [ON LINE `http://rgmia.vu.edu.au/v5n2.html`].

[2] P. Cerone, S.S. Dragomir and J. Roumeliotis, Some Ostrowski type inequalities for n-time differentiable mappings and applications, *Demonstratio Mathematica*, **32** (2) (1999), 697-712

[3] P. Cerone, S.S. Dragomir, J. Roumeliotis and J. Sunde, A new generalistion of the trpezoid formula for n-time differentiable mappings and applications, *Demonstratio Mathematica*, **33**(4) (2000), 719 - 736.

[4] X.-L. Cheng and J. Sun, Note on the perturbed trapezoid inequality, J. Ineq. Pure. & Appl. Math., 3(2002), No. 2, Article 29. [ON LINE: `http://jipam.vu.edu.au/v3n2/046_01.html`]

[5] S.S. Dragomir, Improvements of Ostrowski and Generalised Trapezoid Inequality in Terms of the Upper and Lower Bounds of the First Derivative, RGMIA Res. Rep. Coll., **5**(2002), Supplement, Article 10, [ON LINE: `http://rgmia.vu.edu.au/v5(E).html`]

[6] S.S. Dragomir, Sharp bounds of Čebyšev functional for Stieltjes integrals and application, (in preparation).

[7] A. Ostrowski, On an integral inequality, *Aequat. Math.*, **4**(1970), 358-373.

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