Ostrowski Type Inequalities for Functions whose Derivatives are Convex

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Abstract

An Ostrowski type inequality for absolutely continuous functions whose derivatives are convex and applications are given.

I. Introduction

The following result is known in the literature as Ostrowski's inequality [1].

Theorem 1: Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) with the property that $|f'(t)|\leq M$ for all $t\in(a,b)$. Then

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\leq\left[\frac{1}{4}+\frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right](b-a)\,M \quad (1)$$

for all $x\in[a,b]$.

The constant $\frac{1}{4}$ is the best possible in the sense that it cannot be replaced by a smaller constant.

A simple proof of this fact can be done by using the identity:

$$f(x)=\frac{1}{b-a}\int_a^b f(t)\,dt+\frac{1}{b-a}\int_a^b p(x,t)\,f'(t)\,dt,\quad x\in[a,b], \quad (2)$$

where

$$p(x,t):=\begin{cases} t-a & \text{if}\quad a\leq t\leq x\\ t-b & \text{if}\quad x<t\leq b\end{cases}$$

which also holds for absolutely continuous functions $f:[a,b]\to\mathbb{R}$.

The following Ostrowski type result for absolutely continuous functions holds (see [2], [3] and [4]).

Theorem 2: Let $f:[a,b]\to\mathbb{R}$ be absolutely continuous on $[a,b]$. Then, for all $x\in[a,b]$, we have:

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \quad (3)$$

$$\leq\begin{cases}\left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right](b-a)\,\|f'\|_\infty \text{ if } f'\in L_\infty[a,b];\\ \frac{1}{(p+1)^{\frac{1}{p}}}\left[\left(\frac{x-a}{b-a}\right)^{p+1}+\left(\frac{b-x}{b-a}\right)^{p+1}\right]^{\frac{1}{p}}(b-a)^{\frac{1}{p}}\|f'\|_q\\ \text{if } f'\in L_q[a,b],\ \frac{1}{p}+\frac{1}{q}=1,\ p>1;\\ \left[\frac{1}{2}+\left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right]\|f'\|_1;\end{cases}$$

where $\|\cdot\|_r$ $(r\in[1,\infty])$ are the usual Lebesgue norms on $L_r[a,b]$, i.e.,

$$\|g\|_\infty:=ess\sup_{t\in[a,b]}|g(t)|$$

and

$$\|g\|_r:=\left(\int_a^b|g(t)|^r\,dt\right)^{\frac{1}{r}},\ r\in[1,\infty).$$

The constants $\frac{1}{4}$, $\frac{1}{(p+1)^{\frac{1}{p}}}$ and $\frac{1}{2}$ respectively are sharp in the sense presented in Theorem 1.

The above inequalities can also be obtained from the Fink result in [5] on choosing $n=1$ and performing some appropriate computations.

If one drops the condition of absolute continuity and assumes that f is Hölder continuous, then one may state the following result (see [6]):

Theorem 3: Let $f:[a,b]\to\mathbb{R}$ be of $r-H-$Hölder type, i.e.,

$$|f(x)-f(y)|\leq H|x-y|^r,\ \text{for all}\ x,y\in[a,b], \quad (4)$$

where $r\in(0,1]$ and $H>0$ are fixed. Then for all $x\in[a,b]$ we have the inequality:

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\leq\frac{H}{r+1}\left[\left(\frac{b-x}{b-a}\right)^{r+1}+\left(\frac{x-a}{b-a}\right)^{r+1}\right](b-a)^r. \quad (5)$$

The constant $\frac{1}{r+1}$ is also sharp in the above sense.

Note that if $r=1$, i.e., f is Lipschitz continuous, then we get the following version of Ostrowski's inequality for Lipschitzian functions (with L instead of H) (see [7])

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\leq\left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right](b-a)\,L. \quad (6)$$

Here the constant $\frac{1}{4}$ is also best.
Moreover, if one drops the condition of the continuity of the function, and assumes that it is of bounded variation, then the following result may be stated (see [8]).

Theorem 4: Assume that $f:[a,b]\to\mathbb{R}$ is of bounded variation and denote by $\bigvee_a^b$ its total variation. Then

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\le\left[\frac{1}{2}+\left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right]\bigvee_a^b(f) \quad (7)$$

for all $x\in[a,b]$.
The constant $\frac{1}{2}$ is the best possible.
If we assume more about f, i.e., f is monotonically increasing, then the inequality (7) may be improved in the following manner [9] (see also [10]).

Theorem 5: Let $f:[a,b]\to\mathbb{R}$ be monotonic nondecreasing. Then for all $x\in[a,b]$, we have the inequality:

$$\begin{aligned}&\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \quad (8)\\ &\le\frac{1}{b-a}\left\{[2x-(a+b)]f(x)+\int_a^b sgn(t-x)f(t)\,dt\right\}\\ &\le\frac{1}{b-a}\{(x-a)[f(x)-f(a)]+(b-x)[f(b)-f(x)]\}\\ &\le\left[\frac{1}{2}+\left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right][f(b)-f(a)].\end{aligned}$$

All the inequalities in (8) are sharp and the constant $\frac{1}{2}$ is the best possible.
In this paper we point out a different Ostrowski type inequality assuming some special properties for the derivative of the function f on (a,b).

II. The Results

In [11], S.S. Dragomir pointed out the following identity in representing an absolutely continuous function.

Lemma 6: Let $f:[a,b]\to\mathbb{R}$ be an absolutely continuous function on $[a,b]$. Then for any $x\in[a,b]$, one has the equality:

$$\begin{aligned}f(x)&=\frac{1}{b-a}\int_a^b f(t)\,dt\\ &+\frac{1}{b-a}\int_a^b(x-t)\left(\int_0^1 f'[(1-\lambda)x+\lambda t]\,d\lambda\right)dt. \quad (9)\end{aligned}$$

Proof: For any $t,x\in[a,b]$, $x\ne t$, one has

$$\frac{f(x)-f(t)}{x-t}=\frac{1}{x-t}\int_t^x f'(u)\,du=\int_0^1 f'[(1-\lambda)x+\lambda t]\,d\lambda,$$

showing that

$$f(x)=f(t)+(x-t)\int_0^1 f'[(1-\lambda)x+\lambda t]\,d\lambda \quad (10)$$

for any $t,x\in[a,b]$.
If we integrate (10) over t on $[a,b]$ and divide by $(b-a)$, we deduce the desired identity (9). ■

The following result holds.

Theorem 7: Let $f:[a,b]\to\mathbb{R}$ be a differentiable function such that its derivative f' is convex on (a,b). Then for any $x\in(a,b)$ one has the inequality

$$\begin{aligned}&\frac{1}{2}\cdot\frac{b-x}{b-a}\left[\frac{1}{b-x}\int_x^b f(u)\,du-f(b)-\frac{1}{2}(b-x)f'(x)\right] \quad (11)\\ &\quad+2\frac{(x-a)}{b-a}\left[\frac{2}{x-a}\int_{\frac{a+x}{2}}^x f(u)\,du-f\left(\frac{a+x}{2}\right)\right]\\ &\le f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\\ &\le\frac{1}{2}\cdot\frac{x-a}{b-a}\left[\frac{1}{x-a}\int_a^x f(u)\,du-f(a)-\frac{1}{2}(x-a)f'(x)\right]\\ &\quad+2\frac{(b-x)}{b-a}\left[\frac{2}{b-x}\int_x^{\frac{x+b}{2}} f(u)\,du-f\left(\frac{x+b}{2}\right)\right].\end{aligned}$$

Proof: For any $x\in(a,b)$, one has the identity (see Lemma 6)

$$\begin{aligned}&f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\\ &=\frac{1}{b-a}\left[\int_a^x(x-t)\left(\int_0^1 f'[(1-\lambda)x+\lambda t]\,d\lambda\right)dt\right.\\ &\quad\left.+\int_x^b(x-t)\left(\int_0^1 f'[(1-\lambda)x+\lambda t]\,d\lambda\right)dt\right]. \quad (12)\end{aligned}$$

Since f' is convex, then by the Hermite-Hadamard inequality for f' we have

$$\begin{aligned}f'\left(\frac{x+t}{2}\right)&\le\int_0^1 f'[(1-\lambda)x+\lambda t]\,d\lambda \quad (13)\\ &\le\frac{f'(x)+f'(t)}{2}\end{aligned}$$

for any $t,x\in(a,b)$.
Assume $a\le t\le x$. Then by (13) we get

$$\begin{aligned}&\int_a^x(x-t)f'\left(\frac{x+t}{2}\right)dt \quad (14)\\ &\le\int_a^x(x-t)\left(\int_0^1 f'[(1-\lambda)x+\lambda t]\,d\lambda\right)dt\\ &\le\int_a^x\left[\frac{f'(x)+f'(t)}{2}\right](x-t)\,dt.\end{aligned}$$

Assume $x\le t\le b$. Then by (13) we also get

$$\begin{aligned}&\int_x^b(x-t)\left[\frac{f'(x)+f'(t)}{2}\right]dt \quad (15)\\ &\le\int_x^b(x-t)\left(\int_0^1 f'[(1-\lambda)x+\lambda t]\,d\lambda\right)dt\\ &\le\int_x^b(x-t)f'\left(\frac{x+t}{2}\right)dt.\end{aligned}$$

Summing (14) with (15), dividing with $b-a$ and using the identity (12) we deduce

$$\frac{1}{b-a}\left[\int_a^x (x-t)\,f'\left(\frac{x+t}{2}\right)dt + \int_x^b (x-t)\left[\frac{f'(x)+f'(t)}{2}\right]dt\right] \tag{16}$$
$$\le f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt$$
$$\le \frac{1}{b-a}\left[\int_a^x \left[\frac{f'(x)+f'(t)}{2}\right](x-t)\,dt + \int_x^b (x-t)\,f'\left(\frac{x+t}{2}\right)dt\right].$$

Since

$$\int_a^x (x-t)\,f'\left(\frac{x+t}{2}\right)dt = 2\int_a^x f\left(\frac{x+t}{2}\right)dt - 2f\left(\frac{x+a}{2}\right)(x-a) = 4\int_{\frac{a+x}{2}}^x f(u)\,du - 2f\left(\frac{a+x}{2}\right)(x-a),$$

$$\int_x^b (x-t)\left[\frac{f'(x)+f'(t)}{2}\right]dt = \frac{1}{2}\int_x^b f(t)\,dt - \frac{1}{2}(b-x)\,f(b) - \frac{1}{4}(b-x)^2 f'(x),$$

$$\int_a^x \left[\frac{f'(x)+f'(t)}{2}\right](x-t)\,dt = \frac{1}{2}\int_a^x f(t)\,dt - \frac{1}{2}(x-a)\,f(a) + \frac{1}{4}(x-a)^2 f'(x),$$

and

$$\int_x^b (x-t)\,f'\left(\frac{x+t}{2}\right)dt = 2\int_x^b f\left(\frac{x+t}{2}\right)dt - 2f\left(\frac{x+b}{2}\right)(b-x) = 4\int_x^{\frac{x+b}{2}} f(u)\,du - 2f\left(\frac{x+b}{2}\right)(b-x),$$

then by (16) we deduce the desired result. ■

The following corollary is natural to be considered.

Corollary 8: Assume that $f:[a,b]\to\mathbb{R}$ is as in Theorem 7. Then

$$\frac{1}{4}\left[\frac{2}{b-a}\int_{\frac{a+b}{2}}^b f(u)\,du - f(b) - \frac{1}{4}(b-a)\,f'\left(\frac{a+b}{2}\right)\right] + \frac{4}{b-a}\int_{\frac{3a+b}{4}}^{\frac{a+b}{2}} f(u)\,du - f\left(\frac{3a+b}{4}\right) \tag{17}$$
$$\le f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)\,dt \tag{18}$$
$$\le \frac{1}{4}\left[\frac{2}{b-a}\int_a^{\frac{a+b}{2}} f(u)\,du - f(a) + \frac{1}{4}(b-a)\,f'\left(\frac{a+b}{2}\right)\right] + \frac{4}{b-a}\int_{\frac{a+b}{2}}^{\frac{a+3b}{4}} f(u)\,du - f\left(\frac{a+3b}{4}\right).$$

The proof follows from (11) at the midpoint $x=\frac{a+b}{2}$.

Corollary 9: Assume that $f:[a,b]\to\mathbb{R}$ is as in Theorem 7. Then

$$\frac{8}{5(b-a)}\int_{\frac{a+b}{2}}^b f(u)\,du - \frac{1}{5}f(b) - \frac{1}{10}(b-a)\,f'(a) - \frac{4}{5}f\left(\frac{a+b}{2}\right) \tag{19}$$
$$\le \frac{4}{5}\left[\frac{f(a)+f(b)}{2}\right] - \frac{1}{b-a}\int_a^b f(u)\,du$$
$$\le \frac{8}{5(b-a)}\int_a^{\frac{a+b}{2}} f(u)\,du - \frac{1}{5}f(a) + \frac{1}{10}(b-a)\,f'(b) - \frac{4}{5}f\left(\frac{a+b}{2}\right).$$

Proof: At $x=a$, we have from (11)

$$\frac{1}{2}\left[\frac{1}{b-a}\int_a^b f(u)\,du - f(b) - \frac{1}{2}(b-a)\,f'(a)\right] \tag{20}$$
$$\le f(a) - \frac{1}{b-a}\int_a^b f(u)\,du$$
$$\le 2\left[\frac{2}{b-a}\int_a^{\frac{a+b}{2}} f(u)\,du - f\left(\frac{a+b}{2}\right)\right].$$

At $x=b$,

$$2\left[\frac{2}{b-a}\int_{\frac{a+b}{2}}^b f(u)\,du - f\left(\frac{a+b}{2}\right)\right] \tag{21}$$
$$\le f(b) - \frac{1}{b-a}\int_a^b f(u)\,du$$
$$\le \frac{1}{2}\left[\frac{1}{b-a}\int_a^b f(u)\,du - f(a) + \frac{1}{2}(b-a)\,f'(b)\right].$$

Manipulating (20) and (21) we arrive at (19). ■

III. Inequalities for Special Means

In the examples that follow below, the following definitions of special means will be utilized.

- Arithmetic mean:

$$A = A(a,b) = \frac{a+b}{2};\quad a,b>0.$$

- p – Logarithmic mean:

$$L_p(a,b)=\begin{cases} a, & \text{if } a=b \\ \left[\dfrac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right]^{\frac{1}{p}}, & \text{if } a\neq b \end{cases}.$$

- Logarithmic mean:

$$L(a,b)=\begin{cases} a, & \text{if } a=b \\ \dfrac{b-a}{\ln b-\ln a}, & \text{if } a\neq b \end{cases}.$$

1) Consider a function f with domain $[a,b]\subset(0,\infty)$, $f(x)=x^p$ and $p\in\mathbb{R}$, $p\geq 2$ which is differentiable and its first derivative is convex.
From Corollary 8,

$$\frac{1}{b-a}\int_a^b x^p dx = L_p^p(a,b),$$

notice that

$$\frac{3a+b}{4}=\frac{A(a,b)+a}{2},\qquad \frac{a+3b}{4}=\frac{A(a,b)+b}{2}$$

and from (17)

$$\begin{aligned}
&\frac{1}{2(p+1)(b-a)}\Bigg[b^{p+1}+7A^{p+1}(a,b) \\
&-8\left(\frac{A(a,b)+a}{2}\right)^{p+1}\Bigg]-\left(\frac{A(a,b)+a}{2}\right)^p \\
&-\frac{p(b-a)}{16}A^{p-1}(a,b)-\frac{b^p}{4} \\
&\leq A^p(a,b)-L_p^p(a,b) \\
&\leq \frac{1}{2(p+1)(b-a)}\Bigg[-a^{p+1}-7A^{p+1}(a,b) \\
&+8\left(\frac{A(a,b)+b}{2}\right)^{p+1}\Bigg]-\left(\frac{A(a,b)+b}{2}\right)^p \\
&+\frac{p(b-a)}{16}A^{p-1}(a,b)-\frac{a^p}{4}.
\end{aligned}$$

In a similar fashion, from (19) we have

$$\begin{aligned}
&\frac{8}{5(p+1)(b-a)}\left[b^{p+1}+A^{p+1}(a,b)\right] \\
&-\frac{b^p}{5}-\frac{p(b-a)a^{p-1}}{10}-\frac{4A^p(a,b)}{5} \\
&\leq \frac{2}{5}\left[a^p+b^p\right]-L_p^p(a,b) \\
&\leq \frac{8}{5(p+1)(b-a)}\left[A^{p+1}-a^p\right] \\
&-\frac{a^p}{5}+\frac{p(b-a)b^{p-1}}{10}-\frac{4A^p(a,b)}{5}.
\end{aligned}$$

2) Consider the function $f(x)=-\frac{1}{x}$, $x\neq 0$ then f' is convex on $(a,b)\subset(0,\infty)$ and

$$\frac{1}{b-a}\int_a^b\left(-\frac{1}{x}\right)dx=-L^{-1}(a,b).$$

From (17) and (19) respectively, we have that

$$\begin{aligned}
&\frac{3A(a,b)+7b}{4b(a+A(a,b))}-\left(\frac{b-a}{16}\right)A^{-2}(a,b) \\
&+\frac{1}{2(b-a)}\ln\left[\left(\frac{(a+A(a,b))}{2}\right)^8\frac{1}{bA^7(a,b)}\right] \\
&\leq L^{-1}(a,b)-A^{-1}(a,b) \qquad (22)\\
&\leq \frac{3A(a,b)+7a}{4a(a+A(a,b))}+\left(\frac{b-a}{16}\right)A^{-2}(a,b) \\
&+\frac{1}{2(b-a)}\ln\left[aA^7(a,b)\left(\frac{2}{(b+A(a,b))}\right)^8\right],
\end{aligned}$$

and

$$\begin{aligned}
&\frac{A(a,b)}{5a^2b}(4A(a,b)-3b)+\frac{4}{5}A^{-1}(a,b) \\
&-\frac{8}{5(b-a)}\ln\left[bA^{-1}(a,b)\right] \\
&\leq L^{-1}(a,b)-\frac{4A(a,b)}{5ab} \\
&\leq \frac{A(a,b)}{5a^2b}(4A(a,b)-3a)+\frac{4}{5}A^{-1}(a,b) \\
&-\frac{8}{5(b-a)}\ln\left[\frac{A(a,b)}{a}\right].
\end{aligned}$$

IV. Two Plots

In this section we produce two – three dimensional plots that demonstrate the validity of the inequality (22).
Let

$$\begin{aligned}
m(a,b)=&\frac{3A(a,b)+7b}{4b(a+A(a,b))}-\left(\frac{b-a}{16}\right)A^{-2}(a,b) \\
&+\frac{1}{2(b-a)}\ln\left[\left(\frac{a+A(a,b)}{2}\right)^8\frac{1}{bA^7(a,b)}\right],
\end{aligned}$$

$$h(a,b)=L^{-1}(a,b)-A^{-1}(a,b)$$

and

$$\begin{aligned}
M(a,b)=&\frac{3A(a,b)+7a}{4a(a+A(a,b))}+\left(\frac{b-a}{16}\right)A^{-2}(a,b) \\
&+\frac{1}{2(b-a)}\ln\left[aA^7(a,b)\left(\frac{2}{b+A(a,b)}\right)^8\right];
\end{aligned}$$

then Figure 1 represents the plot of

$$h(a,b)-m(a,b)$$

and Figure 2 represents the plot of

$$M(a,b)-h(a,b)$$

indicating their positivity over the region $(a,b) = (0,1) \times (0,2)$.

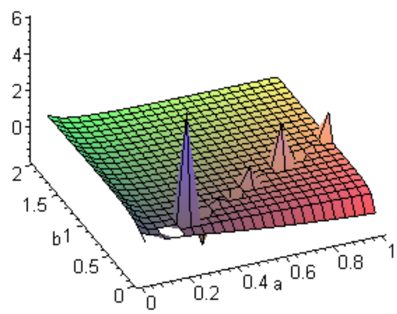

Figure 1: Plot of $h(a,b) - m(a,b)$.

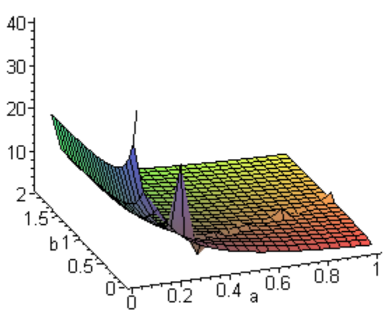

Figure 2: Plot of $M(a,b) - h(a,b)$.

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