PROPERTIES OF SOME SEQUENCES OF MAPPINGS ASSOCIATED TO THE HERMITE-HADAMARD INEQUALITY

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Abstract. The properties of some sequences of functions defined by multiple integrals associated with the Hermite-Hadamard integral inequality for convex functions are studied.

1. Introduction

The following integral inequality

$$f\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_a^b f(t)\,dt\leq\frac{f(a)+f(b)}{2}, \tag{1.1}$$

which holds for any convex function $f:[a,b]\to\mathbb{R}$, is well known in the literature as the Hermite-Hadamard inequality.

There is an extensive amount of literature devoted to this simple and nice result which has many applications in the Theory of Special Means and in Information Theory for divergence measures, from which we would like to refer the reader to the book [5].

The main aim of this paper is to consider some natural sequences of functions defined by multiple integrals and study their properties in relation to the Hermite-Hadamard inequality.

2. Some Sequences of Multiple Integrals

For an integrable mapping $f:[a,b]\to\mathbb{R}$, let us define the sequences of functionals defined by the following multiple integrals:

$$\begin{aligned} L_1(f) &:= \frac{f(a)+f(b)}{2}, \\ L_n(f) &:= \frac{1}{2(b-a)^{n-1}}\int_a^b\cdots\int_a^b\left[f\left(\frac{x_1+\dots+x_{n-1}+b}{n}\right)\right. \\ &\qquad \left.+f\left(\frac{x_1+\dots+x_{n-1}+a}{n}\right)\right]dx_1\dots dx_{n-1} \end{aligned}$$

for $n\geq 2$ and

$$A_n(f):=\frac{1}{(b-a)^n}\int_a^b\cdots\int_a^b f\left(\frac{x_1+\dots+x_n}{n}\right)dx_1\dots dx_n \text{ for } n\geq 1.$$

In [4], the authors proved the following result which connects the functional $A_n(f)$ to the Hermite-Hadamard inequality (1.1).

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Theorem 1. *Let $f:[a,b]\to\mathbb{R}$ be a convex function on $[a,b]$. Then*

$$f\left(\frac{a+b}{2}\right)\leq A_{n+1}(f)\leq A_n(f)\leq\cdots\leq A_2(f)\leq\frac{1}{b-a}\int_a^b f(t)\,dt \tag{2.1}$$

for any $n\in\mathbb{N}$, $n\geq 1$.

The sequence $L_n(f)$ may be also connected to the Hermite-Hadamard inequality through the following result.

Theorem 2. *Assume that $f:[a,b]\to\mathbb{R}$ is convex on $[a,b]$. Then for all $n\geq 2$ one has the inequalities:*

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\leq \frac{1}{2}\left\{f\left[\frac{(n-1)\frac{a+b}{2}+b}{n}\right]+f\left[\frac{(n-1)\frac{a+b}{2}+a}{n}\right]\right\} \\ &\leq L_n(f) \\ &\leq \frac{n-1}{n}\cdot\frac{1}{b-a}\int_a^b f(x)\,dx+\frac{1}{n}\cdot\frac{f(a)+f(b)}{2} \\ &\leq \frac{f(a)+f(b)}{2}. \end{aligned} \tag{2.2}$$

Proof. Using Jensen's inequality for multiple integrals, we have

$$\begin{aligned} &\frac{1}{(b-a)^{n-1}}\int_a^b\cdots\int_a^b f\left(\frac{x_1+\cdots+x_{n-1}+b}{n}\right)dx_1\ldots dx_{n-1} \\ \geq\ & f\left[\frac{1}{(b-a)^{n-1}}\int_a^b\cdots\int_a^b\left[\frac{x_1+\cdots+x_{n-1}+b}{n}\right]dx_1\ldots dx_{n-1}\right] \\ =\ & f\left[\frac{(n-1)\frac{a+b}{2}+b}{n}\right] \end{aligned}$$

and

$$\begin{aligned} &\frac{1}{(b-a)^{n-1}}\int_a^b\cdots\int_a^b f\left(\frac{x_1+\cdots+x_{n-1}+a}{n}\right)dx_1\ldots dx_{n-1} \\ \geq\ & f\left[\frac{1}{(b-a)^{n-1}}\int_a^b\cdots\int_a^b\left[\frac{x_1+\cdots+x_{n-1}+a}{n}\right]dx_1\ldots dx_{n-1}\right] \\ =\ & f\left[\frac{(n-1)\frac{a+b}{2}+a}{n}\right] \end{aligned}$$

which gives, for $n\geq 2$, that

$$\begin{aligned} L_n(f) &\geq \frac{1}{2}\left[f\left(\frac{(n-1)\frac{a+b}{2}+b}{n}\right)+f\left(\frac{(n-1)\frac{a+b}{2}+a}{n}\right)\right] \\ &\geq f\left(\frac{a+b}{2}\right). \end{aligned}$$

By the convexity of f we also have $(n\geq 2)$ that

$$f\left(\frac{x_1+\cdots x_{n-1}+b}{n}\right)\leq\frac{f(x_1)+\cdots+f(x_{n-1})+f(b)}{n}$$

and

$$f\left(\frac{x_1+\cdots+x_{n-1}+a}{n}\right)\leq\frac{f(x_1)+\cdots+f(x_{n-1})+f(a)}{n}.$$

Integrating these inequalities on $[a,b]^{n-1}$, we deduce

$$\begin{aligned}&\frac{1}{(b-a)^{n-1}}\int_a^b\cdots\int_a^b f\left(\frac{x_1+\cdots+x_{n-1}+b}{n}\right)dx_1\dots dx_{n-1}\\ \leq\ &\left(\frac{n-1}{n}\right)\cdot\frac{1}{b-a}\int_a^b f(t)\,dt+\frac{1}{n}f(b)\end{aligned}$$

and

$$\begin{aligned}&\frac{1}{(b-a)^{n-1}}\int_a^b\cdots\int_a^b f\left(\frac{x_1+\cdots+x_{n-1}+a}{n}\right)dx_1\dots dx_{n-1}\\ \leq\ &\left(\frac{n-1}{n}\right)\cdot\frac{1}{b-a}\int_a^b f(t)\,dt+\frac{1}{n}f(a)\end{aligned}$$

giving

$$L_n(f)\leq\frac{1}{2}\left[\frac{2(n-1)}{n}\frac{1}{b-a}\int_a^b f(t)\,dt+\frac{1}{n}(f(a)+f(b))\right].$$

Since

$$\frac{1}{b-a}\int_a^b f(x)\,dx\leq\frac{f(a)+f(b)}{2}.$$

The last part of (2.2) is also proved. ∎

The following lemma holds.

Lemma 1. *Let $f:I\subseteq\mathbb{R}$ be a differentiable function on $\mathring{I}$ ($\mathring{I}$ is the interior of I) and $a,b\in\mathring{I}$ with $a<b$. If f' is integrable on $[a,b]$, then we have the equality:*

$$\begin{aligned}&L_n(f)-A_n(f)\\ =\ &\frac{1}{n}\cdot\frac{1}{(b-a)^n}\int_a^b\ldots\int_a^b f'\left(\frac{x_1+\ldots+x_n}{n}\right)\left(x_n-\frac{a+b}{2}\right)dx_1\ldots dx_n\end{aligned}\tag{2.3}$$

for all $n\geq 1$.

Proof. For $n=1$, we must prove that

$$\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx=\frac{1}{b-a}\int_a^b f'(x)\left(x-\frac{a+b}{2}\right)dx.$$

Indeed, by an integration by parts, we have that:

$$\begin{aligned}\int_a^b f'(x)\left(x-\frac{a+b}{2}\right)dx &= f(x)\left(x-\frac{a+b}{2}\right)\Big|_a^b-\int_a^b f(x)\,dx\\ &=\frac{(b-a)(f(a)+f(b))}{2}-\int_a^b f(x)\,dx\end{aligned}$$

and the required identity is proved.

Let us prove the equality (2.3) for $n\geq 2$.

By an integration by parts, we have:

$$\begin{aligned}
&\int_a^b f'\left(\frac{x_1+...+x_n}{n}\right)\left(x_n-\frac{a+b}{2}\right)dx_n \\
=\;& n\,f\left(\frac{x_1+...+x_n}{n}\right)\left(x_n-\frac{a+b}{2}\right)\Big|_a^b - n\int_a^b f\left(\frac{x_1+...+x_n}{n}\right)dx_n \\
=\;& n\left\{\frac{b-a}{2}\left[f\left(\frac{x_1+...+x_{n-1}+b}{n}\right)+f\left(\frac{x_1+...+x_{n-1}+a}{n}\right)\right]\right. \\
&\left.-\int_a^b f\left(\frac{x_1+...+x_n}{n}\right)dx_n\right\}.
\end{aligned}$$

If we integrate this equality on $[a,b]^{n-1}$, we have that:

$$\begin{aligned}
&\frac{1}{n(b-a)^n}\int_a^b...\int_a^b f'\left(\frac{x_1+...+x_n}{n}\right)\left(x_n-\frac{a+b}{2}\right)dx_1...dx_n \\
=\;& \frac{1}{(b-a)^n}\int_a^b...\int_a^b \frac{b-a}{2}\left[f\left(\frac{x_1+...+x_{n-1}+b}{n}\right)\right. \\
&\left.+f\left(\frac{x_1+...+x_{n-1}+a}{n}\right)\right]dx_1...dx_{n-1} \\
&-\frac{1}{(b-a)^n}\int_a^b...\int_a^b f\left(\frac{x_1+...+x_n}{n}\right)dx_1..dx_n \\
=\;& \frac{1}{2(b-a)^{n-1}}\int_a^b...\int_a^b\left[f\left(\frac{x_1+...+x_{n-1}+b}{n}\right)\right. \\
&\left.+f\left(\frac{x_1+...+x_{n-1}+a}{n}\right)\right]dx_1...dx_{n-1} \\
&-\frac{1}{(b-a)^n}\int_a^b...\int_a^b f\left(\frac{x_1+...+x_n}{n}\right)dx_1..dx_n \\
=\;& L_n(f)-A_n(f)
\end{aligned}$$

and the identity (2.3) is proved. ∎

3. Counterpart Inequalities for $L_n(f)$ and $A_n(f)$

By the use of the lemma in the above seection, we can point out the following estimation results for the sequences defined above.

Theorem 3. *Let $f: I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function defined on the interval I and $a, b \in \mathring{I}$ with $a < b$. Then we have the inequality:*

$$0 \le A_n(f) - f\left(\frac{a+b}{2}\right) \le n\left[L_n(f) - A_n(f)\right] \tag{3.1}$$

for all $n \ge 1$.

Proof. The first inequality in (3.1) follows from Theorem 1.

For $n = 1$ we have

$$\frac{1}{b-a}\int_a^b f(x)\,dx - f\left(\frac{a+b}{2}\right) \le \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx,$$

which is Bullen's inequality (see for example [5, p. 140]).

Since any continuous convex function on $[a,b]$ is the uniform limit of a sequence of differentiable convex functions on (a,b), we can assume, without loss of generality, that f is differentiable convex on I. Thus, we have the inequality

$$f\left(\frac{a+b}{2}\right)-f\left(\frac{x_1+...+x_n}{n}\right)\geq\left(\frac{a+b}{2}-\frac{x_1+...+x_n}{n}\right)f'\left(\frac{x_1+...+x_n}{n}\right)$$

for all $x_1,...,x_n\in[a,b]$.

Integrating on $[a,b]^n$, we get that

$$\begin{aligned}&f\left(\frac{a+b}{2}\right)-A_n(f)\\ \geq\ &\frac{1}{(b-a)^n}\int_a^b...\int_a^b\left(\frac{a+b}{2}-\frac{x_1+...+x_n}{n}\right)f'\left(\frac{x_1+...+x_n}{n}\right)dx_1...dx_n\\ =\ &\frac{1}{(b-a)^n}\int_a^b...\int_a^b\left(\frac{a+b}{2}-x_n\right)f'\left(\frac{x_1+...+x_n}{n}\right)dx_1...dx_n\end{aligned}$$

as

$$\begin{aligned}&\int_a^b...\int_a^b x_1f'\left(\frac{x_1+...+x_n}{n}\right)dx_1...dx_n\\ =\ &...=\int_a^b...\int_a^b x_nf'\left(\frac{x_1+...+x_n}{n}\right)dx_1...dx_n.\end{aligned}$$

Using Lemma 1, we have that

$$\begin{aligned}&\frac{1}{(b-a)^n}\int_a^b...\int_a^b\left(\frac{a+b}{2}-x_n\right)f'\left(\frac{x_1+...+x_n}{n}\right)dx_1...dx_n\\ =\ &n\left(A_n(f)-L_n(f)\right),\end{aligned}$$

and from the above inequality we get (3.1). ∎

Another result of this type is embodied in the following theorem.

Theorem 4. *Let $f:I\subseteq\mathbb{R}\to\mathbb{R}$ be a convex function defined on the interval I and $a,b\in\mathring{I}$ with $a<b$. Then we have the inequality:*

$$0\leq A_n(f)-A_{n+1}(f)\leq\frac{n}{n+1}\left[L_n(f)-A_n(f)\right]\tag{3.2}$$

for all $n\geq1$.

Proof. The first inequality in (3.2) follows from Thorem 1.

For $n=1$ we have to prove that

$$\begin{aligned}0\ &\leq\ \frac{1}{b-a}\int_a^b f(x)\,dx-\frac{1}{(b-a)^2}\int_a^b\int_a^b f\left(\frac{x+y}{2}\right)dxdy\\ &\leq\ \frac{1}{2}\left[\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right],\end{aligned}$$

which is known (see for example [3]).

We can assume, without loss of generality, that f is differentiable and convex on I. Thus, we have the inequality

$$\begin{aligned}
& f\left(\frac{x_1+...+x_{n+1}}{n+1}\right)-f\left(\frac{x_1+...+x_n}{n}\right) \\
\geq & \left(\frac{x_1+...+x_{n+1}}{n+1}-\frac{x_1+...+x_n}{n}\right) f'\left(\frac{x_1+...+x_n}{n}\right) \\
= & \left[\frac{n x_{n+1}-(x_1+...+x_n)}{n(n+1)}\right] f'\left(\frac{x_1+...+x_n}{n}\right),
\end{aligned}$$

for all $x_1,...,x_{n+1} \in [a,b]$.

Integrating on $[a,b]^{n+1}$, we get:

$$\begin{aligned}
& A_{n+1}(f)-A_n(f) \\
\geq & \frac{1}{(b-a)^{n+1} n(n+1)}\left[\int_a^b ... \int_a^b n x_{n+1} f'\left(\frac{x_1+...+x_n}{n}\right) dx_1...dx_{n+1}\right. \\
& \left. -\int_a^b ... \int_a^b (x_1+...+x_n) f'\left(\frac{x_1+...+x_n}{n}\right) dx_1...dx_{n+1}\right] \\
= & \frac{1}{(b-a)^{n+1} n(n+1)}\left[n \cdot \frac{b^2-a^2}{2} \int_a^b ... \int_a^b f'\left(\frac{x_1+...+x_n}{n}\right) dx_1...dx_n\right. \\
& \left. -n(b-a) \int_a^b ... \int_a^b x_n f'\left(\frac{x_1+...+x_n}{n}\right) dx_1...dx_n\right] \\
= & \frac{1}{(b-a)^n (n+1)}\left[\int_a^b ... \int_a^b \left(\frac{a+b}{2}-x_n\right) f'\left(\frac{x_1+...+x_n}{n}\right) dx_1...dx_n\right] \\
& \text{(by Lemma 1)} \\
= & \frac{n}{n+1}\left[A_n(f)-L_n(f)\right],
\end{aligned}$$

and the inequality (3.2) is proved. ∎

Next, we shall point out some estimations for the difference $L_n(f)-A_n(f)$.

Theorem 5. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on $\mathring{I}$ and $a,b \in \mathring{I}$ with $a<b$. If $|f'|^2$ is integrable on $[a,b]$, then we have the inequality:*

$$\begin{aligned}
& |L_n(f)-A_n(f)| \\
\leq & \frac{\sqrt{3}(b-a)}{6n\sqrt{n}}\left[\frac{1}{(b-a)^n}\int_a^b ... \int_a^b \left|f'\left(\frac{x_1+...+x_n}{n}\right)\right|^2 dx_1...dx_n\right]^{\frac{1}{2}}
\end{aligned} \tag{3.3}$$

for all $n \geq 1$.

Proof. If $|f'|^2$ is integrable on $[a,b]$, then f' is integrable on $[a,b]$ and we have the equality (see Lemma 1):

$$\begin{aligned}
& L_n(f)-A_n(f) \\
= & \frac{1}{n} \cdot \frac{1}{(b-a)^n}\int_a^b ... \int_a^b f'\left(\frac{x_1+...+x_n}{n}\right)\left(x_n-\frac{a+b}{2}\right) dx_1...dx_n.
\end{aligned}$$

On the other hand, it is clear that

$$\begin{aligned} & \int_a^b \cdots \int_a^b f'\left(\frac{x_1+\ldots+x_n}{n}\right) x_n \, dx_1 \ldots dx_n \\ = & \int_a^b \cdots \int_a^b f'\left(\frac{x_1+\ldots+x_n}{n}\right)\left(\frac{x_1+\ldots+x_n}{n}\right) dx_1 \ldots dx_n, \end{aligned}$$

and thus

$$\begin{aligned} & L_n(f)-A_n(f) \\ = & \frac{1}{n(b-a)^n} \int_a^b \cdots \int_a^b f'\left(\frac{x_1+\ldots+x_n}{n}\right)\left(\frac{x_1+\ldots+x_n}{n}-\frac{a+b}{2}\right) dx_1 \ldots dx_n \end{aligned}$$

for all $n \geq 1$.

If we apply the Cauchy-Buniakowsky-Schwartz integral inequality, we have

$$\begin{aligned} & \left|L_n(f)-A_n(f)\right| \\ \leq & \frac{1}{n}\left(\frac{1}{(b-a)^n} \int_a^b \cdots \int_a^b \left|f'\left(\frac{x_1+\ldots+x_n}{n}\right)\right|^2 dx_1 \ldots dx_n\right)^{\frac{1}{2}} \\ & \times\left(\frac{1}{(b-a)^n} \int_a^b \cdots \int_a^b \left|\frac{x_1+\ldots+x_n}{n}-\frac{a+b}{2}\right|^2 dx_1 \ldots dx_n\right)^{\frac{1}{2}}. \end{aligned} \tag{3.4}$$

Let us compute

$$U:=\frac{1}{(b-a)^n} \int_a^b \cdots \int_a^b \left(\frac{x_1+\ldots+x_n}{n}-\frac{a+b}{2}\right)^2 dx_1 \ldots dx_n.$$

We have

$$\begin{aligned} U = & \frac{1}{(b-a)^n} \int_a^b \cdots \int_a^b \left[\frac{1}{n^2}\left(x_1^2+\ldots+x_n^2+2 \sum_{1 \leq i<j \leq n} x_i x_j\right)\right] dx_1 \ldots dx_n \\ & -2 \cdot \frac{a+b}{2} \cdot \frac{1}{(b-a)^n} \int_a^b \cdots \int_a^b \left(\frac{x_1+\ldots+x_n}{n}\right) dx_1 \ldots dx_n+\left(\frac{a+b}{2}\right)^2. \end{aligned}$$

However, a simple calculation shows us that

$$\begin{aligned} & \frac{1}{n^2} \cdot \frac{1}{(b-a)^n} \int_a^b \cdots \int_a^b \left(x_1^2+\ldots+x_n^2+2 \sum_{1 \leq i<j \leq n} x_i x_j\right) dx_1 \ldots dx_n \\ = & \frac{1}{n^2}\left[n \cdot \frac{1}{(b-a)} \int_a^b x^2 dx+2 \cdot \frac{n(n-1)}{2}\left(\frac{1}{b-a} \int_a^b x dx\right)^2\right] \\ = & \frac{1}{n}\left[\frac{b^3-a^3}{3(b-a)}+\frac{2(n-1)}{2} \cdot\left(\frac{a+b}{2}\right)^2\right] \end{aligned}$$

and

$$\begin{aligned} & \frac{1}{(b-a)^n} \int_a^b \cdots \int_a^b \left(\frac{x_1+\ldots+x_n}{n}\right) dx_1 \ldots dx_n \\ = & \frac{1}{(b-a)} \int_a^b x dx=\frac{a+b}{2}. \end{aligned}$$

Thus,

$$\begin{aligned} U &= \frac{1}{n}\left[\frac{a^2+ab+b^2}{3}+(n-1)\left(\frac{a+b}{2}\right)^2\right]-\left(\frac{a+b}{2}\right)^2 \\ &= \frac{1}{n}\left[\frac{a^2+ab+b^2}{3}-\left(\frac{a+b}{2}\right)^2\right] \\ &= \frac{1}{12n}(b-a)^2. \end{aligned}$$

Using inequality (3.4) with U as above, we easily obtain inequality (3.3). We shall omit the details. ∎

Corollary 1. *Let $f: I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on $\mathring{I}$ and $a, b \in \mathring{I}$ with $a < b$. If $M := \sup_{x\in[a,b]} |f'(x)| < \infty$, then we have the inequality:*

$$|L_n(f) - A_n(f)| \leq \frac{\sqrt{3}(b-a)M}{6n\sqrt{n}} \tag{3.5}$$

for all $n \geq 1$.

The above corollary allows us to state the following estimation result for convex mappings.

Theorem 6. *Let $f: I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable convex function on I and $a, b \in \mathring{I}$ with $a < b$. If $M := \sup_{x\in[a,b]} |f'(x)| < \infty$, then we have the inequalities:*

$$0 \leq A_n(f) - f\left(\frac{a+b}{2}\right) \leq \frac{\sqrt{3}(b-a)M}{6\sqrt{n}},\ n \geq 1 \tag{3.6}$$

and

$$0 \leq A_n(f) - A_{n+1}(f) \leq \frac{\sqrt{3}(b-a)M}{6(n+1)\sqrt{n}},\ n \geq 1. \tag{3.7}$$

Moreover, we have that:

$$\lim_{n\to\infty} n^p\left[A_n(f) - f\left(\frac{a+b}{2}\right)\right] = 0 \text{ for } 0 \leq p < \frac{1}{2}$$

and

$$\lim_{n\to\infty} n^q\left[A_n(f) - A_{n+1}(f)\right] = 0 \text{ for } 0 \leq q < \frac{3}{2}.$$

4. Counterpart Inequalities for $A_n(f)$

Next, we shall point out some estimation results for the weighted sequence

$$A_n(f,q) = \frac{1}{(b-a)^n}\int_a^b \cdots \int_a^b f\left(\frac{q_1x_1 + \ldots + q_nx_n}{Q_n}\right)dx_1 \ldots dx_n$$

for all $n \geq 1$, where $q_i > 0,\ i = \overline{1,n}$ and $Q_n := \sum_{i=1}^n q_i$.

This sequence is connected to the Hermite-Hadamard inequality through the following result obtained in [1].

Theorem 7. *Let $f: [a,b] \to \mathbb{R}$ be a convex function on $[a,b]$. Then for any $q_i > 0$ $\left(i = \overline{1,n}\right)$ one has the inequalities:*

$$f\left(\frac{a+b}{2}\right) \leq A_n(f) \leq A_n(f,q) \leq \frac{f(a)+f(b)}{2}. \tag{4.1}$$

In what follows we point out other results for this sequence.

Theorem 8. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable convex function on I and $a, b \in \mathring{I}$ with $a < b$. If $|f'|^2$ is integrable on $[a,b]$, then we have the inequality*

$$
\begin{aligned}
0 &\le A_n(f,q) - f\left(\frac{a+b}{2}\right) \\
&\le \frac{\sqrt{3}\left(\sum_{j=1}^{n} q_j^2\right)^{\frac{1}{2}}(b-a)}{6Q_n} \\
&\quad \times \left[\frac{1}{(b-a)^n}\int_a^b \cdots \int_a^b \left| f'\left(\frac{q_1x_1+\ldots+q_nx_n}{Q_n}\right)\right|^2 dx_1 \ldots dx_n\right]^{\frac{1}{2}}
\end{aligned} \tag{4.2}
$$

for $n \ge 1$.

Proof. The first inequality in (4.2) is obvious by (4.1).

By the convexity of f, we can write that

$$
\begin{aligned}
& f\left(\frac{q_1x_1+\ldots+q_nx_n}{Q_n}\right) - f\left(\frac{a+b}{2}\right) \\
\le & \left(\frac{q_1x_1+\ldots+q_nx_n}{Q_n} - \frac{a+b}{2}\right) f'\left(\frac{q_1x_1+\ldots+q_nx_n}{Q_n}\right)
\end{aligned}
$$

for all $x_1, \ldots, x_n \in [a,b]$.

If we integrate over $[a,b]^n$, we obtain

$$
\begin{aligned}
& A_n(f,q) - f\left(\frac{a+b}{2}\right) \\
\le & \frac{1}{(b-a)^n}\int_a^b \cdots \int_a^b \left(\frac{q_1x_1+\ldots+q_nx_n}{Q_n} - \frac{a+b}{2}\right) \\
& \times f'\left(\frac{q_1x_1+\ldots+q_nx_n}{Q_n}\right) dx_1 \ldots dx_n \\
\le & \left[\frac{1}{(b-a)^n}\int_a^b \cdots \int_a^b \left(\frac{q_1x_1+\ldots+q_nx_n}{Q_n} - \frac{a+b}{2}\right)^2 dx_1 \ldots dx_n\right]^{\frac{1}{2}} \\
& \times \left[\frac{1}{(b-a)^n}\int_a^b \cdots \int_a^b \left| f'\left(\frac{q_1x_1+\ldots+q_nx_n}{Q_n}\right)\right|^2 dx_1 \ldots dx_n\right]^{\frac{1}{2}}
\end{aligned} \tag{4.3}
$$

on using the Cauchy-Buniakowsky-Schwartz inequality for the last inequality.

Now, denote

$$
B := \frac{1}{(b-a)^n}\int_a^b \cdots \int_a^b \left(\frac{q_1x_1+\ldots+q_nx_n}{Q_n} - \frac{a+b}{2}\right)^2 dx_1 \ldots dx_n, \ n \ge 1.
$$

Then we have

$$
\begin{aligned}
B &= \frac{1}{Q_n^2}\cdot\frac{1}{(b-a)^n} \\
&\quad\times\int_a^b\cdots\int_a^b\left[q_1^2x_1^2+\dots+q_n^2x_n^2+2\sum_{1\le i<j\le n}q_ix_iq_jx_j\right]dx_1\dots dx_n \\
&\quad-2\cdot\frac{a+b}{2}\cdot\frac{1}{(b-a)^n}\int_a^b\cdots\int_a^b\left(\frac{q_1x_1+\dots+q_nx_n}{Q_n}\right)dx_1\dots dx_n \\
&\quad+\left(\frac{a+b}{2}\right)^2.
\end{aligned}
$$

However,

$$
\begin{aligned}
&\frac{1}{Q_n^2}\cdot\frac{1}{(b-a)^n}\int_a^b\cdots\int_a^b\left[\sum_{j=1}^n q_j^2x_j^2+2\sum_{1\le i<j\le n}q_ix_iq_jx_j\right]dx_1\dots dx_n \\
&= \frac{1}{Q_n^2}\left[\left(\sum_{j=1}^n q_j^2\right)\cdot\frac{1}{b-a}\int_a^b x^2dx+2\sum_{1\le i<j\le n}q_iq_j\left(\frac{1}{b-a}\int_a^b xdx\right)^2\right] \\
&= \frac{1}{Q_n^2}\left[\frac{b^2+ab+a^2}{3}\sum_{j=1}^n q_j^2+2\sum_{1\le i<j\le n}q_iq_j\left(\frac{a+b}{2}\right)^2\right]
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{1}{(b-a)^n}\int_a^b\cdots\int_a^b\left[\frac{q_1x_1+\dots+q_nx_n}{Q_n}\right]dx_1\dots dx_n \\
&= \frac{1}{b-a}\int_a^b xdx=\frac{a+b}{2}.
\end{aligned}
$$

Then we have

$$
\begin{aligned}
B &= \frac{1}{Q_n^2}\left[\sum_{j=1}^n q_j^2\left(\frac{b^2+ab+a^2}{3}\right)+2\sum_{1\le i<j\le n}q_iq_j\left(\frac{a+b}{2}\right)^2\right]-\left(\frac{a+b}{2}\right)^2 \\
&= \frac{1}{Q_n^2}\left[\sum_{j=1}^n q_j^2\left(\frac{b^2+ab+a^2}{3}\right)+2\sum_{1\le i<j\le n}q_iq_j\left(\frac{a+b}{2}\right)^2-Q_n^2\left(\frac{a+b}{2}\right)^2\right].
\end{aligned}
$$

As

$$
Q_n^2=\sum_{j=1}^n q_j^2+2\sum_{1\le i<j\le n}q_iq_j,
$$

then

$$
\begin{aligned}
B &= \frac{1}{Q_n^2}\cdot\sum_{j=1}^n q_j^2\left[\frac{b^2+ab+a^2}{3}-\left(\frac{a+b}{2}\right)^2\right] \\
&= \frac{(b-a)^2\sum_{j=1}^n q_j^2}{12Q_n^2}.
\end{aligned}
$$

Using inequality (4.3), we deduce the desired inequality (4.2). ∎

Corollary 2. *With the above assumptions, given that* $M := \sup_{x\in[a,b]} |f'(x)| < \infty$, *we have the inequality:*

$$0 \leq A_n(f,q) - f\left(\frac{a+b}{2}\right) \leq \frac{\sqrt{3}(b-a)\left(\sum_{j=1}^{n} q_j^2\right)^{\frac{1}{2}}}{6Q_n} M \tag{4.4}$$

for $n \geq 1$.

Remark 1. *Note that if* $\lim_{n\to\infty} \frac{\sum_{j=1}^{n} q_j^2}{Q_n^2} = 0$, *then, from* (4.4), *we recapture the result from* [1].

The following result also holds:

Theorem 9. *Let* $f : I \subseteq \mathbb{R} \to \mathbb{R}$ *be a differentiable convex function on* I *and* $a, b \in \mathring{I}$ *with* $a < b$. *If* $|f'|^2$ *is integrable on* $[a,b]$ *and* $q_i > 0$ $(i \geq 1)$, *then one has the estimation:*

$$\begin{aligned} 0 &\leq A_n(f,q) - A_n(f) \\ &\leq \frac{\sqrt{3}(b-a)}{6}\left[\sum_{j=1}^{n}\left(\frac{q_j}{Q_n} - \frac{1}{n}\right)^2\right]^{\frac{1}{2}} \\ &\quad\times\left[\frac{1}{(b-a)^n}\int_a^b \cdots \int_a^b \left|f'\left(\frac{q_1x_1 + \dots + q_nx_n}{Q_n}\right)\right|^2 dx_1 \dots dx_n\right]^{\frac{1}{2}} \end{aligned} \tag{4.5}$$

for all $n \geq 1$, *where* $Q_n := \sum_{i=1}^{n} q_i$.

Proof. The first inequality follows by Theorem 7.

Using the convexity of f, we have that

$$\begin{aligned} & f\left(\frac{q_1x_1 + \dots + q_nx_n}{Q_n}\right) - f\left(\frac{x_1 + \dots + x_n}{n}\right) \\ \leq & \left(\frac{q_1x_1 + \dots + q_nx_n}{Q_n} - \frac{x_1 + \dots + x_n}{n}\right) f'\left(\frac{q_1x_1 + \dots + q_nx_n}{Q_n}\right) \end{aligned}$$

for all $x_1, \dots, x_n \in [a,b]$.

Integrating on $[a,b]^n$, we obtain

$$\begin{aligned} & A_n(f,q) - A_n(f) \\ \leq & \frac{1}{(b-a)^n}\int_a^b \cdots \int_a^b \left(\frac{q_1x_1 + \dots + q_nx_n}{Q_n} - \frac{x_1 + \dots + x_n}{n}\right) \\ & \times f'\left(\frac{q_1x_1 + \dots + q_nx_n}{Q_n}\right) dx_1 \dots dx_n \\ \leq & \left[\frac{1}{(b-a)^n}\int_a^b \cdots \int_a^b \left(\frac{q_1x_1 + \dots + q_nx_n}{Q_n} - \frac{x_1 + \dots + x_n}{n}\right)^2 dx_1 \dots dx_n\right]^{\frac{1}{2}} \\ & \times\left[\frac{1}{(b-a)^n}\int_a^b \cdots \int_a^b \left|f'\left(\frac{q_1x_1 + \dots + q_nx_n}{Q_n}\right)\right|^2 dx_1 \dots dx_n\right]^{\frac{1}{2}}, \end{aligned} \tag{4.6}$$

by applying the Cauchy-Buniakowsky-Schwartz integral inequality for the last inequality.

Let us define

$$C:=\frac{1}{(b-a)^{n}}\int_{a}^{b}\cdots\int_{a}^{b}\left(\frac{q_{1}x_{1}+\ldots+q_{n}x_{n}}{Q_{n}}-\frac{x_{1}+\ldots+x_{n}}{n}\right)^{2}dx_{1}\ldots dx_{n}.$$

Then we have:

$$\begin{aligned}
C &= \frac{1}{Q_n^2 n^2}\cdot\frac{1}{(b-a)^n} \\
&\quad\times\int_a^b\cdots\int_a^b\left[(nq_1-Q_n)\,x_1+\ldots+(nq_n-Q_n)\,x_n\right]^2dx_1\ldots dx_n \\
&= \frac{1}{n^2Q_n^2}\cdot\frac{1}{(b-a)^n}\int_a^b\cdots\int_a^b\left[\sum_{j=1}^n(nq_j-Q_n)^2x_j^2\right. \\
&\quad\left.+2\sum_{1\le i<j\le n}(nq_i-Q_n)(nq_j-Q_n)\,x_ix_j\right]dx_1\ldots dx_n \\
&= \frac{1}{n^2Q_n^2}\left[\sum_{j=1}^n(nq_j-Q_n)^2\frac{1}{b-a}\int_a^b x^2dx\right. \\
&\quad\left.+2\sum_{1\le i<j\le n}(nq_i-Q_n)(nq_j-Q_n)\left(\frac{1}{b-a}\int_a^b xdx\right)^2\right] \\
&= \frac{1}{n^2Q_n^2}\left[\sum_{j=1}^n(nq_j-Q_n)^2\frac{a^2+ab+b^2}{3}\right. \\
&\quad\left.+2\sum_{1\le i<j\le n}(nq_i-Q_n)(nq_j-Q_n)\left(\frac{a+b}{2}\right)^2\right] \\
&= \frac{1}{n^2Q_n^2}\left[\sum_{j=1}^n(nq_j-Q_n)^2\left[\frac{a^2+ab+b^2}{3}-\left(\frac{a+b}{2}\right)^2\right]\right. \\
&\quad\left.+\left(\frac{a+b}{2}\right)^2\left[\sum_{j=1}^n(nq_i-Q_n)^2+2\sum_{1\le i<j\le n}(nq_i-Q_n)(nq_j-Q_n)\right]\right].
\end{aligned}$$

However, it is easy to see that

$$\begin{aligned}
&\sum_{j=1}^n(nq_j-Q_n)^2+2\sum_{1\le i<j\le n}(nq_i-Q_n)(nq_j-Q_n) \\
&= \left[\sum_{j=1}^n(nq_j-Q_n)\right]^2=0.
\end{aligned}$$

Hence,

$$C=\frac{\sum_{j=1}^n(Q_n-nq_j)^2}{Q_n^2n^2}\cdot\frac{(b-a)^2}{12}.$$

Finally, by using inequality (4.6), we deduce the desired inequality (4.5). ∎

Corollary 3. *With the above assumptions, given that $M := \sup_{x\in[a,b]} |f'(x)| < \infty$, we have the inequality:*

$$0 \leq A_n(f,q) - A_n(f) \leq \frac{\sqrt{3}(b-a)M}{6}\left[\sum_{j=1}^{n}\left(\frac{q_j}{Q_n}-\frac{1}{n}\right)^2\right]^{\frac{1}{2}} \tag{4.7}$$

for all $n \geq 1$.

Remark 2. *If we assume that $q_i > 0$ $(i \geq 1)$ are such that*

$$\lim_{n\to\infty}\frac{\sum_{j=1}^{n}(Q_n - nq_j)^2}{Q_n^2 n^2} = 0,$$

then we have

$$\lim_{n\to\infty}[A_n(f,q) - A_n(f)] = 0.$$

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