A GRÜSS RELATED INTEGRAL INEQUALITY AND APPLICATIONS

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Abstract. A Grüss related integral inequality in the general setting of measurable spaces and abstract Lebesgue integrals is proven. Some particular inequalities and applications for Ostrowski and generalized Trapezoid inequality are mentioned as well.

1. Introduction

Let $(\Omega, \mathcal{A}, \mu)$ be a measurable space consisting of a set Ω, a $\sigma-$ algebra of parts of Ω and a countably additive and positive measure μ on $\mathcal{A}$ with values in $\mathbb{R}\cup\{\infty\}$.

For a $\mu-$measurable function $w : \Omega \to \mathbb{R}$, with $w(x) \geq 0$ for $\mu-$ a.e. $x \in \Omega$, assume $\int_\Omega w(x)\, d\mu(x) > 0$. Consider the Lebesgue space $L_w(\Omega, \mu) := \{f : \Omega \to \mathbb{R},\ f \text{ is } \mu-\text{measurable and } \int_\Omega w(x)|f(x)|\, d\mu(x) < \infty\}$.

If $f, g : \Omega \to \mathbb{R}$ are $\mu-$measurable functions and $f, g, fg \in L_w(\Omega, \mu)$, then we may consider the Čebyšev functional

$$T_w(f,g) := \frac{1}{\int_\Omega w(x)\, d\mu(x)} \int_\Omega w(x) f(x) g(x)\, d\mu(x) - \frac{1}{\int_\Omega w(x)\, d\mu(x)} \int_\Omega w(x) f(x)\, d\mu(x) \times \frac{1}{\int_\Omega w(x)\, d\mu(x)} \int_\Omega w(x) g(x)\, d\mu(x). \tag{1.1}$$

The following result is known in the literature as the Grüss inequality

$$|T_w(f,g)| \leq \frac{1}{4}(\Gamma - \gamma)(\Delta - \delta), \tag{1.2}$$

provided

$$-\infty < \gamma \leq f(x) \leq \Gamma < \infty, \quad -\infty < \delta \leq g(x) \leq \Delta < \infty \tag{1.3}$$

for $\mu-$ a.e. $x \in \Omega$.

The constant $\frac{1}{4}$ is sharp in the sense that it cannot be replaced by a smaller constant.

Note that if $\Omega = \{1, \dots, n\}$ and μ is the discrete measure on Ω, then we obtain the discrete Grüss inequality

$$\left| \frac{1}{W_n} \sum_{i=1}^n w_i x_i y_i - \frac{1}{W_n} \sum_{i=1}^n w_i x_i \cdot \frac{1}{W_n} \sum_{i=1}^n w_i y_i \right| \leq \frac{1}{4}(\Gamma - \gamma)(\Delta - \delta), \tag{1.4}$$

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provided $\gamma \leq x_i \leq \Gamma$, $\delta \leq y_i \leq \Delta$ for each $i \in \{1, \ldots, n\}$ and $w_i \geq 0$ with $W_n := \sum_{i=1}^{n} w_i > 0$.

For some Grüss type inequalities, see the papers [1] – [11] where further references are provided.

2. Some Integral Inequalities

With the above assumption and if $f \in L_w(\Omega, \mu)$ then we may define

$$
\begin{aligned}
D_w(f) &:= D_{w,1}(f) \\
&:= \frac{1}{\int_\Omega w(x)\,d\mu(x)} \int_\Omega w(x) \\
&\quad \times \left| f(x) - \frac{1}{\int_\Omega w(y)\,d\mu(y)} \int_\Omega w(y) f(y)\,d\mu(y) \right| d\mu(x).
\end{aligned} \tag{2.1}
$$

The following fundamental result holds [2].

Theorem 1. *Let $w, f, g : \Omega \to \mathbb{R}$ be $\mu-$measurable functions with $w \geq 0$ $\mu-$ a.e. on Ω and $\int_\Omega w(y)\,d\mu(y) > 0$. If $f, g, fg \in L_w(\Omega, \mu)$ and there exists the constants δ, Δ such that*

$$
-\infty < \delta \leq g(x) \leq \Delta < \infty \quad \text{for} \quad \mu - \text{a.e. } x \in \Omega, \tag{2.2}
$$

then we have the inequality

$$
|T_w(f, g)| \leq \frac{1}{2}(\Delta - \delta) D_w(f). \tag{2.3}
$$

The constant $\frac{1}{2}$ is sharp in the sense that it cannot be replaced by a smaller constant.

For $f \in L_{p,w}(\Omega, \mathcal{A}, \mu) := \left\{ f : \Omega \to \mathbb{R}, \int_\Omega w(x) |f(x)|^p \,d\mu(x) < \infty \right\}$, $p \geq 1$ we may also define (see [2])

$$
\begin{aligned}
D_{w,p}(f) &:= \left[\frac{1}{\int_\Omega w(x)\,d\mu(x)} \int_\Omega w(x) \right. \\
&\quad \left. \times \left| f(x) - \frac{1}{\int_\Omega w(y)\,d\mu(y)} \int_\Omega w(y) f(y)\,d\mu(y) \right|^p d\mu(x) \right]^{\frac{1}{p}} \\
&= \frac{\left\| f - \frac{1}{\int_\Omega w(y) d\mu(y)} \int_\Omega w(y) f(y)\,d\mu(y) \right\|_{w,p}}{\left[\int_\Omega w(x)\,d\mu(x) \right]^{\frac{1}{p}}}
\end{aligned} \tag{2.4}
$$

where $\|\cdot\|_{w,p}$ is the usual $p-$norm on $L_{p,w}(\Omega, \mathcal{A}, \mu)$, i.e.,

$$
\|h\|_{w,p} := \left(\int_\Omega w(x) |h(x)|^p \,d\mu(x) \right)^{\frac{1}{p}}, \quad p \geq 1.
$$

Using Hölder's inequality we get

$$
D_{w,1}(f) \leq D_{w,p}(f) \quad \text{for } p \geq 1,\ f \in L_{p,w}(\Omega, \mathcal{A}, \mu); \tag{2.5}
$$

and, in particular for $p = 2$

$$
D_{w,1}(f) \leq D_{w,2}(f) = \left[\frac{\int_\Omega w(x) f^2(x)\,d\mu(x)}{\int_\Omega w(x)\,d\mu(x)} - \left(\frac{\int_\Omega w(x) f(x)\,d\mu(x)}{\int_\Omega w(x)\,d\mu(x)} \right)^2 \right]^{\frac{1}{2}}, \tag{2.6}
$$

if $f \in L_{2,w}(\Omega, \mathcal{A}, \mu)$.

For $f \in L_{\infty}(\Omega, \mathcal{A}, \mu) := \left\{ f : \Omega \to \mathbb{R},\ \|f\|_{\Omega,\infty} := ess \sup_{\alpha \in \Omega} |f(x)| < \infty \right\}$ we also have

$$D_{w,p}(f) \le D_{w,\infty}(f) := \left\| f - \frac{1}{\int_{\Omega} w(y)\, d\mu(y)} \int_{\Omega} w(y) f(y)\, d\mu(y) \right\|_{\Omega,\infty}. \tag{2.7}$$

The following corollary may be useful in practice.

Corollary 1. *With the assumptions of Theorem 1, we have*

$$\begin{aligned} &|T_w(f,g)| \\ &\le \frac{1}{2}(\Delta - \delta) D_w(f) \\ &\le \frac{1}{2}(\Delta - \delta) D_{w,p}(f) \ \ \textit{if } f \in L_p(\Omega, \mathcal{A}, \mu),\ p > 1; \\ &\le \frac{1}{2}(\Delta - \delta) \cdot \left\| f - \frac{1}{\int_{\Omega} w(y)\, d\mu(y)} \int_{\Omega} w(y) f(y)\, d\mu(y) \right\|_{\Omega,\infty}. \end{aligned} \tag{2.8}$$

Remark 1. *The inequalities in (2.8) are in order of increasing coarseness. If we assume that $-\infty < \gamma \le f(x) \le \Gamma < \infty$ for μ – a.e. $x \in \Omega$, then by the Grüss inequality for $g = f$ we obviously deduce for $p = 2$*

$$\left[\frac{\int_{\Omega} w(x) f^2(x)\, d\mu(x)}{\int_{\Omega} w(x)\, d\mu(x)} - \left(\frac{\int_{\Omega} w(x) f(x)\, d\mu(x)}{\int_{\Omega} w(x)\, d\mu(x)} \right)^2 \right]^{\frac{1}{2}} \le \frac{1}{2}(\Gamma - \gamma) \tag{2.9}$$

and then, by (2.8), we deduce the sequence for inequality

$$\begin{aligned} &|T_w(f,g)| \\ &\le \frac{1}{2}(\Delta - \delta) \frac{1}{\int_{\Omega} w(x)\, d\mu(x)} \int_{\Omega} w(x) \\ &\times \left| f(x) - \frac{1}{\int_{\Omega} w(y)\, d\mu(y)} \int_{\Omega} w(y) f(y)\, d\mu(y) \right| d\mu(x) \\ &\le \frac{1}{2}(\Delta - \delta) \left[\frac{\int_{\Omega} w(x) f^2(x)\, d\mu(x)}{\int_{\Omega} w(x)\, d\mu(x)} - \left(\frac{\int_{\Omega} w(x) f(x)\, d\mu(x)}{\int_{\Omega} w(x)\, d\mu(x)} \right)^2 \right]^{\frac{1}{2}} \\ &\le \frac{1}{4}(\Delta - \delta)(\Gamma - \gamma); \end{aligned} \tag{2.10}$$

for $f, g : \Omega \to \mathbb{R}$, μ – measurable functions and so that $-\infty < \gamma \le f(x) < \Gamma < \infty$, $-\infty < \delta \le g(x) \le \Delta < \infty$ for μ – a.e. $x \in \Omega$. Thus, the inequality (2.4) is a refinement of Grüss' inequality (1.2).

It is well known that if $f \in L_{2,w}(\Omega, \mathcal{A}, \mu)$, then the following Schwartz's type inequality holds:

$$\frac{1}{\int_{\Omega} w(x)\, d\mu(x)} \int_{\Omega} w(x) f^2(x)\, d\mu(x) \ge \left(\int_{\Omega} w(x) f(x)\, d\mu(x) \right)^2. \tag{2.11}$$

Using the above results, we may point out the following result.

Proposition 1. *Assume that the $\mu-$measurable function $f:\Omega\to\mathbb{R}$ satisfies the assumption:*

$$-\infty<\gamma\leq f(x)\leq\Gamma<\infty \quad \text{for} \quad \text{a.e. } x\in\Omega. \tag{2.12}$$

Then one has the inequality

$$\begin{aligned}0 &\leq \frac{1}{\int_\Omega w(x)\,d\mu(x)}\int_\Omega w(x)\,f^2(x)\,d\mu(x) \\ &\quad -\left(\frac{1}{\int_\Omega w(x)\,d\mu(x)}\int_\Omega w(x)\,f(x)\,d\mu(x)\right)^2 \\ &\leq \frac{1}{2}(\Gamma-\gamma)\frac{1}{\int_\Omega w(x)\,d\mu(x)}\int_\Omega w(x) \\ &\quad\times\left|f(x)-\frac{1}{\int_\Omega w(y)\,d\mu(y)}\int_\Omega w(y)\,f(y)\,d\mu(y)\right|d\mu(x) \\ &\left(\leq\frac{1}{4}(\Gamma-\gamma)^2\right).\end{aligned} \tag{2.13}$$

The constant $\frac{1}{2}$ is sharp.

The proof follows by the inequality (2.5) for $g=f$.

In [9], the author obtained the following companion of Grüss inequality.

Theorem 2. *Assume that $\rho,f,g:\Omega\to\mathbb{R}$ are Lebesgue μ measurable on Ω with $\rho,\rho f,\rho g,\rho fg\in L(\Omega,\mu)$. If $\rho(x)\geq0$ for μ – a.e. $x\in\Omega$ with $\int_\Omega\rho(x)\,d\mu(x)>0$ and there exists the real constants m,M,n,N with the property that*

$$-\infty<m\leq f(x)\leq M<\infty,-\infty<n\leq g(x)\leq N<\infty \text{ for } \mu\text{ – a.e. } x\in\Omega \tag{2.14}$$

then we have the inequality

$$\begin{aligned}&\left|\frac{1}{\int_\Omega\rho(x)\,d\mu(x)}\int_\Omega\rho(x)\,f(x)\,g(x)\,d\mu(x)\right. \\ &-\frac{m+M}{2}\cdot\frac{1}{\int_\Omega\rho(x)\,d\mu(x)}\int_\Omega\rho(x)\,g(x)\,d\mu(x) \\ &\left.-\frac{n+N}{2}\cdot\frac{1}{\int_\Omega\rho(x)\,d\mu(x)}\int_\Omega\rho(x)\,f(x)\,d\mu(x)+\frac{m+M}{2}\cdot\frac{n+N}{2}\right| \\ \leq\ &\frac{1}{4}(M-m)(N-n).\end{aligned} \tag{2.15}$$

The constant $\frac{1}{4}$ is best possible.

The following corollary is a natural consequence of this theorem.

Corollary 2. *Assume that $\rho,f:\Omega\to\mathbb{R}$ are Lebesgue μ – measurable on Ω with $\rho,\rho f,\rho f^2\in L(\Omega,\mu)$. If $\rho(x)\geq0$ for μ – a.e. $x\in\Omega$ with $\int_\Omega\rho(x)\,d\mu(x)>0$ and there exists the real constants m,M with the property that*

$$-\infty<m\leq f(x)\leq M<\infty, \text{ for } \mu\text{ – a.e. } x\in\Omega \tag{2.16}$$

then we have the inequality

$$0 \leq \frac{1}{\int_\Omega \rho(x)\, d\mu(x)} \int_\Omega \rho(x) f^2(x)\, d\mu(x) - \left(\frac{1}{\int_\Omega \rho(x)\, d\mu(x)} \int_\Omega \rho(x) f(x)\, d\mu(x)\right)^2 \leq \frac{1}{4}(M-m)^2 - \left(\frac{1}{\int_\Omega \rho(x)\, d\mu(x)} \int_\Omega \rho(x) f(x)\, d\mu(x) - \frac{m+M}{2}\right)^2 \left(\leq \frac{1}{4}(M-m)^2\right). \tag{2.17}$$

The constant $\frac{1}{4}$ is best possible in (2.17).

The main aim of this paper is to establish a refinement of the inequality (2.15) in the spirit of the refinement provided for the Grüss inequality incorporated in Theorem 1. Applications for Ostrowski and Generalized Trapezoid inequalities will be provided as well.

3. New Integral Inequalities

We start to the following lemma that is interesting in itself as well.

Lemma 1. *Assume that $\rho, h, l : \Omega \to \mathbb{R}$ are Lebesgue μ – measurable on Ω with $\rho, \rho h, \rho l, \rho h l \in L(\Omega, \mu)$. If $\rho(x) \geq 0$ for μ-a.e. $x \in \Omega$ with $\int_\Omega \rho(x)\, d\mu(x) > 0$ and there exists the real constants a, A with the property that*

$$-\infty < a \leq h(x) \leq A < \infty, -\infty < b \leq l(x) \leq B < \infty \text{ for } \mu\text{-a.e. } x \in \Omega \tag{3.1}$$

and

$$\int_\Omega \rho(x) l(x)\, d\mu(x) = \int_\Omega \rho(x) h(x)\, d\mu(x) = 0, \tag{3.2}$$

then we have the inequality

$$\left|\frac{1}{\int_\Omega \rho(x)\, d\mu(x)} \int_\Omega \rho(x) h(x) l(x)\, d\mu(x) + \frac{a+A}{2} \cdot \frac{b+B}{2}\right| \leq \frac{1}{2}(A-a) \frac{1}{\int_\Omega \rho(x)\, d\mu(x)} \int_\Omega \rho(x) \left|l(x) - \frac{b+B}{2}\right| d\mu(x). \tag{3.3}$$

The constant $\frac{1}{2}$ is best possible in the sense that it can not be replaced by a smaller one.

Proof. Firstly, let observe, by the assumption (3.2), that

$$\int_\Omega \rho(x) \left(h(x) - \frac{a+A}{2}\right)\left(l(x) - \frac{b+B}{2}\right) d\mu(x) = \int_\Omega \rho(x) h(x) l(x)\, d\mu(x) + \frac{a+A}{2} \cdot \frac{b+B}{2} \int_\Omega \rho(x)\, d\mu(x). \tag{3.4}$$

On the other hand, by (3.1), we have

$$\left|h(x) - \frac{a+A}{2}\right| \leq \frac{A-a}{2}, \text{ for } \mu-\text{a.e. } x \in \Omega$$

and thus

$$\begin{aligned}
& \left|\int_{\Omega}\rho(x)\left(h(x)-\frac{a+A}{2}\right)\left(l(x)-\frac{b+B}{2}\right)d\mu(x)\right| \\
\leq & \int_{\Omega}\rho(x)\left|h(x)-\frac{a+A}{2}\right|\left|l(x)-\frac{b+B}{2}\right|d\mu(x) \\
\leq & \frac{A-a}{2}\cdot\int_{\Omega}\rho(x)\left|l(x)-\frac{b+B}{2}\right|d\mu(x).
\end{aligned}$$

Now, using the equality (3.4), the inequality (3.5) and dividing by $\int_{\Omega}\rho(x)\,d\mu(x)>0$, we deduce the desired inequality (3.3).

To prove the sharpness of the constant $\frac{1}{2}$, we assume that (3.3) holds with a constant $C>0$, i.e.,

$$\begin{aligned}
(3.5)\quad & \left|\frac{1}{\int_{\Omega}\rho(x)\,d\mu(x)}\int_{\Omega}\rho(x)\,h(x)\,l(x)\,d\mu(x)+\frac{a+A}{2}\cdot\frac{b+B}{2}\right| \\
& \leq C(A-a)\,\frac{1}{\int_{\Omega}\rho(x)\,d\mu(x)}\cdot\int_{\Omega}\rho(x)\left|l(x)-\frac{b+B}{2}\right|d\mu(x).
\end{aligned}$$

If we consider the functions $h,l:[\alpha,\beta]\subset\mathbb{R}\to\mathbb{R}$, $l=h$ and

$$h(x)=\begin{cases}-1 & \text{if } x\in\left[\alpha,\frac{\alpha+\beta}{2}\right] \\ \\ 1 & \text{if } x\in\left[\frac{\alpha+\beta}{2},\beta\right],\end{cases}$$

then for $\rho(x)=1$, we have $a=b=-1, A=B=1$

$$\begin{aligned}
\int_{\alpha}^{\beta}h(x)\,dx &= \int_{\alpha}^{\beta}l(x)\,dx=0, \\
\int_{\alpha}^{\beta}h(x)\,l(x)\,dx &= 1;
\end{aligned}$$

and

$$\frac{1}{\beta-\alpha}\cdot\int_{\alpha}^{\beta}\rho(x)\left|l(x)-\frac{b+B}{2}\right|d(x)=1$$

and thus, by (3.3), we deduce $C\geq\frac{1}{2}$. ∎

The above lemma gives us the opportunity to state the following companion of Grüss inequality.

Theorem 3. *Assume that $\rho,f,g:\Omega\to\mathbb{R}$ are Lebesgue μ measurable on Ω with $\rho,\rho f,\rho g,\rho fg\in L(\Omega,\mu)$. If $\rho(x)\geq 0$ for μ-a.e. $x\in\Omega$ with $\int_{\Omega}\rho(x)\,d\mu(x)>0$ and there exists the real constants m,M,n,N with the property that*

$$-\infty<m\leq f(x)\leq M<\infty,-\infty<n\leq g(x)\leq N<\infty \text{ for } \mu\text{-a.e. } x\in\Omega$$

then we have the inequality

$$\begin{aligned}
(3.6) \qquad & \left| \frac{1}{\int_\Omega \rho(x)\, d\mu(x)} \int_\Omega \rho(x) f(x) g(x)\, d\mu(x) \right. \\
& - \frac{m+M}{2} \cdot \frac{1}{\int_\Omega \rho(x)\, d\mu(x)} \int_\Omega \rho(x) g(x)\, d\mu(x) \\
& \left. - \frac{n+N}{2} \cdot \frac{1}{\int_\Omega \rho(x)\, d\mu(x)} \int_\Omega \rho(x) f(x)\, d\mu(x) + \frac{m+M}{2} \cdot \frac{n+N}{2} \right| \\
\leq\ & \frac{1}{2}(M-m) \frac{1}{\int_\Omega \rho(x)\, d\mu(x)} \cdot \int_\Omega \rho(x) \left| g(x) - \frac{n+N}{2} \right| d\mu(x).
\end{aligned}$$

The constant $\frac{1}{2}$ is best possible.

Proof. If we choose in Lemma 1

$$\begin{aligned}
h(x) &= f(x) - \frac{1}{\int_\Omega \rho(y)\, d\mu(y)} \int_\Omega \rho(y) f(y)\, d\mu(y), \\
l(x) &= g(x) - \frac{1}{\int_\Omega \rho(y)\, d\mu(y)} \int_\Omega \rho(y) g(y)\, d\mu(y);
\end{aligned}$$

where $x \in \Omega$, and

$$\begin{aligned}
a &= m - \frac{1}{\int_\Omega \rho(y)\, d\mu(y)} \int_\Omega \rho(y) f(y)\, d\mu(y), \\
A &= M - \frac{1}{\int_\Omega \rho(y)\, d\mu(y)} \int_\Omega \rho(y) f(y)\, d\mu(y), \\
b &= n - \frac{1}{\int_\Omega \rho(y)\, d\mu(y)} \int_\Omega \rho(y) g(y)\, d\mu(y); \\
B &= N - \frac{1}{\int_\Omega \rho(y)\, d\mu(y)} \int_\Omega \rho(y) g d\mu(y);
\end{aligned}$$

then a simple calculation will reveal that

$$\begin{aligned}
& \frac{1}{\int_\Omega \rho(x)\, d\mu(x)} \int_\Omega \rho(x) h(x) l(x)\, d\mu(x) \\
&= \frac{1}{\int_\Omega \rho(x)\, d\mu(x)} \int_\Omega \rho(x) f(x) g(x)\, d\mu(x) - \frac{1}{\int_\Omega \rho(x)\, d\mu(x)} \int_\Omega \rho(x) f(x)\, d\mu(x) \\
& \qquad \cdot \frac{1}{\int_\Omega \rho(x)\, d\mu(x)} \int_\Omega \rho(x) g(x)\, d\mu(x)
\end{aligned}$$

and

$$\begin{aligned}
\frac{a+A}{2} &= \frac{m+M}{2} - \frac{1}{\int_\Omega \rho(y)\, d\mu(y)} \int_\Omega \rho(y) f(y)\, d\mu(y) \\
\frac{b+B}{2} &= \frac{n+N}{2} - \frac{1}{\int_\Omega \rho(y)\, d\mu(y)} \int_\Omega \rho(y) g d\mu(y)
\end{aligned}$$

and since

$$A - a = M - m, B - b = N - n$$

then by (3.3) we deduce the desired inequality (2.15).

The sharpness of the constant follows from Lemma 1 and we omit the details. ∎

For $f \in L_{p,\rho}(\Omega,\mathcal{A},\mu) := \left\{ f:\Omega\to\mathbb{R}, \int_\Omega \rho(x)\,|f(x)|^p\,d\mu(x) < \infty\right\}$, $p \geq 1$ we may also define the quantities

$$\begin{aligned} F_{\rho,p}(f;m,M) &:= \left[\frac{1}{\int_\Omega \rho(x)\,d\mu(x)}\int_\Omega \rho(x)\left|f(x)-\frac{m+M}{2}\right|^p d\mu(x)\right]^{\frac{1}{p}} \\ &= \frac{\left\|f-\frac{m+M}{2}\right\|_{\rho,p}}{\left[\int_\Omega \rho(x)\,d\mu(x)\right]^{\frac{1}{p}}} \end{aligned}$$

where $\|\cdot\|_{\rho,p}$ is the usual $p-$norm on $L_{p,\rho}(\Omega,\mathcal{A},\mu)$, i.e.,

$$\|h\|_{\rho,p} := \left(\int_\Omega \rho(x)\,|h(x)|^p\,d\mu(x)\right)^{\frac{1}{p}}, \quad p \geq 1.$$

Using Hölder's inequality we get

$$F_{\rho,1}(f;m,M) \leq F_{\rho,p}(f;m,M) \quad \text{for } p\geq 1,\ f\in L_{p,\rho}(\Omega,\mathcal{A},\mu);$$

For $f\in L_\infty(\Omega,\mathcal{A},\mu) := \left\{ f:\Omega\to\mathbb{R},\ \|f\|_{\Omega,\infty} := ess\sup_{\alpha\in\Omega}|f(x)| < \infty\right\}$ we also have

$$F_{\rho,p}(f;m,M) \leq F_{\rho,\infty}(f;m,M) := \left\|f-\frac{m+M}{2}\right\|_{\rho,\infty}.$$

The following corollary may be useful in practice.

Corollary 3. *With the assumptions of Theorem 3, we have*

$$\begin{aligned} &\left|\frac{1}{\int_\Omega \rho(x)\,d\mu(x)}\int_\Omega \rho(x)\,f(x)\,g(x)\,d\mu(x)\right. \\ &-\frac{m+M}{2}\cdot\frac{1}{\int_\Omega \rho(x)\,d\mu(x)}\int_\Omega \rho(x)\,g(x)\,d\mu(x) \\ &\left.-\frac{n+N}{2}\cdot\frac{1}{\int_\Omega \rho(x)\,d\mu(x)}\int_\Omega \rho(x)\,f(x)\,d\mu(x)+\frac{m+M}{2}\cdot\frac{n+N}{2}\right| \\ &\leq \frac{1}{2}(M-m)\,F_\rho(g;n,N) \ \text{ if } f\in L_{1,\rho}(\Omega,\mathcal{A},\mu) \\ &\leq \frac{1}{2}(M-m)\,F_{\rho,p}(g;n,N) \ \text{ if } f\in L_{p,\rho}(\Omega,\mathcal{A},\mu) \\ &\leq \frac{1}{2}(M-m)\,F_{\rho,\infty}(g;n,N) \leq \frac{1}{4}(M-m)(N-n) \ \text{ if } f\in L_{p,\rho}(\Omega,\mathcal{A},\mu). \end{aligned}$$

The following corollary providing a counterpart for Schwarz's inequality, is a natural consequence of this theorem

Corollary 4. *Assume that* $\rho, f:\Omega\to\mathbb{R}$ *are Lebesgue* μ *measurable on* Ω *with* $\rho,\rho f,\rho f^2\in L(\Omega,\mu)$. *If* $\rho(x)\geq 0$ *for* μ *- a.e.* $x\in\Omega$ *with* $\int_\Omega \rho(x)\,d\mu(x) > 0$ *and there exists the real constants* m, M *with the property that*

$$-\infty < m \leq f(x) \leq M < \infty,\ \text{for } \mu \text{ - a.e. } x\in\Omega$$

then we have the inequality

$$
\begin{aligned}
(3.7)\qquad 0 &\leq \frac{1}{\int_{\Omega}\rho(x)\,d\mu(x)}\int_{\Omega}\rho(x)\,f^{2}(x)\,d\mu(x) \\
&\quad -\left(\frac{1}{\int_{\Omega}\rho(x)\,d\mu(x)}\int_{\Omega}\rho(x)\,f(x)\,d\mu(x)\right)^{2} \\
&\leq \frac{1}{2}(M-m)\frac{1}{\int_{\Omega}\rho(x)\,d\mu(x)}\cdot\int_{\Omega}\rho(x)\left|f(x)-\frac{m+M}{2}\right|d\mu(x) \\
&\quad -\left(\frac{1}{\int_{\Omega}\rho(x)\,d\mu(x)}\int_{\Omega}\rho(x)\,f(x)\,d\mu(x)-\frac{m+M}{2}\right)^{2}.
\end{aligned}
$$

The constant $\frac{1}{2}$ is best possible in (3.7).

4. Particular Inequalities

The following particular inequalities are of interest and may be used in applications.

(1) Let $f,g:[a,b]\to\mathbb{R}$ be Lebesgue measurable functions. If $f,g,fg\in L[a,b]$, where $L[a,b]$ is the usual Lebsgue space, and

$$
(4.1)\qquad -\infty<m\leq f(x)\leq M<\infty,-\infty<n\leq g(x)\leq N<\infty \quad \text{for}\ \ \mu-\text{a.e. } x\in[a,b],
$$

then we have the inequalities

$$
\begin{aligned}
&\left|\frac{1}{b-a}\int_{a}^{b}f(x)\,g(x)\,dx-\frac{n+N}{2}\cdot\frac{1}{b-a}\int_{a}^{b}f(x)\,dx\right. \\
&\left.-\frac{m+M}{2}\cdot\frac{1}{b-a}\int_{a}^{b}g(x)\,dx+\frac{m+M}{2}\cdot\frac{n+N}{2}\right| \\
\leq\ & \frac{1}{2}(M-m)\frac{1}{b-a}\int_{a}^{b}\left|g(x)-\frac{n+N}{2}\right|dx \\
\leq\ & \frac{1}{2}(M-m)\left[\frac{1}{b-a}\int_{a}^{b}\left|g(x)-\frac{n+N}{2}\right|^{p}dx\right]^{\frac{1}{p}},p>1 \\
\leq\ & \frac{1}{2}(M-m)\,ess\sup_{x\in[a,b]}\left|g(x)-\frac{n+N}{2}\right| \\
\leq\ & \frac{1}{4}(M-m)(N-n).
\end{aligned}
$$

The constant $\frac{1}{2}$ in front of $(M-m)$ is sharp in the inequality (4.2).

The following counterpart of Schwartz's inequality holds

$$0 \leq \frac{1}{b-a}\int_a^b f^2(x)\,dx - \left(\frac{1}{b-a}\int_a^b f(x)\,dx\right)^2 \tag{4.2}$$
$$\leq \frac{1}{2}(M-m)\frac{1}{b-a}\int_a^b \left|f(x)-\frac{m+M}{2}\right|dx - \left(\frac{1}{b-a}\int_a^b f(x)\,dx - \frac{m+M}{2}\right)^2$$
$$\left(\leq \frac{1}{4}(M-m)^2\right),$$

provided $-\infty < m \leq f(x) \leq M < \infty$ for a.e. $x \in [a,b]$. The constant $\frac{1}{4}$ is sharp.

2. Let $\bar{\mathbf{a}} = (a_1, \dots, a_n)$, $\bar{\mathbf{b}} = (b_1, \dots, b_n)$, $\bar{\mathbf{p}} = (p_1, \dots, p_n)$ be $n-$tuples of real numbers with $p_i \geq 0$ $(i \in \{1, \dots, n\})$ and $\sum_{i=1}^n p_i = 1$. If

$$a \leq a_i \leq A, b \leq b_i \leq B, \qquad i \in \{1, \dots, n\}, \tag{4.3}$$

then one has the inequality

$$\left|\sum_{i=1}^n p_i a_i b_i - \frac{b+B}{2}\cdot\sum_{i=1}^n p_i a_i - \frac{a+A}{2}\cdot\sum_{i=1}^n p_i b_i + \frac{b+B}{2}\cdot\frac{a+A}{2}\right| \tag{4.4}$$
$$\leq \frac{1}{2}(A-a)\sum_{i=1}^n p_i\left|b_i - \frac{b+B}{2}\right|$$
$$\leq \frac{1}{2}(A-a)\left[\sum_{i=1}^n p_i\left|b_i - \frac{b+B}{2}\right|^p\right]^{\frac{1}{p}}, p>1$$
$$\leq \frac{1}{2}(A-a)\max_{i=1,\dots,n}\left|b_i - \frac{b+B}{2}\right| \leq \frac{1}{4}(A-a)(B-b).$$

The constant $\frac{1}{2}$ in front of $(A-a)$ is sharp in all the above inequalities.

The following counterpart of Schwartz's inequality also holds

$$0 \leq \sum_{i=1}^n p_i a_i^2 - \left(\sum_{i=1}^n p_i a_i\right)^2 \tag{4.5}$$
$$\leq \frac{1}{2}(A-a)\sum_{i=1}^n p_i\left|b_i - \frac{b+B}{2}\right| - \left(\sum_{i=1}^n p_i a_i - \frac{a+A}{2}\right)^2$$
$$\left(\leq \frac{1}{4}(A-a)^2\right),$$

provided $a \leq a_i \leq A$ for each $i \in \{1, \dots, n\}$. The constant $\frac{1}{2}$ in front of $(A-a)$ is sharp in the second inequality (4.5).

5. Applications for Ostrowski's Inequality

If $\varphi : [a,b] \to \mathbb{R}$ is an absolutely continuous function on $[a,b]$ such that $\varphi' \in L_\infty[a,b]$, then the following inequality is known in the literature as Ostrowski's inequality

$$\left|\varphi(x)-\frac{1}{b-a}\int_a^b \varphi(t)\,dt\right| \leq \left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right]\|\varphi'\|_\infty (b-a), \quad x\in[a,b], \tag{5.1}$$

where $\|\varphi'\|_\infty := ess \sup_{\alpha\in[a,b]} |\varphi'(x)|$. The constant $\frac{1}{4}$ is best possible.

A simple proof of this fact, may be accomplished by the use of the identity

$$\varphi(x)=\frac{1}{b-a}\int_a^b \varphi(t)\,dt+\frac{1}{b-a}\int_a^b K(x,t)\,\varphi'(t)\,dt, \tag{5.2}$$

where the kernel $K:[a,b]^2\to\mathbb{R}$ is defined by

$$K(x,t):=\begin{cases} t-a & \text{if } \ a\leq t\leq x \\ t-b & \text{if } \ a\leq x<t\leq b. \end{cases} \tag{5.3}$$

We will now use the following inequality

$$\begin{aligned} &\left|\frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx-\frac{n+N}{2}\cdot\frac{1}{b-a}\int_a^b f(x)\,dx\right. \\ &\left.-\frac{m+M}{2}\cdot\frac{1}{b-a}\int_a^b g(x)\,dx+\frac{m+M}{2}\cdot\frac{n+N}{2}\right| \\ \leq\ & \frac{1}{2}(M-m)\frac{1}{b-a}\int_a^b\left|g(x)-\frac{n+N}{2}\right|dx \end{aligned}$$

provided $-\infty<m\leq g(x)\leq M<\infty, -\infty<n\leq g(x)\leq N<\infty$ for a.e. $x\in[a,b]$ and $f\in L_\infty[a,b]$, to obtain the next result concerning a perturbed version of Ostrowski's inequality.

Theorem 4. *Assume that $\varphi:[a,b]\to\mathbb{R}$ is an absolutely continuous function on $[a,b]$ such that $\varphi':[a,b]\to\mathbb{R}$ satisfies the condition*

$$-\infty<\gamma\leq\varphi'(x)\leq\Gamma<\infty \ \ \textit{for} \ \ a.e.\ x\in[a,b]. \tag{5.4}$$

Then we have the inequality

$$\left|\varphi(x)-\frac{1}{b-a}\int_a^b\varphi(t)\,dt-\left(x-\frac{a+b}{2}\right)[\varphi;a,b]\right|\leq\frac{1}{8}(b-a)(\Gamma-\gamma) \tag{5.5}$$

for any $x\in[a,b]$, where $[\varphi;a,b]=\frac{\varphi(b)-\varphi(a)}{b-a}$ is the divided difference. The constant $\frac{1}{8}$ is best possible in the sense that it cannot be replaced by a smaller one.

Proof. We apply inequality (5.4) for the choices $f(t) = K(x,t)$ defined by (5.3), $g(t) = \varphi'(t)$, $t \in [a,b]$ to get

$$\begin{aligned}(5.6)\quad & \left|\frac{1}{b-a}\int_a^b K(x,t)\,\varphi'(t)\,dt - \frac{\gamma+\Gamma}{2}\cdot\frac{1}{b-a}\int_a^b K(x,t)\,dt\right. \\ & \left. -\left(x-\frac{a+b}{2}\right)\frac{1}{b-a}\int_a^b \varphi'(t)\,dt + \frac{\gamma+\Gamma}{2}\left(x-\frac{a+b}{2}\right)\right| \\ & \leq \frac{1}{2}(\Gamma-\gamma)\cdot\frac{1}{b-a}\int_a^b\left|K(x,t)-\left(x-\frac{a+b}{2}\right)\right|dt\end{aligned}$$

since, obviously,

$$x-b \leq K(x,t) \leq x-a.$$

We obviously have:

$$\frac{1}{b-a}\int_a^b \varphi'(t)\,dt = \frac{\varphi(b)-\varphi(a)}{b-a}.$$

Also (see [2])

$$\begin{aligned}(5.7)\quad I(x) &:= \frac{1}{b-a}\int_a^b\left|K(x,t)-\left(x-\frac{a+b}{2}\right)\right|dt \\ &= \frac{1}{b-a}\left[\int_a^x\left|t-a-x+\frac{a+b}{2}\right|dt + \int_x^b\left|t-b-x+\frac{a+b}{2}\right|dt\right] \\ &= \frac{1}{b-a}\left[\int_a^x\left|t-x+\frac{b-a}{2}\right|dt + \int_x^b\left|t-x+\frac{b-a}{2}\right|dt\right].\end{aligned}$$

Straight forward substitution of $u = t-x+\frac{b-a}{2}$ and $v = t-x-\frac{b-a}{2}$ gives

$$\begin{aligned}I(x) &= \frac{1}{b-a}\left[\int_{\frac{a+b}{2}-x}^{\frac{b-a}{2}}|u|\,du + \int_{-\frac{b-a}{2}}^{\frac{a+b}{2}-x}|v|\,dv\right] \\ &= \frac{1}{b-a}\int_{-\frac{b-a}{2}}^{\frac{b-a}{2}}|u|\,du = \frac{2}{b-a}\int_0^{\frac{b-a}{2}}udu = \frac{b-a}{4}.\end{aligned}$$

Substitution of the above into (5.7) produces (5.5). ∎

Remark 2. *The above inequality was firstly proved in a different manner in* [3].

6. Application for Generalised Trapezoid Inequality

If $\varphi : [a,b] \to \mathbb{R}$ is an absolutely continuous function on $[a,b]$ so that $\varphi' \in L_\infty[a,b]$, then the following inequality is known as the generalised trapezoid inequality

$$\begin{aligned}(6.1)\quad & \left|(x-a)\varphi(a)+(b-x)\varphi(b)-\int_a^b\varphi(t)\,dt\right| \\ & \leq \left[\frac{1}{4}(b-a)^2+\left(x-\frac{a+b}{2}\right)^2\right]\|\varphi'\|_\infty\end{aligned}$$

for any $x \in [a,b]$. The constant $\frac{1}{4}$ is best possible.

A simple proof of this fact may be done by using the identity

$$\int_a^b \varphi(t)\,dt = (x-a)\,\varphi(a) + (b-x)\,\varphi(b) + \int_a^b (x-t)\,\varphi'(t)\,dt. \tag{6.2}$$

Utilising the inequality (5.4) we may point out the following perturbed version of (6.1).

Theorem 5. *Assume that $\varphi : [a,b] \to \mathbb{R}$ is an absolutely continuous function on $[a,b]$ so that $\varphi' : [a,b] \to \mathbb{R}$ satisfies the condition (5.4). Then we have the inequality*

$$\begin{aligned}(6.3)\quad &\left|\frac{1}{b-a}\int_a^b \varphi(t)\,dt\right. \\ &\left.-\left[\left(\frac{x-a}{b-a}\right)\varphi(a) + \left(\frac{b-x}{b-a}\right)\varphi(b)\right] - \left(x-\frac{a+b}{2}\right)[\varphi;a,b]\right| \\ &\leq \frac{1}{8}(b-a)(\Gamma-\gamma)\end{aligned}$$

for any $x \in [a,b]$. The constant $\frac{1}{8}$ is best possible in the sense that it cannot be replaced by a smaller one.

Proof. We apply inequality (5.4) for the choices $f(t) = x-t$, $g(t) = \varphi'(t)$, $t \in [a,b]$, to get

$$\begin{aligned}(6.4)\quad &\left|\frac{1}{b-a}\int_a^b (x-t)\,\varphi'(t)\,dt - \frac{\gamma+\Gamma}{2}\frac{1}{b-a}\int_a^b (x-t)\,dt\right. \\ &\left.-\left(x-\frac{a+b}{2}\right)\frac{1}{b-a}\int_a^b \varphi'(t)\,dt + \frac{\gamma+\Gamma}{2}\left(x-\frac{a+b}{2}\right)\right| \\ &\leq \frac{1}{2}(\Gamma-\gamma)\frac{1}{b-a}\int_a^b \left|(x-t)-\left(x-\frac{a+b}{2}\right)\right|dt.\end{aligned}$$

Since

$$\begin{aligned}\frac{1}{b-a}\int_a^b (x-t)\,dt &= \left(x-\frac{a+b}{2}\right), \\ \frac{1}{b-a}\int_a^b \varphi'(t)\,dt &= \frac{1}{b-a}\int_a^b \varphi'(t)\,dt\end{aligned}$$

and

$$\begin{aligned}\frac{1}{b-a}\int_a^b \left|(x-t)-\left(x-\frac{a+b}{2}\right)\right|dt &= \frac{1}{b-a}\int_a^b \left|x-t-x+\frac{a+b}{2}\right|dt \\ &= \frac{1}{b-a}\int_a^b \left|t+\frac{a+b}{2}\right|dt \\ &= \frac{b-a}{4},\end{aligned}$$

from (6.4) we deduce the desired inequality (6.3). ∎

Remark 3. *The above inequality (6.3) was firstly proved in* [2] *by a different argument. In the same paper the authors have shown that the constant $\frac{1}{8}$ is sharp in the sense that it cannot be replaced by a smaller constant.*

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