NEW ESTIMATES OF THE ČEBYŠEV FUNCTIONAL FOR STIELTJES INTEGRALS AND APPLICATIONS

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Abstract. In this paper, some new estimates of the Čebyšev functional considered in [1] are provided. Applications for quadrature formulae in approximating the Riemann-Stieltjes integral are also given.

1. Introduction

In [1], the author has considered the following Čebyšev's functional for the Stieltjes integral

$$T(f,g;u) := \frac{1}{u(b)-u(a)}\int_a^b f(t)\,g(t)\,du(t) - \frac{1}{u(b)-u(a)}\int_a^b f(t)\,du(t)\cdot\frac{1}{u(b)-u(a)}\int_a^b g(t)\,du(t), \tag{1.1}$$

where $f,g\in C[a,b]$ (are continuous on $[a,b]$) and $u\in BV[a,b]$ (is of bounded variation on $[a,b]$) with $u(b)\neq u(a)$.

If there exists the constants m,M such that

$$m\le f(t)\le M \quad \text{for each} \quad t\in[a,b], \tag{1.2}$$

then the following Grüss type inequality holds (see [1])

$$|T(f,g;u)| \le \frac{1}{2}(M-m)\frac{1}{|u(b)-u(a)|} \times\left\|g-\frac{1}{u(b)-u(a)}\int_a^b g(s)\,du(s)\right\|_\infty \bigvee_a^b(u). \tag{1.3}$$

The constant $\frac{1}{2}$ cannot be replaced be a smaller constant.

If we restrict the class for u assuming it is monotonic nondecreasing on $[a,b]$ with $u(b)>u(a)$, then the following better result holds

$$|T(f,g;u)| \le \frac{1}{2}(M-m)\frac{1}{u(b)-u(a)} \times\int_a^b\left|g(t)-\frac{1}{u(b)-u(a)}\int_a^b g(s)\,du(s)\right|du(t). \tag{1.4}$$

Here the constant $\frac{1}{2}$ is also sharp in the sense mentioned above.

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It is known that if $p:[a,b]\to\mathbb{R}$ is Riemann integrable on $[a,b]$ and $v:[a,b]\to\mathbb{R}$ is Lipschitzian, then the Riemann Stieltjes integral $\int_a^b p(t)\,dv(t)$ exists and one has the inequality

$$\left|\int_a^b p(t)\,dv(t)\right|\leq L\int_a^b |p(t)|\,dt, \tag{1.5}$$

where $L>0$ is the Lipschitz constant for v. This fact enables us to consider the above Čebyšev functional for $f,g\in R[a,b]$ (Riemann integrable on $[a,b]$) and $u\in Lip_L[a,b]$ (Lipschitzian with the constant $L>0$).

In [1], the author obtained the following inequality as well

$$\begin{aligned}|T(f,g;u)|\leq{}&\frac{1}{2}L(M-m)\frac{1}{|u(b)-u(a)|}\\&\times\int_a^b\left|g(t)-\frac{1}{u(b)-u(a)}\int_a^b g(s)\,du(s)\right|dt,\end{aligned} \tag{1.6}$$

provided $f,g\in R[a,b]$, $u\in Lip_L[a,b]$, $u(b)\neq u(a)$ and f satisfies the condition (1.2). The constant $\frac{1}{2}$ cannot be replaced by a smaller constant in (1.6).

In this paper some further results for the Čebyšev functional (1.1) are obtained. Applications for quadrature rules are also emphasized.

For some recent inequalities for Stieltjes integral see [2]-[5].

2. The Results

The following result holds.

Theorem 1. *Let $f,g:[a,b]\to\mathbb{R}$ be such that f is of $r-H-$Hölder type on $[a,b]$, i.e.,*

$$|f(t)-f(s)|\leq H|t-s|^r \quad \text{for any} \quad t,s\in[a,b], \tag{2.1}$$

and g is continuous on $[a,b]$. If $u:[a,b]\to\mathbb{R}$ is of bounded variation on $[a,b]$ with $u(a)\neq u(b)$, then we have the inequality

$$\begin{aligned}|T(f,g;u)|\leq{}&\frac{H(b-a)^r}{2^r}\cdot\frac{1}{|u(b)-u(a)|}\\&\times\left\|g-\frac{1}{u(b)-u(a)}\int_a^b g(s)\,du(s)\right\|_\infty\bigvee_a^b(u),\end{aligned} \tag{2.2}$$

where $\bigvee_a^b(u)$ denotes the total variation of u on $[a,b]$.

Proof. It is easy to see, by simple computation with the Stieltjes integral, that the following equality

$$\begin{aligned}T(f,g;u)={}&\frac{1}{u(b)-u(a)}\int_a^b\left[f(t)-f\left(\frac{a+b}{2}\right)\right]\\&\times\left[g(t)-\frac{1}{u(b)-u(a)}\int_a^b g(s)\,du(s)\right]du(t)\end{aligned} \tag{2.3}$$

holds.

Using the known inequality

$$\left|\int_a^b p(t)\,dv(t)\right| \le \sup_{t\in[a,b]} |p(t)| \bigvee_a^b (v) \tag{2.4}$$

provided $p \in C[a,b]$ and $v \in BV[a,b]$, we have, by (2.3), that

$$\begin{aligned}
|T(f,g;u)| &\le \sup_{t\in[a,b]} \left|\left[f(t) - f\left(\frac{a+b}{2}\right)\right]\left[g(t) - \frac{1}{u(b)-u(a)}\int_a^b g(s)\,du(s)\right]\right| \\
&\qquad \times \frac{1}{|u(b)-u(a)|}\bigvee_a^b (u) \\
&\le \sup_{t\in[a,b]} \left|f(t) - f\left(\frac{a+b}{2}\right)\right| \left\|g - \frac{1}{u(b)-u(a)}\int_a^b g(s)\,du(s)\right\|_\infty \\
&\qquad \times \frac{1}{|u(b)-u(a)|}\bigvee_a^b (u) \\
&\le L\left(\frac{b-a}{2}\right)^r \left\|g - \frac{1}{u(b)-u(a)}\int_a^b g(s)\,du(s)\right\|_\infty \\
&\qquad \times \frac{1}{|u(b)-u(a)|}\bigvee_a^b (u),
\end{aligned}$$

and the inequality (2.2) is proved. ∎

The following corollary may be useful in applications.

Corollary 1. *Let f be Lipschitzian with the constant $L > 0$, i.e.,*

$$|f(t) - f(s)| \le L|t-s| \quad \text{for any} \quad t,s \in [a,b], \tag{2.5}$$

and u, g are as in Theorem 1. Then we have the inequality

$$\begin{aligned}
|T(f,g;u)| &\le \frac{1}{2}\frac{L(b-a)}{|u(b)-u(a)|} \\
&\qquad \times \left\|g - \frac{1}{u(b)-u(a)}\int_a^b g(s)\,du(s)\right\|_\infty \bigvee_a^b (u).
\end{aligned} \tag{2.6}$$

The constant $\frac{1}{2}$ cannot be replaced by a smaller constant.

Proof. The inequality (2.6) follows by (2.2) for $r = 1$. It remains to prove only the sharpness of the constant $\frac{1}{2}$.

Consider the functions $f = g$, where $f : [a,b] \to \mathbb{R}$, $f(t) = t$ and $u : [a,b] \to \mathbb{R}$, given by

$$u(t) = \begin{cases} -1 & \text{if } \ t = a, \\ 0 & \text{if } \ t \in (a,b), \\ 1 & \text{if } \ t = b. \end{cases} \tag{2.7}$$

Then, f is Lipschitzian with the constant $L = 1$, g is continuous and u is of bounded variation.

If we assume that the inequality (2.6) holds with a constant $C>0$, i.e.,

$$|T(f,g;u)| \leq CL(b-a)\left\|g-\frac{1}{u(b)-u(a)}\int_a^b g(s)\,du(s)\right\|_\infty \bigvee_a^b(u), \tag{2.8}$$

and since

$$\begin{aligned}\frac{1}{u(b)-u(a)}\int_a^b f(t)\,g(t)\,du(t) &= \frac{1}{2}\int_a^b t^2 du(t)\\ &= \frac{1}{2}\left[t^2u(t)\,\Big|_a^b - 2\int_a^b tu(t)\,dt\right]\\ &= \frac{b^2+a^2}{2},\end{aligned}$$

$$\begin{aligned}\frac{1}{u(b)-u(a)}\int_a^b f(t)\,du(t) &= \frac{1}{u(b)-u(a)}\int_a^b g(t)\,du(t)\\ &= \frac{1}{2}\int_a^b tdu(t)\\ &= \frac{1}{2}\left[tu(t)\,\Big|_a^b - \int_a^b u(t)\,dt\right]\\ &= \frac{b+a,}{2}\end{aligned}$$

$$\left\|g-\frac{1}{u(b)-u(a)}\int_a^b g(s)\,du(s)\right\|_\infty = \sup_{t\in[a,b]}\left|t-\frac{a+b}{2}\right| = \frac{b-a}{2}$$

and $\bigvee_a^b(u)=2$, then, by (2.8), we have

$$\left|\frac{b^2+a^2}{2}-\left(\frac{a+b}{2}\right)^2\right| \leq C\frac{(b-a)}{2}\frac{b-a}{2}\cdot 2,$$

giving $C\geq\frac{1}{2}$. ∎

The following result concerning monotonic function $u:[a,b]\to\mathbb{R}$ also holds.

Theorem 2. *Assume that f and g are as in Theorem 1. If $u:[a,b]\to\mathbb{R}$ is monotonic nondecreasing on $[a,b]$ with $u(b)>u(a)$, then we have the inequalities:*

$$\begin{aligned}|T(f,g;u)| &\leq \frac{H}{u(b)-u(a)}\int_a^b\left|t-\frac{a+b}{2}\right|^r\\ &\qquad\times\left|g(t)-\frac{1}{u(b)-u(a)}\int_a^b g(s)\,du(s)\right|du(t)\\ &\leq \frac{H(b-a)^r}{2^r[u(b)-u(a)]}\\ &\qquad\times\int_a^b\left|g(t)-\frac{1}{u(b)-u(a)}\int_a^b g(s)\,du(s)\right|du(t).\end{aligned} \tag{2.9}$$

Proof. Using the known inequality

$$\left|\int_a^b p(t)\,dv(t)\right| \leq \int_a^b |p(t)|\,dv(t), \tag{2.10}$$

provided $p \in C[a,b]$ and v is monotonic nondecreasing on $[a,b]$, we have, by (2.3), the following estimate:

$$\begin{aligned}
|T(f,g;u)| &\leq \frac{1}{u(b)-u(a)} \int_a^b \left| \left(f(t) - f\left(\frac{a+b}{2}\right)\right)\right. \\
&\quad \times \left.\left(g(t) - \frac{1}{u(b)-u(a)} \int_a^b g(s)\,du(s)\right)\right| du(t) \\
&\leq \frac{H}{u(b)-u(a)} \int_a^b \left| t - \frac{a+b}{2}\right|^r \\
&\quad \times \left| g(t) - \frac{1}{u(b)-u(a)} \int_a^b g(s)\,du(s)\right| du(t) \\
&\leq \frac{H}{u(b)-u(a)} \sup_{t\in[a,b]} \left| t - \frac{a+b}{2}\right|^r \\
&\quad \times \int_a^b \left| g(t) - \frac{1}{u(b)-u(a)} \int_a^b g(s)\,du(s)\right| du(t)
\end{aligned}$$

which simply provides (2.9). ∎

The particular case of Lipschitzian functions that is relevant for applications is embodied in the following corollary.

Corollary 2. *Assume that f is $L-$Lipschitzian, g is continuous and u is monotonic nondecreasing on $[a,b]$ with $u(b) > u(a)$. Then we have the inequalities*

$$\begin{aligned}
|T(f,g;u)| &\leq \frac{L}{u(b)-u(a)} \int_a^b \left| t - \frac{a+b}{2}\right| \\
&\quad \times \left| g(t) - \frac{1}{u(b)-u(a)} \int_a^b g(s)\,du(s)\right| du(t) \\
&\leq \frac{1}{2}\cdot\frac{L(b-a)}{u(b)-u(a)} \\
&\quad \times \int_a^b \left| g(t) - \frac{1}{u(b)-u(a)} \int_a^b g(s)\,du(s)\right| du(t).
\end{aligned} \tag{2.11}$$

The first inequality is sharp. The constant $\frac{1}{2}$ in the second inequality cannot be replaced by a smaller constant.

Proof. The inequality (2.11) follows by (2.9) on choosing $r=1$. Assume that (2.11) holds with the constants $D,E>0$, i.e.,

$$\begin{aligned}(2.12)\qquad &|T(f,g;u)| \\ &\leq \frac{LD}{u(b)-u(a)}\int_a^b \left|t-\frac{a+b}{2}\right| \\ &\qquad\times\left|g(t)-\frac{1}{u(b)-u(a)}\int_a^b g(s)\,du(s)\right| du(t) \\ &\leq \frac{LE(b-a)}{u(b)-u(a)}\int_a^b \left|g(t)-\frac{1}{u(b)-u(a)}\int_a^b g(s)\,du(s)\right| du(t).\end{aligned}$$

Consider the functions $f=g$, where $f:[a,b]\to\mathbb{R}$, $f(t)=t$ and u is as given by (2.7). Then, obviously, f is Lipschitzian with the constant $L=1$, g is continuous and u is monotonic nondecreasing on $[a,b]$.

Since, we know, for these functions

$$T(f,g;u)=\frac{(b-a)^2}{4},$$

and

$$\begin{aligned}&\int_a^b \left|t-\frac{a+b}{2}\right|\left|g(t)-\frac{1}{u(b)-u(a)}\int_a^b g(s)\,du(s)\right| du(t) \\ &=\int_a^b \left(t-\frac{a+b}{2}\right)^2 du(t) \\ &=\left(t-\frac{a+b}{2}\right)^2 u(t)\bigg|_a^b - 2\int_a^b \left(t-\frac{a+b}{2}\right) u(t)\,dt \\ &=\left(\frac{b-a}{2}\right)^2+\left(\frac{b-a}{2}\right)^2=\frac{(b-a)^2}{2},\end{aligned}$$

$$\begin{aligned}&\int_a^b \left|g(t)-\frac{1}{u(b)-u(a)}\int_a^b g(s)\,du(s)\right| du(t) \\ &=\int_a^b \left|t-\frac{a+b}{2}\right| du(t) \\ &=\int_a^{\frac{a+b}{2}} \left(\frac{a+b}{2}-t\right) du(t)+\int_{\frac{a+b}{2}}^b \left(t-\frac{a+b}{2}\right) du(t) \\ &=\left[u(t)\left(\frac{a+b}{2}-t\right)\right]\bigg|_a^{\frac{a+b}{2}}+\int_a^{\frac{a+b}{2}} u(t)\,dt \\ &\qquad+\left[\left(t-\frac{a+b}{2}\right)u(t)\right]\bigg|_{\frac{a+b}{2}}^b-\int_{\frac{a+b}{2}}^b u(t)\,dt \\ &=b-a,\end{aligned}$$

then by (2.12) we deduce

$$\frac{(b-a)^2}{4} \leq \frac{D}{2} \cdot \frac{(b-a)^2}{2} \leq \frac{E(b-a)^2}{2}$$

giving $D \geq 1$ and $E \geq \frac{1}{2}$. ∎

Another natural possibility to obtain bounds for the functional $T(f,g;u)$, where u is Lipschitzian with the constant $K > 0$, is embodied in the following theorem.

Theorem 3. *Assume that $f : [a,b] \to \mathbb{R}$ is of $r-H-$Hölder type on $[a,b]$. If $g : [a,b] \to \mathbb{R}$ is Riemann integrable on $[a,b]$ and $u : [a,b] \to \mathbb{R}$ is Lipschitzian with the constant $K > 0$ and $u(a) \neq u(b)$, then one has the inequalities:*

$$\begin{aligned} & |T(f,g;u)| \\ & \leq \frac{HK}{|u(b)-u(a)|} \int_a^b \left| t - \frac{a+b}{2} \right|^r \\ & \quad \times \left| g(t) - \frac{1}{u(b)-u(a)} \int_a^b g(s)\,du(s) \right| dt \\ & \leq \begin{cases} \frac{HK(b-a)^{r+1}}{2^r(r+1)|u(b)-u(a)|} \left\| g - \frac{1}{u(b)-u(a)} \int_a^b g(s)\,du(s) \right\|_\infty ; \\ \\ \frac{HK(b-a)^{r+\frac{1}{q}}}{2^r(qr+1)^{\frac{1}{q}}|u(b)-u(a)|} \left\| g - \frac{1}{u(b)-u(a)} \int_a^b g(s)\,du(s) \right\|_p \\ \hfill \text{if } p > 1,\ \frac{1}{p} + \frac{1}{q} = 1; \\ \\ \frac{HK(b-a)^r}{2^r|u(b)-u(a)|} \left\| g - \frac{1}{u(b)-u(a)} \int_a^b g(s)\,du(s) \right\|_1 . \end{cases} \end{aligned} \tag{2.13}$$

Proof. Using the inequality (1.5) and the identity (2.3), we have successively

$$\begin{aligned} |T(f,g;u)| & \leq \frac{K}{|u(b)-u(a)|} \int_a^b \left| f(t) - f\left(\frac{a+b}{2}\right) \right| \\ & \quad \times \left| g(t) - \frac{1}{u(b)-u(a)} \int_a^b g(s)\,du(s) \right| dt \\ & \leq \frac{KH}{|u(b)-u(a)|} \int_a^b \left| t - \frac{a+b}{2} \right|^r \\ & \quad \times \left| g(t) - \frac{1}{u(b)-u(a)} \int_a^b g(s)\,du(s) \right| dt \end{aligned} \tag{2.14}$$

and the first inequality in (2.13) is proved.

Since

$$\begin{aligned} & \int_a^b \left| t - \frac{a+b}{2} \right|^r \left| g(t) - \frac{1}{u(b)-u(a)} \int_a^b g(s)\,du(s) \right| dt \\ & \leq \left\| g - \frac{1}{u(b)-u(a)} \int_a^b g(s)\,du(s) \right\|_\infty \int_a^b \left| t - \frac{a+b}{2} \right|^r dt \\ & = \frac{(b-a)^{r+1}}{2^r(r+1)} \left\| g - \frac{1}{u(b)-u(a)} \int_a^b g(s)\,du(s) \right\|_\infty , \end{aligned}$$

then by (2.14) we deduce the first part in the second inequality in (2.13).

By Hölder's integral inequality we have

$$\begin{aligned}
&\int_a^b \left|t-\frac{a+b}{2}\right|^r \left|g(t)-\frac{1}{u(b)-u(a)}\int_a^b g(s)\,du(s)\right| dt \\
&\le \left(\int_a^b \left|t-\frac{a+b}{2}\right|^{qr} dt\right)^{\frac{1}{q}} \left(\int_a^b \left|g(t)-\frac{1}{u(b)-u(a)}\int_a^b g(s)\,du(s)\right|^p dt\right)^{\frac{1}{p}} \\
&= \left[\frac{(b-a)^{qr+1}}{2^{qr}(qr+1)}\right]^{\frac{1}{q}} \left\|g-\frac{1}{u(b)-u(a)}\int_a^b g(s)\,du(s)\right\|_p \\
&= \frac{(b-a)^{r+\frac{1}{q}}}{2^r(qr+1)^{\frac{1}{q}}} \left\|g-\frac{1}{u(b)-u(a)}\int_a^b g(s)\,du(s)\right\|_p .
\end{aligned}$$

Using (2.14), we deduce the second part of the second inequality in (2.13).

Finally, since

$$\left|t-\frac{a+b}{2}\right|^r \le \left(\frac{b-a}{2}\right)^r, \quad t\in[a,b],$$

we deduce

$$\begin{aligned}
&\int_a^b \left|t-\frac{a+b}{2}\right|^r \left|g(t)-\frac{1}{u(b)-u(a)}\int_a^b g(s)\,du(s)\right| dt \\
&\qquad\qquad \le \frac{(b-a)^r}{2^r}\left\|g-\frac{1}{u(b)-u(a)}\int_a^b g(s)\,du(s)\right\|_1
\end{aligned}$$

and the theorem is completely proved. ∎

Corollary 3. *If f is Lipschitzian with the constant L and g and u are as in Theorem 3, then we have the inequalities:*

$$\begin{aligned}
&|T(f,g;u)| \\
&\le \frac{LK}{|u(b)-u(a)|}\int_a^b \left|t-\frac{a+b}{2}\right| \\
&\quad\times\left|g(t)-\frac{1}{u(b)-u(a)}\int_a^b g(s)\,du(s)\right| dt \\
&\le \begin{cases}
\frac{LK(b-a)^2}{4|u(b)-u(a)|}\left\|g-\frac{1}{u(b)-u(a)}\int_a^b g(s)\,du(s)\right\|_\infty; \\[2ex]
\frac{LK(b-a)^{1+\frac{1}{q}}}{2(q+1)^{\frac{1}{q}}|u(b)-u(a)|}\left\|g-\frac{1}{u(b)-u(a)}\int_a^b g(s)\,du(s)\right\|_p \\
\qquad\qquad\qquad \text{if } p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\[2ex]
\frac{LK(b-a)}{2|u(b)-u(a)|}\left\|g-\frac{1}{u(b)-u(a)}\int_a^b g(s)\,du(s)\right\|_1 .
\end{cases}
\end{aligned} \tag{2.15}$$

The first inequality in (2.15) is sharp.

The constants $\frac{1}{4}$ and $\frac{1}{2}$ in the second branch of the second inequality cannot be replaced by smaller constants, respectively.

Proof. The inequality (2.15) follows obviously from (2.13) on choosing $r=1$.

Now, assume that the following inequalities hold

$$
\begin{aligned}
(2.16)\qquad & |T(f,g;u)| \\
&\leq \frac{CLK}{|u(b)-u(a)|}\int_a^b \left|t-\frac{a+b}{2}\right| \\
&\quad\times\left|g(t)-\frac{1}{u(b)-u(a)}\int_a^b g(s)\,du(s)\right| dt \\
&\leq \begin{cases} \dfrac{DLK(b-a)^2}{|u(b)-u(a)|}\left\|g-\dfrac{1}{u(b)-u(a)}\displaystyle\int_a^b g(s)\,du(s)\right\|_\infty; \\[2ex] \dfrac{ELK(b-a)^{1+\frac{1}{q}}}{(q+1)^{\frac{1}{q}}|u(b)-u(a)|}\left\|g-\dfrac{1}{u(b)-u(a)}\displaystyle\int_a^b g(s)\,du(s)\right\|_p \\ \hfill \text{if } p>1,\ \frac{1}{p}+\frac{1}{q}=1; \end{cases}
\end{aligned}
$$

with $C,D,E>0$.

Consider the functions $f,g,u:[a,b]\to\mathbb{R}$, defined by $f(t)=t-\frac{a+b}{2}$, $u(t)=t$ and

$$
g(t)=\begin{cases} -1 & \text{if } t\in\left[a,\frac{a+b}{2}\right], \\ 1 & \text{if } t\in\left(\frac{a+b}{2},b\right]. \end{cases}
$$

Then both f and u are Lipschitzian with the constant $L=K=1$ and g is Riemann integrable on $[a,b]$.

We obviously have

$$
\begin{aligned}
|T(f,g;u)| &= \frac{1}{b-a}\int_a^b f(t)g(t)\,dt-\frac{1}{b-a}\int_a^b f(t)\,dt\cdot\frac{1}{b-a}\int_a^b g(t)\,dt \\
&= \frac{1}{b-a}\left[\int_a^{\frac{a+b}{2}}\left(\frac{a+b}{2}-t\right)dt+\int_{\frac{a+b}{2}}^b\left(t-\frac{a+b}{2}\right)dt\right] \\
&= \frac{b-a}{4},
\end{aligned}
$$

$$
\begin{aligned}
\int_a^b\left|t-\frac{a+b}{2}\right|\left|g(t)-\frac{1}{u(b)-u(a)}\int_a^b g(s)\,du(s)\right|dt &= \int_a^b\left|t-\frac{a+b}{2}\right|dt \\
&= \frac{(b-a)^2}{4}
\end{aligned}
$$

$$
\left\|g-\frac{1}{u(b)-u(a)}\int_a^b g(s)\,du(s)\right\|_\infty=\|g\|_\infty=1
$$

and

$$
\left\|g-\frac{1}{u(b)-u(a)}\int_a^b g(s)\,du(s)\right\|_p=\|g\|_p=(b-a)^{\frac{1}{p}}.
$$

Consequently, by (2.16), one has

$$\frac{b-a}{4} \leq \frac{C}{b-a} \frac{(b-a)^2}{4} \leq \begin{cases} \dfrac{D(b-a)^2}{b-a} \cdot 1 \\[2ex] \dfrac{E(b-a)^2}{(q+1)^{\frac{1}{q}}(b-a)} \end{cases}$$

giving

$$\frac{1}{4} \leq \frac{C}{4} \leq \begin{cases} D \\[2ex] \dfrac{E}{(q+1)^{\frac{1}{q}}}, \quad q > 1. \end{cases}$$

From the first inequality we obtain $C \geq 1$. Also, we get $D \geq \frac{1}{4}$ and $E \geq \frac{(q+1)^{\frac{1}{q}}}{4}$. Letting $q \to 1+$, we deduce $E \geq \frac{1}{2}$ and the corollary is proved. ∎

3. A Quadrature Formula

Let us consider the partition of the interval $[a,b]$ given by

$$I_n : a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b. \tag{3.1}$$

Denote $v(I_n) := \max\{h_i | i = 0,\dots,n-1\}$ where $h_i := x_{i+1} - x_i$, $i = 0,\dots,n-1$.

Consider now the quadrature rule

$$S_n(f,g;u,I_n) := \sum_{i=0}^{n-1} \frac{1}{u(x_{i+1}) - u(x_i)} \int_{x_i}^{x_{i+1}} f(t)\,du(t) \cdot \int_{x_i}^{x_{i+1}} g(t)\,du(t) \tag{3.2}$$

provided $f,g \in C[a,b]$, $u \in BV[a,b]$ and $u(x_{i+1}) \neq u(x_i)$, $i = 0,\dots,n-1$.

We may now state the following result in approximating the Stieltjes integral

$$\int_a^b f(t)\,g(t)\,du(t).$$

Theorem 4. *Let $f,g : [a,b] \to \mathbb{R}$ be such that f is of $r-H-$Hölder type on $[a,b]$ (see Theorem 1), g is continuous on $[a,b]$, I_n is as above and $u : [a,b] \to \mathbb{R}$ is of bounded variation on $[a,b]$ with $u(x_{i+1}) \neq u(x_i)$, $i = 0,\dots,n-1$. Then we have the representation*

$$\int_a^b f(t)\,g(t)\,du(t) = S_n(f,g;u,I_n) + R_n(f,g;u,I_n), \tag{3.3}$$

where the quadrature $S_n(f,g;u,I_n)$ is as defined in (3.2) and the remainder $R_n(f,g;u,I_n)$ satisfies the estimate

$$\begin{aligned} |R_n(f,g;u,I_n)| &\leq \frac{H}{2^r}[v(f,I_n)]^r \\ &\quad \times \max_{i=\overline{0,n-1}} \left\| g - \frac{1}{u(x_{i+1}) - u(x_i)} \int_{x_i}^{x_{i+1}} g(s)\,du(s) \right\|_{[x_i,x_{i+1}],\infty} \bigvee_a^b (u). \end{aligned} \tag{3.4}$$

Proof. Applying the inequality (2.2) on the interval $[x_i, x_{i+1}]$ to get

$$\begin{aligned}(3.5)\quad & \left|\int_{x_i}^{x_{i+1}} f(t)\,g(t)\,du(t)\right. \\ & \left. - \frac{1}{u(x_{i+1}) - u(x_i)} \int_{x_i}^{x_{i+1}} f(t)\,du(t) \cdot \int_{x_i}^{x_{i+1}} g(t)\,du(t)\right| \\ & \leq \frac{Hh_i^r}{2^r} \left\| g - \frac{1}{u(x_{i+1}) - u(x_i)} \int_{x_i}^{x_{i+1}} g(t)\,du(t) \right\|_{[x_i,x_{i+1}],\infty} \bigvee_{x_i}^{x_{i+1}} (u),\end{aligned}$$

for each $i \in \{0, \dots, n-1\}$.

Summing the inequalities (3.5) over i from 0 to $n-1$, and using the generalised triangle inequality, we have

$$\begin{aligned}(3.6)\quad & |R_n(f,g;u,I_n)| \\ & \leq \frac{H}{2^r} \sum_{i=0}^{n-1} h_i^r \left\| g - \frac{1}{u(x_{i+1}) - u(x_i)} \int_{x_i}^{x_{i+1}} g(t)\,du(t) \right\|_{[x_i,x_{i+1}],\infty} \bigvee_{x_i}^{x_{i+1}} (u) \\ & \leq \frac{H}{2^r} [v(f,I_n)]^n \max_{\underline{i=0,n-1}} \left\| g - \frac{1}{u(x_{i+1}) - u(x_i)} \int_{x_i}^{x_{i+1}} g(t)\,du(t) \right\|_{[x_i,x_{i+1}],\infty} \\ & \qquad \times \sum_{i=0}^{n-1} \bigvee_{x_i}^{x_{i+1}} (u) \\ & = \frac{H}{2^r} [v(f,I_n)]^n \max_{\underline{i=0,n-1}} \left\| g - \frac{1}{u(x_{i+1}) - u(x_i)} \int_{x_i}^{x_{i+1}} g(s)\,du(s) \right\|_{[x_i,x_{i+1}],\infty} \\ & \qquad \times \bigvee_a^b (u),\end{aligned}$$

and the inequality (3.4) is obtained. ∎

Remark 1. *Similar results may be stated if one uses Theorem 2 and Theorem 3. We omit the details.*

4. Some Particular Cases

For $f, g, w : [a,b] \to \mathbb{R}$, integrable and with the property that $\int_a^b w(t)\,dt \neq 0$, consider the weighted Čebyšev functional

$$\begin{aligned}(4.1)\quad T_w(f,g) :=\ & \frac{1}{\int_a^b w(t)\,dt} \int_a^b w(t)\,f(t)\,g(t)\,dt \\ & - \frac{1}{\int_a^b w(t)\,dt} \int_a^b w(t)\,f(t)\,dt \cdot \frac{1}{\int_a^b w(t)\,dt} \int_a^b w(t)\,g(t)\,dt.\end{aligned}$$

1. If $f,g,w:[a,b]\to\mathbb{R}$ are continuous and f is of $r-H-$Hölder type (see Theorem 1), then one has the inequality

$$|T_w(f,g)|\leq\frac{H|b-a|^r}{2^r}\cdot\frac{1}{\left|\int_a^b w(s)\,ds\right|}\times\left\|g-\frac{1}{\int_a^b w(s)\,ds}\int_a^b g(s)w(s)\,ds\right\|_{[a,b],\infty}\int_a^b|w(s)|\,ds. \tag{4.2}$$

The proof follows by Theorem 1 on choosing $u(t)=\int_a^t w(s)\,ds$.
2. If f,g,w are as in **1** and $w(s)\geq 0$ for $s\in[a,b]$, then one has the inequality

$$\begin{aligned}|T_w(f,g)|&\leq\frac{H}{\int_a^b w(s)\,ds}\int_a^b\left|t-\frac{a+b}{2}\right|^r\\&\quad\times\left|g(t)-\frac{1}{\int_a^b w(s)\,ds}\int_a^b g(s)w(s)\,ds\right|w(s)\,ds\\&\leq\frac{H(b-a)^r}{2^r\int_a^b w(s)\,ds}\\&\quad\times\int_a^b\left|g(t)-\frac{1}{\int_a^b w(s)\,ds}\int_a^b g(s)w(s)\,ds\right|w(s)\,ds.\end{aligned}\tag{4.3}$$

The proof follows by Theorem 2 on choosing $u(t)=\int_a^t w(s)\,ds$.
3. If f is of $r-H-$Hölder type, g are Riemann integrable on $[a,b]$ and w is continuous on $[a,b]$, then one has the inequality

$$\begin{aligned}&|T_w(f,g)|\\&\leq\frac{H\|w\|_{[a,b],\infty}}{\left|\int_a^b w(s)\,ds\right|}\int_a^b\left|t-\frac{a+b}{2}\right|^r\left|g(t)-\frac{1}{\int_a^b w(s)\,ds}\int_a^b g(s)w(s)\,ds\right|dt\\&\leq\begin{cases}\dfrac{H\|w\|_{[a,b],\infty}(b-a)^{r+1}}{2^r(r+1)\left|\int_a^b w(s)\,ds\right|}\left\|g-\dfrac{1}{\int_a^b w(s)\,ds}\int_a^b g(s)w(s)\,ds\right\|_{[a,b],\infty};\\[2ex]\dfrac{H\|w\|_{[a,b],\infty}(b-a)^{r+\frac{1}{q}}}{2^r(qr+1)^{\frac{1}{q}}\left|\int_a^b w(s)\,ds\right|}\left\|g-\dfrac{1}{\int_a^b w(s)\,ds}\int_a^b g(s)w(s)\,ds\right\|_{[a,b],p},\\ \qquad\qquad p>1,\ \frac{1}{p}+\frac{1}{q}=1;\\[2ex]\dfrac{H\|w\|_{[a,b],\infty}(b-a)^{r}}{2^r\left|\int_a^b w(s)\,ds\right|}\left\|g-\dfrac{1}{\int_a^b w(s)\,ds}\int_a^b g(s)w(s)\,ds\right\|_{[a,b],1}.\end{cases}\end{aligned}\tag{4.4}$$

The proof follows by Theorem 3 on choosing $u(t)=\int_a^t w(s)\,ds$.

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