SHARP BOUNDS OF ČEBYŠEV FUNCTIONAL FOR STIELTJES INTEGRALS AND APPLICATIONS

S.S. DRAGOMIR

Abstract. Sharp bounds of the Čebyšev functional for the Stieltjes integrals similar to the Grüss one and applications for quadrature rules are given.

1. Introduction

Consider the *weighted Čebyšev functional*

$$T_w(f,g) := \frac{1}{\int_a^b w(t)\,dt}\int_a^b w(t) f(t) g(t)\,dt - \frac{1}{\int_a^b w(t)\,dt}\int_a^b w(t) f(t)\,dt \cdot \frac{1}{\int_a^b w(t)\,dt}\int_a^b w(t) g(t)\,dt \tag{1.1}$$

where $f,g,w:[a,b]\to\mathbb{R}$ and $w(t)\geq 0$ for a.e. $t\in[a,b]$ are measurable functions such that the involved integrals exist and $\int_a^b w(t)\,dt>0$.

In [1], the authors obtained, among others, the following inequalities:

$$\begin{aligned}
&|T_w(f,g)| \\
&\leq \frac{1}{2}(M-m)\frac{1}{\int_a^b w(t)\,dt}\int_a^b w(t)\left|g(t)-\frac{1}{\int_a^b w(s)\,ds}\int_a^b w(s)g(s)\,ds\right|dt \\
&\leq \frac{1}{2}(M-m)\left[\frac{1}{\int_a^b w(t)\,dt}\int_a^b w(t)\right. \\
&\qquad \times\left.\left|g(t)-\frac{1}{\int_a^b w(s)\,ds}\int_a^b w(s)g(s)\,ds\right|^p dt\right]^{\frac{1}{p}} \quad (p>1) \\
&\leq \frac{1}{2}(M-m)\, ess\sup_{t\in[a,b]}\left|g(t)-\frac{1}{\int_a^b w(s)\,ds}\int_a^b w(s)g(s)\,ds\right|
\end{aligned} \tag{1.2}$$

provided

$$-\infty < m \leq f(t) \leq M < \infty \quad \text{for a.e. } t\in[a,b] \tag{1.3}$$

and the corresponding integrals are finite. The constant $\frac{1}{2}$ is sharp in all the inequalities in (1.2) in the sense that it cannot be replaced by a smaller constant.

In addition, if

$$-\infty < n \leq g(t) \leq N < \infty \quad \text{for a.e. } t\in[a,b], \tag{1.4}$$

Date: June 18, 2002.
1991 *Mathematics Subject Classification.* Primary 26D15; Secondary 41A55.
Key words and phrases. Čebyšev functional, Grüss type inequality, Stieltjes integral.

then the following refinement of the celebrated Grüss inequality is obtained:

$$
\begin{aligned}
(1.5)\quad & |T_w(f,g)| \\
&\leq \frac{1}{2}(M-m)\frac{1}{\int_a^b w(t)\,dt}\int_a^b w(t)\left|g(t)-\frac{1}{\int_a^b w(s)\,ds}\int_a^b w(s)\,g(s)\,ds\right|dt \\
&\leq \frac{1}{2}(M-m)\left[\frac{1}{\int_a^b w(t)\,dt}\int_a^b w(t)\right. \\
&\qquad\qquad \left.\times\left|g(t)-\frac{1}{\int_a^b w(s)\,ds}\int_a^b w(s)\,g(s)\,ds\right|^2 dt\right]^{\frac{1}{2}} \\
&\leq \frac{1}{4}(M-m)(N-n).
\end{aligned}
$$

Here, the constants $\frac{1}{2}$ and $\frac{1}{4}$ are also sharp in the sense mentioned above.

In this paper, we extend the above results for Riemann-Stieltjes integrals. A quadrature formula is also considered.

For this purpose, we introduce the following Čebyšev functional for the Stieltjes integral

$$
\begin{aligned}
(1.6)\quad T(f,g;u) &:= \frac{1}{u(b)-u(a)}\int_a^b f(t)\,g(t)\,du(t) \\
&\quad -\frac{1}{u(b)-u(a)}\int_a^b f(t)\,du(t)\cdot\frac{1}{u(b)-u(a)}\int_a^b g(t)\,du(t),
\end{aligned}
$$

where $f,g\in C[a,b]$ (are continuous on $[a,b]$) and $u\in BV[a,b]$ (is of bounded variation on $[a,b]$) with $u(b)\neq u(a)$.

For some recent inequalities for Stieltjes integral see [2]-[5].

2. The Results

The following result holds.

Theorem 1. *Let $f,g:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$ and $u:[a,b]\to\mathbb{R}$ with $u(a)\neq u(b)$. Assume also that there exists the real constants m,M such that*

$$
(2.1)\qquad m\leq f(t)\leq M \quad \text{for each } t\in[a,b].
$$

If u is of bounded variation on $[a,b]$, then we have the inequality

$$
\begin{aligned}
(2.2)\quad |T(f,g;u)| &\leq \frac{1}{2}(M-m)\frac{1}{|u(b)-u(a)|} \\
&\quad\times\left\|g-\frac{1}{u(b)-u(a)}\int_a^b g(s)\,du(s)\right\|_\infty \bigvee_a^b(u),
\end{aligned}
$$

where $\bigvee_a^b(u)$ denotes the total variation of u in $[a,b]$. The constant $\frac{1}{2}$ is sharp, in the sense that it cannot be replaced by a smaller constant.

Proof. It is easy to see, by simple computation with the Stieltjes integral, that the following equality

$$T(f,g;u)=\frac{1}{u(b)-u(a)}\int_a^b\left[f(t)-\frac{m+M}{2}\right]\times\left[g(t)-\frac{1}{u(b)-u(a)}\int_a^b g(s)\,du(s)\right]du(t) \tag{2.3}$$

holds.

Using the known inequality

$$\left|\int_a^b p(t)\,dv(t)\right|\le\sup_{t\in[a,b]}|p(t)|\bigvee_a^b(v), \tag{2.4}$$

provided $p\in C[a,b]$ and $v\in BV[a,b]$, we have, by (2.3), that

$$\begin{aligned}|T(f,g;u)|&\le\sup_{t\in[a,b]}\left|\left[f(t)-\frac{m+M}{2}\right]\left[g(t)-\frac{1}{u(b)-u(a)}\int_a^b g(s)\,du(s)\right]\right|\\&\quad\cdot\frac{1}{|u(b)-u(a)|}\bigvee_a^b(u)\\&\quad\left(\text{since }\left|f(t)-\frac{m+M}{2}\right|\le\frac{M-m}{2}\text{ for any }t\in[a,b]\right)\\&\le\frac{M-m}{2}\left\|g-\frac{1}{u(b)-u(a)}\int_a^b g(s)\,du(s)\right\|_\infty\cdot\frac{1}{|u(b)-u(a)|}\bigvee_a^b(u)\end{aligned}$$

and the inequality (2.2) is proved.

To prove the sharpness of the constant $\frac{1}{2}$ in the inequality (2.2), we assume that it holds with a constant $C>0$, i.e.,

$$|T(f,g;u)|\le C(M-m)\frac{1}{|u(b)-u(a)|}\times\left\|g-\frac{1}{u(b)-u(a)}\int_a^b g(s)\,du(s)\right\|_\infty\bigvee_a^b(u). \tag{2.5}$$

Let us consider the functions $f=g$, $f:[a,b]\to\mathbb{R}$, $f(t)=t$, $t\in[a,b]$ and $u:[a,b]\to\mathbb{R}$ given by

$$u(t)=\begin{cases}-1 & \text{if } t=a,\\ 0 & \text{if } t\in(a,b),\\ 1 & \text{if } t=b.\end{cases} \tag{2.6}$$

Then f,g are continuous on $[a,b]$, u is of bounded variation on $[a,b]$ and

$$\frac{1}{u(b)-u(a)}\int_a^b f(t)\,g(t)\,du(t) = \frac{1}{2}\int_a^b t^2 du(t) = \frac{1}{2}\left[t^2 u(t)\big|_a^b - 2\int_a^b tu(t)\,dt\right] = \frac{b^2+a^2}{2},$$

$$\frac{1}{u(b)-u(a)}\int_a^b f(t)\,du(t) = \frac{1}{u(b)-u(a)}\int_a^b g(t)\,du(t) = \frac{1}{2}\int_a^b tdu(t) = \frac{1}{2}\left[tu(t)\big|_a^b - \int_a^b u(t)\,dt\right] = \frac{b+a}{2},$$

$$\left\|g-\frac{1}{u(b)-u(a)}\int_a^b g(s)\,du(s)\right\|_\infty = \sup_{t\in[a,b]}\left|t-\frac{a+b}{2}\right| = \frac{b-a}{2}$$

and

$$\bigvee_a^b(u) = 2, \quad M=b, \quad m=a.$$

Inserting these values in (2.5), we get

$$\left|\frac{a^2+b^2}{2}-\frac{(a+b)^2}{4}\right| \le C(b-a)\cdot\frac{1}{2}\cdot\frac{(b-a)}{2}\cdot 2,$$

giving $C \ge \frac{1}{2}$, and the theorem is thus proved. ∎

The corresponding result for monotonic function u is incorporated in the following theorem.

Theorem 2. *Assume that f and g are as in Theorem 1. If $u:[a,b]\to\mathbb{R}$ is monotonic nondecreasing on $[a,b]$, then one has the inequality:*

$$|T(f,g;u)| \le \frac{1}{2}(M-m)\frac{1}{u(b)-u(a)} \times \int_a^b\left|g(t)-\frac{1}{u(b)-u(a)}\int_a^b g(s)\,du(s)\right|du(t). \tag{2.7}$$

The constant $\frac{1}{2}$ is sharp in the sense that it cannot be replaced by a smaller constant.

Proof. Using the known inequality

$$\left|\int_a^b p(t)\,dv(t)\right| \le \int_a^b |p(t)|\,dv(t), \tag{2.8}$$

provided $p \in C[a,b]$ and v is a monotonic nondecreasing function on $[a,b]$, we have (by the use of equality (2.3)) that

$$\begin{aligned}|T(f,g;u)| &\leq \frac{1}{u(b)-u(a)}\int_a^b \left|f(t)-\frac{m+M}{2}\right| \\ &\qquad \times\left|g(t)-\frac{1}{u(b)-u(a)}\int_a^b g(s)\,du(s)\right| du(t) \\ &\leq \frac{1}{2}(M-m)\frac{1}{u(b)-u(a)}\int_a^b \left|g(t)-\frac{1}{u(b)-u(a)}\int_a^b g(s)\,du(s)\right| du(t).\end{aligned}$$

Now, assume that the inequality (2.7) holds with a constant $D>0$, instead of $\frac{1}{2}$, i.e.,

$$\begin{aligned}|T(f,g;u)| &\leq D(M-m)\frac{1}{u(b)-u(a)} \\ &\qquad \times\int_a^b \left|g(t)-\frac{1}{u(b)-u(a)}\int_a^b g(s)\,du(s)\right| du(t).\end{aligned} \tag{2.9}$$

If we choose the same function as in the proof of Theorem 1, we observe that f,g are continuous and u is monotonic nondecreasing on $[a,b]$. Then, for these functions, we have

$$T(f,g;u)=\frac{a^2+b^2}{2}-\frac{(a+b)^2}{4}=\frac{(b-a)^2}{4},$$

$$\begin{aligned}&\int_a^b \left|g(t)-\frac{1}{u(b)-u(a)}\int_a^b g(s)\,du(s)\right| du(t) \\ &=\int_a^b \left|t-\frac{a+b}{2}\right| du(t) \\ &=\int_a^{\frac{a+b}{2}}\left(\frac{a+b}{2}-t\right)du(t)+\int_{\frac{a+b}{2}}^b\left(t-\frac{a+b}{2}\right)du(t) \\ &=\left[u(t)\left(\frac{a+b}{2}-t\right)\right]\Bigg|_a^{\frac{a+b}{2}}+\int_a^{\frac{a+b}{2}}u(t)\,dt \\ &\qquad+\left[\left(t-\frac{a+b}{2}\right)u(t)\right]\Bigg|_{\frac{a+b}{2}}^b-\int_{\frac{a+b}{2}}^b u(t)\,dt \\ &=b-a,\end{aligned}$$

and then, by (2.9) we get

$$\frac{(b-a)^2}{4}\leq D(b-a)\frac{1}{2}(b-a)$$

giving $D\geq\frac{1}{2}$, and the theorem is completely proved. ∎

The case when u is a Lipschitzian function is embodied in the following theorem.

Theorem 3. *Assume that $f,g:[a,b]\to\mathbb{R}$ are Riemann integrable functions on $[a,b]$ and f satisfies the condition (2.1). If $u:(a,b)\to\mathbb{R}$ $(u(b)\neq u(a))$ is Lipschitzian with the constant L, then we have the inequality*

$$|T(f,g;u)|\leq\frac{1}{2}L(M-m)\frac{1}{|u(b)-u(a)|}\times\int_a^b\left|g(t)-\frac{1}{u(b)-u(a)}\int_a^b g(s)\,du(s)\right|dt. \tag{2.10}$$

The constant $\frac{1}{2}$ cannot be replaced by a smaller constant.

Proof. It is well known that if $p:[a,b]\to\mathbb{R}$ is Riemann integrable on $[a,b]$ and $v:[a,b]\to\mathbb{R}$ is Lipschitzian with the constant L, then the Riemann-Stieltjes integral $\int_a^b p(t)\,dv(t)$ exists and

$$\left|\int_a^b p(t)\,dv(t)\right|\leq L\int_a^b|p(t)|\,dt. \tag{2.11}$$

Using this fact and the identity (2.3), we deduce

$$\begin{aligned}|T(f,g;u)|&\leq\frac{L}{|u(b)-u(a)|}\int_a^b\left|f(t)-\frac{m+M}{2}\right|\\&\qquad\times\left|g(t)-\frac{1}{u(b)-u(a)}\int_a^b g(s)\,du(s)\right|dt\\&\leq\frac{1}{2}(M-m)\frac{L}{|u(b)-u(a)|}\int_a^b\left|g(t)-\frac{1}{u(b)-u(a)}\int_a^b g(s)\,du(s)\right|dt\end{aligned}$$

and the inequality (2.10) is proved.

Now, assume that (2.10) holds with a constant $E>0$ instead of $\frac{1}{2}$, i.e.,

$$|T(f,g;u)|\leq EL(M-m)\frac{1}{|u(b)-u(a)|}\times\int_a^b\left|g(t)-\frac{1}{u(b)-u(a)}\int_a^b g(s)\,du(s)\right|dt. \tag{2.12}$$

Consider the function $f=g$, $f:[a,b]\to\mathbb{R}$ with

$$f(t)=\begin{cases}-1 & \text{if } t\in\left[a,\frac{a+b}{2}\right]\\ 1 & \text{if } t\in\left(\frac{a+b}{2},b\right]\end{cases}$$

and $u:[a,b]\to\mathbb{R}$, $u(t)=t$. Then, obviously, f and g are Riemann integrable on $[a,b]$ and u is Lipschitzian with the constant $L=1$.

Since

$$\frac{1}{u(b)-u(a)}\int_a^b f(t)\,g(t)\,du(t)=\frac{1}{b-a}\int_a^b dt=1,$$

$$\frac{1}{u(b)-u(a)}\int_a^b f(t)\,du(t)=\frac{1}{u(b)-u(a)}\int_a^b g(t)\,du(t)=0,$$

$$\int_a^b\left|g(t)-\frac{1}{u(b)-u(a)}\int_a^b g(s)\,du(s)\right|dt=\int_a^b dt=b-a$$

and

$$M=1,\quad m=1$$

then, by (2.12), we deduce $E\geq\frac{1}{2}$, and the theorem is completely proved. ∎

3. A Quadrature Formula

Let us consider the partition of the interval $[a,b]$ given by

$$I_n: a=x_0<x_1<\cdots<x_{n-1}<x_n=b. \tag{3.1}$$

Denote $v(I_n):=\max\left\{h_i|i=\overline{0,n-1}\right\}$ where $h_i:=x_{i+1}-x_i$, $i=\overline{0,n-1}$.

If $f:[a,b]\to\mathbb{R}$ is continuous on $[a,b]$ and if we define

$$M_i:=\sup_{t\in[x_i,x_{i+1}]}f(t),\quad m_i:=\inf_{t\in[x_i,x_{i+1}]}f(t),\text{ and}$$

$$v(f,I_n)=\max_{i=\overline{0,n-1}}(M_i-m_i),$$

then, obviously, by the continuity of f on $[a,b]$, for any $\varepsilon>0$, we may find a division I_n with norm $v(I_n)<\delta$ such that $v(f,I_n)<\varepsilon$.

Consider now the quadrature rule

$$S_n(f,g;u,I_n):=\sum_{i=0}^{n-1}\frac{1}{u(x_{i+1})-u(x_i)}\int_{x_i}^{x_{i+1}}f(t)\,du(t)\cdot\int_{x_i}^{x_{i+1}}g(t)\,du(t) \tag{3.2}$$

provided $f,g\in C[a,b]$, $u\in BV[a,b]$ and $u(x_{i+1})\neq u(x_i)$, $i=0,\ldots,n-1$.

We may now state the following result in approximating the Stieltjes integral

$$\int_a^b f(t)\,g(t)\,du(t).$$

Theorem 4. *Let $f,g\in C[a,b]$ and $u\in BV[a,b]$. If I_n is a division of the interval $[a,b]$ and $u(x_{i+1})\neq u(x_i)$, $i=0,\ldots,n-1$, then we have:*

$$\int_a^b f(t)\,g(t)\,du(t)=S_n(f,g;u,I_n)+R_n(f,g;u,I_n), \tag{3.3}$$

where $S_n(f,g;u,I_n)$ is as defined in (3.2) and the remainder $R_n(f,g;u,I_n)$ satisfies the estimate

$$|R_n(f,g;u,I_n)|\leq\frac{1}{2}v(f,I_n)\times\max_{i=\overline{0,n-1}}\left\|g-\frac{1}{u(x_{i+1})-u(x_i)}\int_{x_i}^{x_{i+1}}g(s)\,du(s)\right\|_{[x_i,x_{i+1}],\infty}\bigvee_a^b(u). \tag{3.4}$$

The constant $\frac{1}{2}$ is sharp in (3.4) in the sense that it cannot be replaced by a smaller constant.

Proof. Applying the inequality (2.2) on the intervals $[x_i, x_{i+1}]$, $i = 0, \dots, n-1$, we have

$$\left| \int_{x_i}^{x_{i+1}} f(t)\, g(t)\, du(t) - \frac{1}{u(x_{i+1}) - u(x_i)} \int_{x_i}^{x_{i+1}} f(t)\, du(t) \cdot \int_{x_i}^{x_{i+1}} g(t)\, du(t) \right| \leq \frac{1}{2}(M_i - m_i) \sup_{t \in [x_i, x_{i+1}]} \left| g(t) - \frac{1}{u(x_{i+1}) - u(x_i)} \int_{x_i}^{x_{i+1}} g(s)\, du(s) \right| \bigvee_{x_i}^{x_{i+1}}(u). \tag{3.5}$$

Summing the inequalities (3.5) over i from 0 to $n-1$, and using the generalised triangle inequality, we have

$$\begin{aligned} |R_n(f, g; u, I_n)| &\leq \frac{1}{2} \sum_{i=0}^{n-1} (M_i - m_i) \left\| g - \frac{1}{u(x_{i+1}) - u(x_i)} \int_{x_i}^{x_{i+1}} g(s)\, du(s) \right\|_{[x_i, x_{i+1}], \infty} \times \bigvee_{x_i}^{x_{i+1}}(u) \\ &\leq \frac{1}{2} v(f, I_n) \max_{i = \overline{0, n-1}} \left\| g - \frac{1}{u(x_{i+1}) - u(x_i)} \int_{x_i}^{x_{i+1}} g(s)\, du(s) \right\|_{[x_i, x_{i+1}], \infty} \times \sum_{i=0}^{n-1} \bigvee_{x_i}^{x_{i+1}}(u) \\ &= \frac{1}{2} v(f, I_n) \max_{i = \overline{0, n-1}} \left\| g - \frac{1}{u(x_{i+1}) - u(x_i)} \int_{x_i}^{x_{i+1}} g(s)\, du(s) \right\|_{[x_i, x_{i+1}], \infty} \times \bigvee_a^b(u), \end{aligned} \tag{3.6}$$

and the estimate (3.4) is obtained. ∎

Remark 1. *Similar results may be stated for either u monotonic or Lipschitzian. We omit the details.*

4. Some Particular Cases

For $f, g, w : [a, b] \to \mathbb{R}$, integrable and with the property that $\int_a^b w(t)\, dt \neq 0$, reconsider the weighted Čebyšev functional

$$T_w(f, g) := \frac{1}{\int_a^b w(t)\, dt} \int_a^b w(t) f(t) g(t)\, dt - \frac{1}{\int_a^b w(t)\, dt} \int_a^b w(t) f(t)\, dt \cdot \frac{1}{\int_a^b w(t)\, dt} \int_a^b w(t) g(t)\, dt. \tag{4.1}$$

1. If $f, g, w : [a, b] \to \mathbb{R}$ are continuous and there exists the real constants m, M such that

$$m \leq f(t) \leq M \text{ for each } t \in [a, b], \tag{4.2}$$

then one has the inequality

$$|T_w(f,g)| \le \frac{1}{2}(M-m)\frac{1}{\left|\int_a^b w(s)\,ds\right|} \times \left\| g - \frac{1}{\int_a^b w(s)\,ds}\int_a^b g(s)\,w(s)\,ds \right\|_{[a,b],\infty} \int_a^b |w(s)|\,ds. \tag{4.3}$$

The proof follows by Theorem 1 on choosing $u(t) = \int_a^t w(s)\,ds$.

2. If f, g, w are as in **1** and $w(s) \ge 0$ for $s \in [a,b]$, then one has the inequality

$$|T_w(f,g)| \le \frac{1}{2}(M-m)\frac{1}{\int_a^b w(s)\,ds} \times \int_a^b \left| g(t) - \frac{1}{\int_a^b w(s)\,ds}\int_a^b g(s)\,w(s)\,ds \right| w(s)\,ds. \tag{4.4}$$

The proof follows by Theorem 2 on choosing $u(t) = \int_a^t w(s)\,ds$.

3. If f, g are Riemann integrable on $[a,b]$ and f satisfies (4.2), and w is continuous on $[a,b]$, then one has the inequality

$$|T_w(f,g)| \le \frac{1}{2}\|w\|_{[a,b],\infty}(M-m)\frac{1}{\left|\int_a^b w(s)\,ds\right|} \times \int_a^b \left| g(t) - \frac{1}{\int_a^b w(s)\,ds}\int_a^b g(s)\,w(s)\,ds \right| ds. \tag{4.5}$$

The proof follows by Theorem 3 on choosing $u(t) = \int_a^t w(s)\,ds$.

References

[1] P. CERONE and S.S. DRAGOMIR, A refinement of the Grüss inequality and applications, *RGMIA Res. Rep. Coll.,* **5**(2) (2002), Article 14. [ONLINE: `http://rgmia.vu.edu.au/v5n2.html`].

[2] S.S. DRAGOMIR, On the Ostrowski's inequality for Riemann-Stieltjes integral and applications, *Korean J. Comput. & Appl. Math.*, **7**(3) (2000), 611-627.

[3] S.S. DRAGOMIR, Some inequalities for Riemann-Stieltjes integral and applications, *Optimisation and Related Topics*, Editor: A. Rubinov, Kluwer Academic Publishers, (2000), 197-235.

[4] S.S. DRAGOMIR, Some inequalities of midpoint and trapezoid type for the Riemann-Stieltjes integral, *Nonlinear Analysis*, **47**(4) (2001), 2333-2340

[5] S.S. DRAGOMIR, On the Ostrowski inequality for Riemann-Stieltjes integral where f is of Hölder type and u is of bounded variation and applications, *J. KSIAM*, **5**(1) (2001), 35-45.

School of Communications and Informatics, Victoria University of Technology, P.O. Box 14428, MCMC 8001,, Victoria, Australia.

E-mail address: `sever@matilda.vu.edu.au`

URL: `http://rgmia.vu.edu.au/SSDragomirWeb.html`