Sharp Integral Inequalities of the Hermite–Hadamard Type

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We consider a family of two-point quadrature formulae and establish sharp estimates for the remainders under various regularity conditions. Improved forms of certain integral inequalities due to Hermite–Hadamard, Iyengar, Milovanović–Pečarić, and others are obtained as special cases. Our results may also be interpreted as analogues to a theorem of Ostrowski on the deviation of a functions from its averages. Furthermore, we generalize a result of Fink concerning L^p estimates for the remainder of the trapezoidal rule. When p is equal to 1, 2, or ∞, we obtain explicit values for the best constants in Fink's bounds.

Key Words: Hermite–Hadamard inequality, two-point quadrature, Lipschitz classes, L^p estimates

1. INTRODUCTION

An extensive literature deals with inequalities between an integral

$$\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t$$

and its trapezoidal approximation $\frac{1}{2}(f(a)+f(b))$ or its midpoint approximation $f(\frac{1}{2}(a+b))$; see [2], [3], or [7, Chap. XV]. A classical result of Hermite and Hadamard states that

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t \le \frac{f(a)+f(b)}{2} \tag{1}$$

when f is a real-valued convex function.

It is assumed throughout this paper that all functions f are real-valued and therefore we shall tacitly include this property in our hypotheses.

For a function f defined on an interval $[a,b]$, we write $f \in \mathrm{Lip}_M(\kappa)$ with $M > 0$ and $\kappa \in (0,1]$, and say that f satisfies a *Lipschitz condition* of *order* κ with the *Lipschitz constant* M, if

$$|f(t_2) - f(t_1)| \le M\,|t_2 - t_1|^{\kappa} \qquad \text{for all} \quad t_1, t_2 \in [a,b].$$

For notational convenience, the class $\mathrm{Lip}_M(1)$ is simply denoted by Lip_M.

Recently Dragomir, Cho and Kim [1] proved the following result.

THEOREM A. *Let f be a function defined on an interval $[a,b]$ and belonging to Lip_M. Then*

$$\left|\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t - f\left(\frac{a+b}{2}\right)\right| \le \frac{M}{4}(b-a) \tag{2}$$

and

$$\left|\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t - \frac{f(a)+f(b)}{2}\right| \le \frac{M}{3}(b-a). \tag{3}$$

For the trapezoidal approximation, an attractive inequality was found by Iyengar in 1938; see [6] or [7, p. 471, Theorem 1].

THEOREM B. *Let f be a differentiable function on $[a,b]$ with $|f'(t)| \le M$. Then*

$$\left|\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t - \frac{f(a)+f(b)}{2}\right| \le \frac{M}{4}(b-a) - \frac{\big(f(b)-f(a)\big)^2}{4M(b-a)}. \tag{4}$$

Here it is remarkable that a non-negative term is subtracted on the right-hand side. Another inequality of that type was obtained by Milovanović and Pečarić in 1976; see [7, p. 472, Theorem 4].

THEOREM C. *Let f be a differentiable function on $[a,b]$ with $f' \in \mathrm{Lip}_M$. Suppose that $f'(a) = f'(b) = 0$. Then*

$$\left|\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t - \frac{f(a)+f(b)}{2}\right| \le \frac{M}{24}(b-a)^2 - \frac{\big(f(b)-f(a)\big)^2}{2M(b-a)^2}. \tag{5}$$

Note that, if f is twice differentiable and $|f''(t)| \le M$ on $[a,b]$, then f' belongs to Lip_M. If $n > 2$ and f has an n-th derivative with $\left|f^{(n)}(t)\right| \le M$, then we cannot have a bound in terms of M only. We must also involve some of the derivatives $f^{(\nu)}$ $(2 \le \nu < n)$ since the trapezoidal rule is not exact for all polynomials of degree $n-1$. Fink [3, p. 308, Theorem I] imposed constraints on the derivatives at the end points of the interval and obtained the following result.

Theorem D. *Let f be n times continuously differentiable on $[a,b]$, and suppose that $f^{(\nu)}(a) = f^{(\nu)}(b) = 0$ for $\nu = 1,\dots,n-1$. Then, for each p in $[1,\infty]$, there exists a smallest number $R(n,p)$ such that*

$$\left|\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t - \frac{f(a)+f(b)}{2}\right| \le \frac{R(n,p)}{n!}\,\left\|f^{(n)}\right\|_p, \tag{6}$$

where

$$\left\|f^{(n)}\right\|_p = \left(\int_a^b \left|f^{(n)}(t)\right|^p \,\mathrm{d}t\right)^{1/p} \qquad (1 \le p < \infty)$$

and $\|f^{(n)}\|_\infty = \operatorname{ess\,sup}_{a\le t\le b}\left|f^{(n)}(t)\right|$.

Fink described the constants $R(n,p)$ by an approximation problem in the dual norm $\|\cdot\|_q$, where $p^{-1}+q^{-1}=1$. He computed the following explicit values [3, p. 308, Corollary 5]

$$R(1,1) = \frac{1}{2}\,,\ \ R(1,p) = \frac{(b-a)^{1-1/p}}{2(1+q)^{1/q}}\ \ (1<p<\infty),\ \ R(1,\infty) = \frac{b-a}{4}\,, \tag{7}$$

$$R(2,1) = \frac{b-a}{8}\,,\qquad R(2,2) = \frac{(b-a)^{3/2}}{6\sqrt{5}}\,,\qquad R(2,\infty) = \frac{(b-a)^2}{16}\,, \tag{8}$$

and

$$R(3,\infty) = \frac{(b-a)^3}{64}\,, \tag{9}$$

and established upper bounds when $n \ge 3$. Here we have given revised values for $R(2,2)$ and $R(3,\infty)$ since the computation in [3] contained an inaccurracy.

In 1938, Ostrowski [8] proved the following result which estimates the approximation of an integral by the midpoint rule.

Theorem E. *Let f be a differentiable function on (a,b) and let $|f'(t)| \le M$ for $t \in (a,b)$. Then, for each $x \in (a,b)$,*

$$\left|\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t - f(x)\right| \le \left[\frac{1}{4} + \left(\frac{x-\frac{1}{2}(a+b)}{b-a}\right)^2\right](b-a)M. \quad (10)$$

This result was generalized and refined by Fink [3].

In this paper we study, for each real number $x \in \left[a, \frac{1}{2}(a+b)\right]$, the more general quadrature formula

$$\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t = \frac{1}{2}\Big(f(x) + f(a+b-x)\Big) + E(f;x) \qquad (11)$$

with $E(f;x)$ being the remainder. Our motivation for this choice comes from the following observations.

Considering x as a parameter, we observe that (11) defines a family of quadrature formulae which contains the trapezoidal rule and the midpoint rule as the boundary cases $x = a$ and $x = \frac{1}{2}(a+b)$, respectively. It also includes any other quadrature formula with two symmetric nodes; for example, it includes the two-point Maclaurin formula and the two-point Gaussian formula. We shall establish estimates for $E(f;x)$ which generalize Theorems A–D, include these theorems as the special case $x = a$, or lead to improvements in two respects. In some cases we can not only relax the hypotheses on f, but we can also diminish the constant in the estimate of $E(f;a)$. All of our results are sharp.

Another important motivation for (11) comes from the fact that any function f on $[a,b]$ can be split into

$$f(t) = f_{\mathrm{e}}(t) + f_{\mathrm{o}}(t),$$

where

$$f_{\mathrm{e}}(t) = \frac{f(t) + f(a+b-t)}{2} \qquad (12)$$

is its *even part* and

$$f_{\mathrm{o}}(t) = \frac{f(t) - f(a+b-t)}{2} \qquad (13)$$

is its *odd part*. Hence estimates for the remainder $E(f;x)$ in (11) may be seen as Ostrowski type inequalities (10) for the even part f_{e} of f. It should be noted that

$$\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t \;=\; \frac{1}{b-a}\int_a^b f_{\mathrm{e}}(t)\,\mathrm{d}t\,.$$

2. STATEMENT OF THE RESULTS

Theorem 2.1. *Let f be a function defined on $[a,b]$ and belonging to* $\mathrm{Lip}_M(\kappa)$ *with $\kappa \in (0,1]$. Then, for each $x \in [a, \frac{1}{2}(a+b)]$, the remainder in* (11) *satisfies*

$$|E(f;x)| \;\le\; \frac{M}{b-a}\cdot\frac{(2x-2a)^{\kappa+1}+(a+b-2x)^{\kappa+1}}{2^{\kappa}(\kappa+1)}\,. \tag{14}$$

This inequality is sharp for each admissible x. Equality is attained if and only if $f = \pm M f_* + c$ with $c \in \mathbb{R}$ and

$$f_*(t) \;:=\; \begin{cases} (x-t)^{\kappa} & \text{for} \quad a \le t \le x \\ (t-x)^{\kappa} & \text{for} \quad x \le t \le \frac{1}{2}(a+b) \\ f_*(a+b-t) & \text{for} \quad \frac{1}{2}(a+b) \le t \le b\,. \end{cases}$$

Setting $\kappa = 1$ and $x = \frac{1}{2}(a+b)$, we recover the estimate (2) of Theorem A. However, setting $\kappa = 1$ and $x = a$, we find that the estimate (3) of Theorem A is not sharp; for a sharp bound we have to replace the 3 in the denominator on the right-hand side by 4. All the results in [1] which are derived from (3) can be improved accordingly.

Although the estimate (14) is sharp, we can establish an improvement in the spirit of Iyengar's Theorem B. We restrict ourselves to $\kappa = 1$ since otherwise the result would be in terms of the solution of a transcendental equation.

Theorem 2.2. *Let f be a function defined on $[a,b]$ and belonging to* Lip_M. *Then, for each $x \in [a, \frac{1}{2}(a+b)]$, the remainder in* (11) *satisfies*

$$|E(f;x)| \;\le\; \frac{M}{4}\cdot\frac{(2x-2a)^2+(a+b-2x)^2}{b-a} \;-\; \frac{\big(f(a+b-x)-f(x)\big)^2}{4M(b-a)}\,. \tag{15}$$

This inequality is sharp for each admissible x. Equality is attained if and only if $f = \pm M f_*(\delta;\cdot) + c$ with $c \in \mathbb{R}$ and

$$f_*(\delta;t) := \begin{cases} x-t & \text{for} \quad a \le t \le x \\ t-x & \text{for} \quad x \le t \le \frac{1}{2}(a+b+\delta) \\ a+b-x-t+\delta & \text{for} \quad \frac{1}{2}(a+b+\delta) \le t \le a+b-x \\ t-a-b+x+\delta & \text{for} \quad a+b-x \le t \le b\,, \end{cases}$$

where δ is any real number satisfying $|\delta| \le a+b-2x$.

Note that for $x = a$, we obtain the conclusion of Theorem B under a weaker hypothesis. In particular, we see that Iyengar's inequality is sharp in a somewhat larger class of functions. We also obtain a further improvement of the estimate (3) of Theorem A.

In terms of the even and the odd part of f (see (12)–(13)), we may state the inequality (15) as

$$\left|\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t - f_{\mathrm{e}}(x)\right| \le M\frac{\big(x-\frac{1}{2}(a+b)\big)^2 + (x-a)^2}{b-a} - \frac{\big(f_{\mathrm{o}}(x)\big)^2}{M(b-a)}$$

for $x \in [a,b]$. This should be compared with Ostrowski's inequality (10).

Sometimes, for a Lipschitzian function f a more refined condition,

$$\ell(t_2-t_1) \le f(t_2)-f(t_1) \le L(t_2-t_1) \qquad (a \le t_1 < t_2 \le b) \tag{16}$$

with $\ell < L$, is known. Observe in particular that if $f \in \mathrm{Lip}_M$ is non-decreasing, then (16) holds with $L = M$ and $\ell = 0$, and if it is non-increasing, then it holds with $L = 0$ and $\ell = -M$.

If (16) is satisfied, then the function

$$g(t) := f(t) - \tfrac{1}{2}(L+\ell)t$$

belongs to the class Lip_M with $M = \frac{1}{2}(L-\ell)$. Moreover, $E(g;x) = E(f;x)$ for all $x \in [a,b]$. Therefore, applying Theorem 2.2 to g, we can derive a refined estimate for $E(f;x)$ in terms of ℓ and L. We leave it to the reader to state a refinement of Theorem 2.2 for monotonic functions.

If a function f is convex on an open interval that contains $[a,b]$, then it satisfies (16); see [5, p. 3, Corollary 1.1.6]. We may call ℓ and L the *extremal slopes* of f on $[a,b]$.

THEOREM 2.3. *Let f be a convex function on an open interval that contains $[a,b]$, and let ℓ and L be a lower and an upper bound for the* *slopes of* f *on* $[a,b]$*. Then, for each* $x \in \left[a, \frac{1}{2}(a+b)\right]$*, the remainder in* (11) *satisfies*

$$-\frac{1}{8}(L-\ell)(3a+b-4x)_+ \;\le\; E(f;x) \;\le\; \frac{(x-a)^2}{2(b-a)}(L-\ell), \tag{17}$$

where $t_+ := \frac{1}{2}(t+|t|)$*.*

Equality is attained in the upper estimate when $x \neq a$ *and* $f = f_* + c$ *with* $c \in \mathbb{R}$*,*

$$f_*(t) \;:=\; \begin{cases} \ell t + x(\mu-\ell) & \text{for} \quad a \le t \le x \\ \lambda t & \text{for} \quad x \le t \le a+b-x \\ Lt + (a+b-x)(\mu - L) & \text{for} \quad a+b-x \le t \le b\,, \end{cases}$$

and any $\lambda \in [\ell, L]$*. For* $x = a$ *equality is attained when* $\lambda \in [\ell, L]$ *and* $f(t) = \lambda t + c$ *with* $c \in \mathbb{R}$*.*

Equality is attained in the lower estimate when $x \in [a, \frac{1}{4}(3a+b)]$ *and* $f = f_* + c$ *with* $c \in \mathbb{R}$ *and*

$$f_*(t) \;:=\; \begin{cases} \ell t & \text{for} \quad a \le t \le \frac{1}{2}(a+b) \\ Lt - \frac{1}{2}(a+b)(L-\ell) & \text{for} \quad \frac{1}{2}(a+b) \le t \le b\,. \end{cases}$$

For $x \in [\frac{1}{4}(3a+b), \frac{1}{2}(a+b)]$ *equality is attained when* $\lambda \in [\ell, L]$ *and* $f(t) = \lambda t + c$ *with* $c \in \mathbb{R}$*.*

Theorem 2.3 shows that $E(f;x) \ge 0$ for $x \in [\frac{1}{4}(3a+b), \frac{1}{2}(a+b)]$. The extremal values $x = \frac{1}{4}(3a+b)$ and $x = \frac{1}{2}(a+b)$ give the Maclaurin formula and the midpoint rule, respectively.

For $x = a$ and $x = \frac{1}{2}(a+b)$, Theorem 2.3 implies the Hermite–Hadamard inequality (1) under the slightly stronger hypothesis that f is convex on an interval larger than $[a,b]$. But by considering first $[a+\varepsilon, b-\varepsilon]$ and letting $\varepsilon \to 0+$, we can easily deduce the result in its full generality.

Now we consider differentiable functions f. Our first result is in terms of $f'(x) - f'(a+b-x)$. Therefore it may be reformulated as an Ostrowski type inequality in terms of $f'_{\mathrm{e}}(x)$.

Theorem 2.4. *Let* f *be a differentiable function defined on* $[a,b]$ *with* $f' \in \mathrm{Lip}_M$*. For* $x \in [a, \frac{1}{2}(a+b)]$*, define*

$$\Gamma := \frac{a+b-2x}{b-a} \qquad \textit{and} \qquad \Delta \;:=\; \frac{|f'(x) - f'(a+b-x)|}{(a+b-2x)M}$$

Then the remainder in (11) *satisfies*

$$\begin{aligned}|E(f;x)| \;\le\; & \frac{M(x-a)^3}{3(b-a)} + \frac{M(b-a)^2}{32}\left(1-\Delta^2\right)\Gamma^3 \\ & + \frac{M(b-a)^2}{96}\,\Gamma\Delta\left|12-24\Gamma+(3+\Delta^2)\Gamma^2\right|.\end{aligned} \tag{18}$$

This inequality is sharp for each admissible x. Equality is attained for $f(t) = \pm M\int f_*'(\delta;t)\,\mathrm{d}t + c_1 t + c_0$, where $c_0, c_1 \in \mathbb{R}$,

$$f_*'(\delta;t) := \begin{cases} x-t-\delta & \text{for} \quad a \le t \le \frac{1}{4}(a+b+2x-2\delta) \\ t-\frac{1}{2}(a+b) & \text{for} \quad \frac{1}{4}(a+b+2x-2\delta) \le t \le \frac{1}{2}(a+b) \\ -f_*'(a+b-t) & \text{for} \quad \frac{1}{2}(a+b) \le t \le b \end{cases}$$

and $\delta = \left(x-\frac{1}{2}(a+b)\right)\Delta\cdot\operatorname{sgn}(12-24\Gamma+3\Gamma^2+\Delta^2\Gamma^2)$.

Since $f' \in \mathrm{Lip}_M$, we must have $0 \le \Delta \le 1$. By standard calculus, we can discuss the behaviour of the right-hand side of (18) and determine the maximum value in dependence of x. This leads us to the following result.

Corollary 2.1. *Let f be a differentiable function defined on $[a,b]$ with $f' \in \mathrm{Lip}_M$. If $a \le x \le \frac{1}{4}(3a+b)$, then*

$$\begin{aligned}|E(f;x)| \;\le\; \frac{M}{12(b-a)}\Big\{ & 4(x-a)^3 + 6(a+b-2x)(x-a)^2 - (a+b-2x)^3 \\ & + 2\Big[(a+b-2x)^2 - 4(x-a)^2\Big]^{3/2}\Big\}.\end{aligned}$$

This inequality is sharp for each admissible x. Equality is attained for $f(t) = \pm M\int f_*'(t)\,\mathrm{d}t + c_1 t + c_0$ with $c_0, c_1 \in \mathbb{R}$ and

$$f_*'(t) := \begin{cases} \frac{1}{2}(a+b) - 2\gamma - t & \text{for} \quad a \le t \le \frac{1}{2}(a+b) - \gamma \\ t - \frac{1}{2}(a+b) & \text{for} \quad \frac{1}{2}(a+b) - \gamma \le t \le \frac{1}{2}(a+b) + \gamma \\ \frac{1}{2}(a+b) + 2\gamma - t & \text{for} \quad \frac{1}{2}(a+b) + \gamma \le t \le b\,, \end{cases}$$

where $\gamma = \frac{1}{2}\sqrt{(b-a)(3a+b-4x)}$.

If $\frac{1}{4}(3a+b) \le x \le \frac{1}{2}(a+b)$, then

$$|E(f;x)| \le \frac{M}{12(b-a)}\left[4(x-a)^3 + 6(a+b-2x)(x-a)^2 - (a+b-2x)^3\right].$$

This inequality is sharp for each admissible x. Equality is attained for $f(t) = \pm\frac{1}{2}Mt^2 + c_1 t + c_0$ with $c_0, c_1 \in \mathbb{R}$.

If f has a bounded second derivative and $x = a$ (trapezoidal rule) or $x = \frac{1}{2}(a+b)$ (midpoint rule), then we recover two standard results of Numerical Analysis.

As it is immediately seen, a function that furnishes equality in (18) can always be chosen such that $d := f(a+b-x) - f(x)$ has a prescribed value. In other words, the bound of Theorem 2.4 cannot be improved if the value of d is known. As such, the situation is quite different from that of Theorem C which we now improve and generalize.

THEOREM 2.5. *Let f be a differentiable function defined on $[a,b]$ with $f' \in \mathrm{Lip}_M$. Let $x \in [a, \frac{1}{2}(a+b)]$, and suppose that $f'(x) = f'(a+b-x) = 0$. Then the remainder in (11) satisfies*

$$|E(f;x)| \le \frac{1}{b-a}\left[\frac{M}{3}(x-a)^3 + \frac{M}{32}(a+b-2x)^3 - \frac{\big(f(a+b-x)-f(x)\big)^2}{2M(a+b-2x)}\right]. \tag{19}$$

This inequality is sharp for each $x \in [a, \frac{1}{2}(a+b)]$. Equality is attained for $f(t) = \pm M \int f_*'(t)\,\mathrm{d}t + c$ with $c \in \mathbb{R}$ and

$$f_*'(t) := \begin{cases} x - t & \text{for } a \le t \le \frac{1}{4}(a+b+2x) - \delta =: x_1 \\ t - \frac{1}{2}(a+b) + 2\delta & \text{for } x_1 \le t \le \frac{1}{4}(3a+3b-2x) - \delta =: x_2 \\ a+b-x-t & \text{for } x_2 \le t \le b\,, \end{cases}$$

where δ is any real number satisfying $|\delta| \le \frac{1}{4}(a+b-2x)$.

For $x = a$, we obtain an improved and sharp version of Theorem C. When f satisfies the hypotheses of Theorem 2.5, then it also satisfies the hypotheses of Theorem 2.4 with $\Delta = 0$. But in this case, Theorem 2.5 gives a better result than Theorem 2.4.

We now want to take advantage of a possible higher regularity of f and establish results related to Theorem D. The following theorem may also be seen as a generalization of the Euler–Maclaurin formula.

THEOREM 2.6. *Let f be a function defined on $[a,b]$ and having there a piecewise continuous n-th derivative. Let Q_n be any monic polynomial of* *degree n such that $Q_n(t) \equiv (-1)^n Q_n(a+b-t)$. Define*

$$K_n(t) := \begin{cases} (t-a)^n & \text{for} \quad a \le t \le x \\ Q_n(t) & \text{for} \quad x < t \le a+b-x \\ (t-b)^n & \text{for} \quad a+b-x < t \le b\,. \end{cases}$$

Then, for the remainder in (11), *we have*

$$E(f;x) = \sum_{\nu=1}^{n-1} \left[\frac{(x-a)^{\nu+1}}{(\nu+1)!} - \frac{Q_n^{(n-\nu-1)}(x)}{n!} \right] \frac{f^{(\nu)}(a+b-x) + (-1)^\nu f^{(\nu)}(x)}{b-a}$$
$$+ \frac{(-1)^n}{n!\,(b-a)} \int_a^b K_n(t) f^{(n)}(t)\,\mathrm{d}t\,. \qquad (20)$$

By appropriate choices of the polynomial Q_n, we can deduce from Theorem 2.6 various results in the spirit of Theorem D.

COROLLARY 2.2. *Let f be $n-1$ times differentiable on $[a,b]$ with $f^{(n-1)}$ belonging to Lip_M. Define*

$$c_{n\nu} := \prod_{j=0}^{\nu} \frac{1}{2} \left(1 + \frac{n-\nu}{n-j} \right)$$

and

$$\phi_{n\nu}(x) := (x-a)^{\nu+1} - c_{n\nu} \left(x - \frac{a+b}{2} \right)^{\nu+1} \qquad \left(a \le x \le \frac{a+b}{2} \right).$$

Then the remainder in (11) *satisfies*

$$\left| E(f;x) - \sum_{\nu=1}^{n-1} \phi_{n\nu}(x)\, \frac{f^{(\nu)}(a+b-x) - (-1)^\nu f^{(\nu)}(x)}{(\nu+1)!\,(b-a)} \right|$$
$$\le \frac{2M}{b-a} \left[\frac{(x-a)^{n+1}}{(n+1)!} + \frac{(a+b-2x)^{n+1}}{2^{2n+1} n!} \right].$$

This inequality is sharp for each admissible x. Equality is attained for

$$f(t) := \pm M f_*(t) + c_0 + c_1 t + \cdots + c_{n-1} t^{n-1}$$

with $c_0,\dots,c_{n-1}\in\mathbb{R}$ *and*

$$f_*(t) := \begin{cases} \dfrac{(t-x)^n}{n!} & \text{for } a\le t\le x \\ \displaystyle\int_x^t \frac{(t-\xi)^{n-1}}{(n-1)!}\operatorname{sgn} U_n\left(\frac{2\xi-a-b}{a+b-2x}\right)\mathrm{d}\xi & \text{for } x\le t\le a+b-x \\ \dfrac{(a+b-x-t)^n}{n!} & \text{for } a+b-x\le t\le b, \end{cases}$$

where U_n *is the* n*-th Chebyshev polynomial of the second kind.*

The following corollary generalizes Theorem D.

COROLLARY 2.3. *Let* f *be a function defined on* $[a,b]$ *and having there a piecewise continuous* n*-th derivative. Let* $x\in[a,\frac{1}{2}(a+b)]$*, and suppose that* $f^{(\nu)}(x)=f^{(\nu)}(a+b-x)=0$ *for* $\nu=1,\dots,n-1$*. Then, for each* p *in* $[1,\infty]$*, there exists a smallest number* $R(n,p,x)$ *such that for the remainder in* (11)

$$|E(f;x)| \le \frac{R(n,p,x)}{n!}\left\|f^{(n)}\right\|_p . \tag{21}$$

In particular, we have:

$$R(n,\infty,x) = \frac{1}{b-a}\left[\frac{2(x-a)^{n+1}}{n+1}+\frac{(a+b-2x)^{n+1}}{4^n}\right] \tag{22}$$

and $f=c_0+c_1f_*$ *with* $c_0,c_1\in\mathbb{R}$ *and* f_* *as defined in Corollary* 2.2 *yields equality in* (21)*;*

$$R(n,2,x) = \frac{1}{(b-a)\sqrt{n+\frac{1}{2}}}\left[(x-a)^{2n+1}+\frac{(a+b-2x)^{2n+1}}{2\binom{2n}{n}^2}\right]^{1/2} \tag{23}$$

and $f=c_0+c_1f_*$ *with* $c_0,c_1\in\mathbb{R}$ *and*

$$f_*(t) := \begin{cases} \dfrac{(t-a)^{2n}}{(2n)!}-\displaystyle\sum_{\nu=1}^{n}\frac{(t-x)^{n-\nu}(x-a)^{n+\nu}}{(n-\nu)!\,(n+\nu)!} & \text{for } a\le t\le x \\ \dfrac{1}{(2n)!}\left[\left(\dfrac{a+b}{2}-t\right)^2-\left(\dfrac{a+b}{2}-x\right)^2\right]^n & \text{for } x\le t\le \frac{1}{2}(a+b) \\ f_*(a+b-t) & \text{for } \frac{1}{2}(a+b)\le t\le b \end{cases}$$

yields equality in (21); *if*

$$a \le x \le \frac{\left(2^{2-1/n}+1\right)a+b}{2^{2-1/n}+2}\,, \tag{24}$$

then

$$R(n,1,x) = \frac{(a+b-2x)^n}{2^{2n-1}(b-a)}\,. \tag{25}$$

For $x = a$, our last result gives explicit values of the error constants $R(n,p)$ for $p = 1, 2, \infty$ in Theorem D.

Corollary 2.4. *In the situation of Theorem* D, *we have:*

$$R(n,1) = \frac{(b-a)^{n-1}}{2^{2n-1}}\,, \tag{26}$$

$$R(n,2) = \frac{(n!)^2\,(b-a)^{n-1/2}}{(2n)!\,\sqrt{2n+1}}\,, \tag{27}$$

$$R(n,\infty) = \frac{(b-a)^n}{4^n}\,. \tag{28}$$

3. TECHNIQUES AND LEMMAS

For the convenience of the reader we shall first collect some technical results, which will be used in the proofs of our Theorems. As we will see, we make decisive use of the following observation, which we state as a remark for later reference.

Remark 3.1. Let $f \in \mathrm{Lip}_M$, and suppose that the graph of f passes through the point (ξ, η). Then

$$\varphi(\xi,\eta;t) := \eta - M\,|t-\xi| \le f(t) \le \eta + M\,|t-\xi| =: \psi(\xi,\eta;t). \tag{29}$$

The functions $\varphi(\xi,\eta;\,\cdot\,)$ and $\psi(\xi,\eta;\,\cdot\,)$ themselves belong to Lip_M. Moreover, if we know k points $(\xi_1,\eta_1),\ldots,(\xi_k,\eta_k)$ on the graph of f, then the estimate (29) can be refined. In fact, defining

$$\varphi(t) := \max_{1\le j\le k} \varphi(\xi_j,\eta_j;t) \quad \text{and} \quad \psi(t) := \min_{1\le j\le k} \psi(\xi_j,\eta_j;t),$$

we have

$$\varphi(t) \le f(t) \le \psi(t),$$

and again φ and ψ belong to Lip_M.

The following observation, which we state as a lemma, will be very useful. Roughly spoken it implies that, if an estimate for the remainder of a quadrature formula holds for all functions f which are piecewise continuously differentiable and satisfy $|f'(t)| \le M$, then it also holds for all functions f from the wider class Lip_M.

Lemma 3.1. *Let g be a piecewise continuous function on $[a,b]$ such that $\int_a^b |g(t)|\,\mathrm{d}t \le K$. Let $f \in \mathrm{Lip}_M$, and consider a partition*

$$a = t_0 < t_1 < \cdots < t_N = b \tag{30}$$

of the interval $[a,b]$. Define $\widetilde{f} : [a,b] \to \mathbb{R}$ by

$$\left.\begin{array}{l} \widetilde{f}(t) := \dfrac{t-t_j}{t_{j-1}-t_j}\, f(t_{j-1}) + \dfrac{t_{j-1}-t}{t_{j-1}-t_j}\, f(t_j) \qquad \text{for } t \in [t_{j-1}, t_j] \\ \qquad\qquad (j = 1, \ldots, N). \end{array}\right\} \tag{31}$$

Then $\widetilde{f}$ is piecewise continuously differentiable, $\left|\widetilde{f}'(t)\right| \le M$ at all points t in $[a,b] \setminus \{t_0, \ldots, t_N\}$, and

$$\left| f(t) - \widetilde{f}(t) \right| \le \frac{M}{2} \max_{1 \le j \le N} (t_j - t_{j-1}) \qquad (a \le t \le b),$$

$$\left| \int_a^b g(t) f(t)\,\mathrm{d}t - \int_a^b g(t)\widetilde{f}(t)\,\mathrm{d}t \right| \le \frac{MK}{2} \max_{1 \le j \le N} (t_j - t_{j-1}).$$

Proof. Obviously, the function $\widetilde{f}$ is piecewise linear. Moreover, if $t \in (t_{j-1}, t_j)$ for some $j \in \{1, \ldots, N\}$, then $\widetilde{f}'(t)$ exists and, because of the Lipschitz condition for f,

$$\left|\widetilde{f}'(t)\right| = \left| \frac{f(t_{j-1}) - f(t_j)}{t_{j-1} - t_j} \right| \le M.$$

Now let t be any point in $[a,b]$. Clearly t lies in some interval $[t_{j-1}, t_j]$ and therefore

$$\left| f(t) - \widetilde{f}(t) \right| = \left| \frac{t-t_j}{t_{j-1}-t_j}\Big(f(t) - f(t_{j-1})\Big) + \frac{t_{j-1}-t}{t_{j-1}-t_j}\Big(f(t) - f(t_j)\Big) \right|.$$

Hence, making again use of the Lipschitz condition, we find that

$$\left|f(t)-\widetilde{f}(t)\right| \le 2M\frac{|t-t_j|\cdot|t-t_{j-1}|}{|t_j-t_{j-1}|} \le \frac{M}{2}\,|t_j-t_{j-1}|\,.$$

With this, the proof is easily completed. ▮

The following result is a special case of the representation of functionals by Peano kernels. It may be directly verified by integration by parts on appropriate subintervals.

Lemma 3.2. *Let f be piecewise continuously differentiable on $[a,b]$. Then*

$$E(f;x) \;=\; \frac{1}{b-a}\int_a^b K(t)f'(t)\,\mathrm{d}t, \tag{32}$$

where

$$K(t) := \begin{cases} a-t & \text{for} \quad a\le t\le x \\ \frac{1}{2}(a+b)-t & \text{for} \quad x<t\le(a+b-x) \\ b-t & \text{for} \quad (a+b-x)<t\le b. \end{cases} \tag{33}$$

Finally, we shall use an interesting property of the Chebyshev polynomials of the second kind. It seems that this property has not been mentioned in the literature so far, and therefore it may need a proof. For the following lemma, we should realize that, if f is a piecewise continuous function on an interval $[a,b]$, then a primitive of order k, as given by

$$f_k(t) \;:=\; \int_a^t\left(\int_a^{t_k}\left(\ldots\int_a^{t_2} f(t_1)\,\mathrm{d}t_1\ldots\right)\mathrm{d}t_{k-1}\right)\mathrm{d}t_k\,,$$

can be expressed by a single integral as

$$f_k(t) \;=\; \int_a^t \frac{(t-\xi)^{k-1}}{(k-1)!}\,f(\xi)\,\mathrm{d}\xi\,.$$

Lemma 3.3. *Let*

$$F_k(t) \;:=\; \int_{-1}^t \frac{(t-\xi)^{k-1}}{(k-1)!}\,\operatorname{sgn}U_n(\xi)\,\mathrm{d}\xi\,,$$

where U_n is the n-th Chebyshev polynomial of the second kind. Then

$$F_k(-1) \;=\; F_k(1) \;=\; 0 \qquad \textit{for all} \quad k = 1, 2, ...,.$$

Proof. That $F_k(-1) = 0$, is a trivial consequence of the definition of F_k.

As we have mentioned in the paragraph preceding the lemma, F_k is a primitive of order k of $\operatorname{sgn} U_n$. Since

$$x_\nu \;:=\; -\cos\frac{\nu\pi}{n+1} \qquad (\nu = 1, \dots, n)$$

are the zeros of U_n in increasing order, we find that

$$F_k(t) \;=\; (-1)^n \left(\frac{(t+1)^k}{k!} \;+\; 2\sum_{\nu=1}^{n} (-1)^\nu \frac{(t-x_\nu)_+^k}{k!} \right).$$

Noting that

$$1 - x_\nu \;=\; 1 + \cos\frac{\nu\pi}{n+1} \;=\; 2\cos^2\!\left(\frac{\nu\pi}{2(n+1)}\right),$$

we obtain

$$F_k(1) \;=\; \frac{2^k(-1)^n}{k!}\left[1 + 2\sum_{\nu=1}^{n}(-1)^\nu \cos^{2k}\!\left(\frac{\nu\pi}{2(n+1)}\right)\right].$$

Using the identity

$$\cos^{2k}\theta \;=\; \frac{1}{2^{2k}}\sum_{j=0}^{2k}\binom{2k}{j}\cos\bigl(2(k-j)\theta\bigr) \qquad (\theta \in \mathbb{R}),$$

which is easily deduced from formula 1.320(5) in [4, p. 31], we may rewrite $F_k(1)$ as

$$F_k(1) \;=\; \frac{2^k(-1)^n}{k!}\left[1 \;+\; \frac{1}{2^{2k-1}}\sum_{\nu=1}^{n}(-1)^\nu \sum_{j=0}^{2k}\binom{2k}{j}\cos\!\left(\frac{(k-j)\nu\pi}{n+1}\right)\right].$$

On the right-hand side, the order of summation may be interchanged. Employing the identity (see e.g., [4, p. 37, formula 1.343(1)])

$$\sum_{\nu=1}^{n}(-1)^\nu \cos\nu\theta \;=\; -\frac{1}{2} \;+\; \frac{(-1)^n\cos\bigl((2n+1)\theta/2\bigr)}{2\cos(\theta/2)} \qquad (\theta \in \mathbb{R})$$

and noting that

$$\cos\left(\frac{(2n+1)(k-j)\pi}{2(n+1)}\right) = (-1)^{k-j}\cos\left(\frac{(k-j)\pi}{2(n+1)}\right),$$

we therefore obtain

$$\begin{aligned} F_k(1) &= \frac{2^k(-1)^n}{k!}\left[1 + \frac{1}{2^{2k-1}}\sum_{j=0}^{2k}\binom{2k}{j}\left(-\frac{1}{2} + \frac{(-1)^{n+k-j}}{2}\right)\right] \\ &= \frac{2^k(-1)^n}{k!}\left[1 - \frac{(1+1)^{2k}}{2^{2k}} + (-1)^{n+k}\frac{(1-1)^{2k}}{2^{2k}}\right], \end{aligned}$$

which is obviously zero. This completes the proof. ▮

4. PROOFS

Proof (Proof of Theorem 2.1). Set $c := \frac{1}{2}(a+b)$. Then, as a consequence of the Lipschitz condition,

$$\begin{aligned} |E(f;x)| &= \frac{1}{b-a}\left|\int_a^c \big(f(t)-f(x)\big)\,\mathrm{d}t + \int_c^b \big(f(t)-f(a+b-x)\big)\,\mathrm{d}t\right| \\ &\leq \frac{M}{b-a}\left[\int_a^c |t-x|^\kappa\,\mathrm{d}t + \int_c^b |t-a-b+x|^\kappa\,\mathrm{d}t\right] \\ &= \frac{2M}{b-a}\cdot\frac{(x-a)^{\kappa+1} + \left(\frac{1}{2}(a+b)-x\right)^{\kappa+1}}{\kappa+1}, \end{aligned}$$

which is the bound in (14). The statement on the occurrence of equality is easily verified. ▮

Proof (Proof of Theorem 2.2). Let $u, v \in \mathbb{R}$, and denote by $\mathcal{F}_M(u,v)$ the class of all functions which belong to Lip_M on $[a,b]$ and satisfy $f(x) = u$ and $f(a+b-x) = v$. Hence the graph of each $f \in \mathcal{F}_M(u,v)$ passes through the points (x,u) and $(a+b-x,v)$. In view of Remark 3.1 with $k=2$, we therefore have

$$\varphi(t) := u - Mf_*\left(\frac{u-v}{M};t\right) \leq f(t) \leq u + Mf_*\left(\frac{v-u}{M};t\right) =: \psi(t)$$

for each $f \in \mathcal{F}_M(u,v)$ and all $t \in [a,b]$, where f_* is the function specified in the theorem. Moreover, φ and ψ themselves belong to $\mathcal{F}_M(u,v)$. Thus, for any $f \in \mathcal{F}_M(u,v)$,

$$\begin{aligned} |E(f;x)| \;&\le\; \sup_{g\in\mathcal{F}_M(u,v)} \left| \frac{1}{b-a}\int_a^b g(t)\,\mathrm{d}t - \frac{u+v}{2} \right| \\ &=\; \max\left\{ |E(\varphi;x)|\;,\;|E(\psi;x)| \right\}. \end{aligned}$$

A simple calculation shows that $|E(\varphi;x)| = |E(\psi;x)|$ and that this value is equal to the right-hand side of (15). This proves (15) and verifies the statement on the occurrence of equality. ∎

Proof (Proof of Theorem 2.3). If f satisfies the hypotheses of Theorem 2.3, then the associated function $\widetilde{f}$, as defined by (31), is again convex and

$$\ell \;\le\; \widetilde{f}'(t) \;\le\; L \qquad \text{for} \quad t \in [a,b] \setminus \{t_0,\dots,t_N\}.$$

Furthermore, we may choose the partition (30) such that x and $a+b-x$ are amongst the points $t_0,\dots,t_N$, so that

$$\widetilde{f}(x) = f(x) \qquad \text{and} \qquad \widetilde{f}(a+b-x) = f(a+b-x).$$

Now it is easily seen by employing Lemma 3.1 with $g(t) \equiv 1$ and considering sufficiently refined partitions that it is enough to prove the theorem under the additional assumption that f is piecewise continuously differentiable. But then

$$\ell \;\le\; f'_-(t_1) \;\le\; f'_+(t_2) \;\le\; L \qquad \text{for} \quad a \le t_1 \le t_2 \le b, \tag{34}$$

where

$$f'_\pm(x) \;:=\; \lim_{\varepsilon\to 0+} f'(x \pm \varepsilon).$$

Moreover, Lemma 3.2 applies. Discussing $K(t)f'(t)$ on the four subintervals

$$\big(a,x\big), \quad \big(x,\tfrac{1}{2}(a+b)\big), \quad \big(\tfrac{1}{2}(a+b),(a+b-x)\big), \quad \big((a+b-x),b\big)$$

under the side condition (34), we find that the integral in (32) becomes largest when

$$f'(t) \;=\; \begin{cases} \ell & \text{for} \quad a \le t \le x \\ \lambda & \text{for} \quad x < t \le a+b-x \\ L & \text{for} \quad a+b-x < t \le b \end{cases}$$

with any $\lambda \in [\ell, L]$. This gives the upper bound in (17).

Similarly we note that $E(f;x)$ becomes smallest for a function f such that

$$f'(t) = \begin{cases} \lambda_1 & \text{for} \quad a \le t \le \frac{1}{2}(a+b) \\ \lambda_2 & \text{for} \quad \frac{1}{2}(a+b) \le t \le b, \end{cases}$$

where $\ell \le \lambda_1 \le \lambda_2 \le L$. Calculating $E(f;x)$ for these functions f, we find that the minimum value depends on x. If $x \in [a, \frac{1}{4}(3a+b)]$, then it is attained for $\lambda_1 = \ell$ and $\lambda_2 = L$, while for $x \in [\frac{1}{4}(3a+b), \frac{1}{2}(a+b)]$, it is attained when $\lambda_1 = \lambda_2 \in [\ell, L]$. This leads to the lower estimate in (17). The proof also reveals the cases of equality. ▮

Proof (Proof of Theorem 2.4). Denote by $\mathcal{F}_M(\Delta)$ the class of all functions f on $[a,b]$ such that $f' \in \mathrm{Lip}_M$ and

$$\frac{|f'(x) - f'(a+b-x)|}{(a+b-2x)M} = \Delta.$$

We have to determine

$$A := \sup_{f \in \mathcal{F}_M(\Delta)} |E(f;x)|.$$

First we note that, if $f \in \mathcal{F}_M(\Delta)$, then its even part f_{e}, as defined by (12), also belongs to $\mathcal{F}_M(\Delta)$ and $E(f;x) = E(f_{\mathrm{e}};x)$. Hence we may restrict our considerations to the subclass $\mathcal{F}_{M,\mathrm{e}}(\Delta)$ consisting of all even functions in $\mathcal{F}_M(\Delta)$.

If $f \in \mathcal{F}_{M,\mathrm{e}}(\Delta)$, then f' is an odd function on $[a,b]$. In view of Lemma 3.2, we therefore have

$$A = \sup_{f \in \mathcal{F}_{M,\mathrm{e}}(\Delta)} \frac{2}{b-a} \left| \int_a^{(a+b)/2} K(t) f'(t) \, \mathrm{d}t \right|.$$

Next we note that every $f \in \mathcal{F}_{M,\mathrm{e}}(\Delta)$ satisfies

$$f'\big(\tfrac{1}{2}(a+b)\big) = 0 \qquad \text{and} \qquad f'(x) = \pm \frac{a+b-2x}{2} M\Delta. \tag{35}$$

For the following discussion, it suffices to consider the case that $f'(x)$ is non-negative. Then (35) fixes two points on the graph of f'. In view of Remark 3.1, this allows us to establish within $\mathcal{F}_{M,\mathrm{e}}(\Delta)$ a majorant and a minorant for f'. In particular, we find that

$$MK(t)\Phi(t) \le K(t)f'(t) \le MK(t)\Psi(t) \qquad \text{for} \quad t \in [a, \tfrac{1}{2}(a+b)],$$

where

$$\Phi(t) := \begin{cases} x - t + d & \text{for} \quad a \le t \le \frac{1}{4}(a+b+2x+2d) \\ t - \frac{1}{2}(a+b) & \text{for} \quad \frac{1}{4}(a+b+2x+2d) < t \le \frac{1}{2}(a+b) \end{cases}$$

and

$$\Psi(t) := \begin{cases} t - x + d & \text{for} \quad a \le t \le \frac{1}{4}(a+b+2x-2d) \\ \frac{1}{2}(a+b) - t & \text{for} \quad \frac{1}{4}(a+b+2x-2d) < t \le \frac{1}{2}(a+b) \end{cases}$$

with $d := \bigl(\frac{1}{2}(a+b) - x\bigr)\Delta$. Hence

$$A = \max\left\{ \frac{2M}{b-a} \left| \int_a^{(a+b)/2} K(t)\Phi(t)\,\mathrm{d}t \right|, \frac{2M}{b-a} \left| \int_a^{(a+b)/2} K(t)\Psi(t)\,\mathrm{d}t \right| \right\}.$$

The two integrals on the right-hand side can be calculated explicitly. Each of them may be bigger than the other, depending on x and Δ. Carrying out the details, we arrive at the conclusion of Theorem 2.4. ▮

Proof (Proof of Corollary 2.1). As we have pointed out in § 2, Corollary 2.1 can be deduced from Theorem 2.4, but a dircet proof may be simpler. In view of Lemmas 3.1 and 3.2, it suffices to prove the corollary for functions f which have piecewise a continuous second derivative such that $|f''(t)| \le M$. For that class of functions, we may use the representation of $E(f;x)$ by means of the second Peano kernel. It says that

$$E(f;x) = \frac{1}{b-a} \int_a^b K_2(t) f''(t)\,\mathrm{d}t,$$

where

$$K_2(t) := \begin{cases} \frac{1}{2}(t-a)^2 & \text{for} \quad a \le t \le x \\ \frac{1}{2}[t^2 - (a+b)t + (b-a)x + a^2] & \text{for} \quad x < t \le a+b-x \\ \frac{1}{2}(t-b)^2 & \text{for} \quad a+b-x < t \le b. \end{cases}$$

Now we see that $|E(f;x)|$ becomes largest when $f''(t) = M\mathrm{sgn}K_2(t)$. The proof is easily completed by determining the sign of $K_2(t)$ in dependence of x and t. ▮

Proof (Proof of Theorem 2.5). Let us denote by $\mathcal{F}'_M(\Delta)$ be class of all functions which are differentiable on $[a,b]$ with f' belonging to Lip_M and which satisfy

$$f(a+b-x)-f(x)=\Delta \quad \text{and} \quad f'(x)=f'(a+b-x)=0.$$

We want to determine for each $x \in [a, \frac{1}{2}(a+b)]$ the supremum of $|E(f;x)|$ over all $f \in \mathcal{F}'_M(\Delta)$. Using Lemma 3.2, we find by a short reflection that

$$\sup_{f \in \mathcal{F}'_M(\Delta)} |E(f;x)| \;=\; S_1 + S_2 + S_3, \tag{36}$$

where

$$S_1 \;=\; \sup_{f \in \mathcal{F}'_M(\Delta)} \left| \frac{1}{b-a} \int_a^x (a-t) f'(t)\,\mathrm{d}t \right|,$$

$$S_2 \;=\; \sup_{f \in \mathcal{F}'_M(\Delta)} \left| \frac{1}{b-a} \int_x^{a+b-x} \big(\tfrac{1}{2}(a+b)-t\big) f'(t)\,\mathrm{d}t \right|,$$

$$S_3 \;=\; \sup_{f \in \mathcal{F}'_M(\Delta)} \left| \frac{1}{b-a} \int_{a+b-x}^{b} (b-t) f'(t)\,\mathrm{d}t \right|.$$

In view of Remark 3.1, it is easily seen that

$$S_1 \;=\; S_3 \;=\; \frac{M}{b-a} \int_a^x (a-t)(t-x)\,\mathrm{d}t \;=\; \frac{(x-a)^3 M}{6(b-a)}\,. \tag{37}$$

The calculation of S_2 is much more difficult. Performing the substitution

$$t \;\longmapsto\; x + \frac{a+b-2x}{2}\,(t+1)$$

and introducing

$$g(t) \;:=\; \frac{2}{(a+b-2x)M}\, f'\!\left(x + \frac{a+b-2x}{2}\,(t+1)\right),$$

we find that

$$\int_x^{a+b-x} \big(\tfrac{1}{2}(a+b)-t\big) f'(t)\,\mathrm{d}t \;=\; -M \left(\frac{a+b-2x}{2}\right)^3 \int_{-1}^{1} t g(t)\,\mathrm{d}t.$$

In this equation, the side condition that $f \in \mathcal{F}'_M(\Delta)$ means equivalently that g is defined on $[-1,1]$ and satisfies

$$g \in \mathrm{Lip}_1, \quad g(-1)=g(1)=0, \quad \text{and} \quad \int_{-1}^{1} g(t)\,\mathrm{d}t = D \tag{38}$$

with

$$D := \frac{\Delta}{M}\left(\frac{2}{a+b-2x}\right)^2 \tag{39}$$

which we may suppose to be non-negative, replacing g by $-g$ otherwise. We may also suppose that $\int_{-1}^{1} tg(t)\mathrm{d}t$ is non-negative, replacing g by $g(-\,\cdot)$ otherwise, which is again a function satisfying (38). Thus S_2 can be obtained as

$$S_2 = \frac{M}{b-a}\left(\frac{a+b-2x}{2}\right)^3 \Omega, \tag{40}$$

where Ω is the solution of the following optimization problem:

$$\text{Maximize} \quad \Phi(g) := \int_{-1}^{1} tg(t)\,\mathrm{d}t$$
$$\text{under the constraints (38).}$$

Now we have to introduce some notations. For any g, we define

$$g_+(t) := \tfrac{1}{2}\big(|g(t)| + g(t)\big) \qquad \text{and} \qquad g_-(t) := \tfrac{1}{2}\big(|g(t)| - g(t)\big).$$

These functions are non-negative, their supports are disjoint, and

$$g(t) = g_+(t) - g_-(t).$$

Furthermore, for a non-negative function g, we define

$$\mathcal{A}(g) := \big\{(t,u) \,:\, 0 \le u \le g(t), \;\; -1 \le t \le 1\big\}$$

and denote by $|\mathcal{A}(g)|$ the area of $\mathcal{A}(g)$.

With these notations, we have

$$D = |\mathcal{A}(g_+)| - |\mathcal{A}(g_-)|$$

and

$$\Phi(g) = \iint_{\mathcal{A}(g_+)} t\,\mathrm{d}(t,u) - \iint_{\mathcal{A}(g_-)} t\,\mathrm{d}(t,u).$$

Recall that for a measurable set $\mathcal{B}$ in the (t,u)-plane, the integral

$$\frac{1}{|\mathcal{B}|}\iint_{\mathcal{B}} t\,\mathrm{d}(t,u)$$

is the abscissa of the center of gravity of $\mathcal{B}$.

In order to increase Φ, we want to modify g such that the conditions (38) are preserved, the areas of $\mathcal{A}(g_+)$ and $\mathcal{A}(g_-)$ remain fixed, but the abscissa of the center of gravity of $\mathcal{A}(g_+)$ increases while that of $\mathcal{A}(g_-)$ decreases.

Now let g be any function satisfying (38). Our first manipulation may be called the *shift to the left* (respectively, *to the right*) *of an interval of zeros.*

Suppose that for some subinterval $[\xi,\eta]$ of $[-1,1]$, we have $g_+(t)=0$ for all $t\in[\xi,\eta]$, but g_+ does not vanish identically on $[-1,\xi]$. Then we define

$$\widetilde{g}_+(t) := \begin{cases} 0 & \text{for} \quad -1 \le t \le -1+\eta-\xi \\ g_+(t-\eta+\xi) & \text{for} \quad -1+\eta-\xi \le t \le \eta \\ g_+(t) & \text{for} \quad \eta \le t \le 1. \end{cases}$$

We note that $\widetilde{g}_+$ satisfies the first two conditions in (38),

$$\int_{-1}^{1} \widetilde{g}_+(t)\,\mathrm{d}t \;=\; |\mathcal{A}(g_+)|\,,$$

and the length of the supports of g_+ and $\widetilde{g}_+$ is the same. Moreover, the abscissa of the center of gravity of $\mathcal{A}(\widetilde{g}_+)$ is at least as large as that of $\mathcal{A}(g_+)$.

Using this construction, we shift successively all the intervals of zeros to the left, starting with those of length at least $1/2$ (if there are any), continuing with those of length at least $1/3$, $1/4$, ..., and so on. This process either terminates after a finite number of steps or it provides a converging sequence of functions. Analogously we shift all the intervals of zeros of the function g_- to the right. Altogether we arrive at functions g_+^* and g_-^* with the following properties:

(1) the support of g_-^* is located to the left of that of g_+^*;
(2) $|\mathcal{A}(g_+)| = \big|\mathcal{A}(g_+^*)\big|$ and $|\mathcal{A}(g_-)| = \big|\mathcal{A}(g_-^*)\big|$;
(3) $g^* := g_+^* - g_-^*$ satisfies the constraints (38);
(4) $\Phi(g^*) \ge \Phi(g)$.

For the next step, we write $\alpha := \sqrt{|\mathcal{A}(g_-)|}$ and $\beta := \sqrt{|\mathcal{A}(g_+)|}$, and introduce the function

$$G(t) := \begin{cases} -1-t & \text{for} \quad -1 \le t \le -1+\alpha \\ t+1-2\alpha & \text{for} \quad -1+\alpha \le t \le -1+2\alpha \\ 0 & \text{for} \quad -1+2\alpha \le t \le 1-2\beta \\ t-1+2\beta & \text{for} \quad 1-2\beta \le t \le 1-\beta \\ 1-t & \text{for} \quad 1-\beta \le t \le 1. \end{cases} \tag{41}$$

It satisfies the conditions (38). Moreover, the sets $\mathcal{A}(G_-)$ and $\mathcal{A}(G_+)$ are triangles of area α^2 and β^2, respectively. In view of Remark 3.1, we also see that on the interval $[1-\beta,\, 1]$ any function satisfying (38) is majorized by G. Comparing the graphs of G and g^*, we find that there exist pairwise disjoint sets $\mathcal{B}_1$, $\mathcal{B}_2$, and $\mathcal{D}$ such that

$$\mathcal{A}(G_+) \;=\; \mathcal{B}_1 \cup \mathcal{D}, \qquad \mathcal{A}(g^*_+) \;=\; \mathcal{B}_2 \cup \mathcal{D},$$

$|\mathcal{B}_1| = |\mathcal{B}_2|$ and each point of $\mathcal{B}_1$ has an abscissa which is larger than the abscissa of any point of $\mathcal{B}_2$. Therefore

$$\frac{1}{|\mathcal{B}_2|}\iint_{\mathcal{B}_2} t\,\mathrm{d}(t,u) \;\le\; \frac{1}{|\mathcal{B}_1|}\iint_{\mathcal{B}_1} t\,\mathrm{d}(t,u)\,,$$

and so

$$\iint_{\mathcal{A}(g^*_+)} t\,\mathrm{d}(t,u) \;\le\; \iint_{\mathcal{A}(G_+)} t\,\mathrm{d}(t,u)\,.$$

Analogously we conclude that

$$\iint_{\mathcal{A}(g^*_-)} t\,\mathrm{d}(t,u) \;\ge\; \iint_{\mathcal{A}(G_-)} t\,\mathrm{d}(t,u)\,.$$

Combining these inequalities, we obtain

$$\Phi(G) \;\ge\; \Phi(g^*).$$

Hence it is enough to maximize Φ over all functions (41) with admissible values for α and β; in particular, $\beta^2-\alpha^2 = D$, as a consequence of (38).

Amongst these functions, there is exactly one, say G^*, which has no interval of zeros. It is obtained for

$$\alpha \;=\; \frac{1-D}{2} \qquad \text{and} \qquad \beta \;=\; \frac{1+D}{2}\,,$$

and may be described as

$$G^*(t) \;=\; \begin{cases} -1-t & \text{for} \quad -1 \le t \le -\frac{1}{2}(1+D) \\ t+D & \text{for} \quad -\frac{1}{2}(1+D) \le t \le \frac{1}{2}(1-D) \\ 1-t & \text{for} \quad \frac{1}{2}(1-D) \le t \le 1. \end{cases}$$

We now claim that, if G, as defined by (41), has an interval of zeros, then

$$\Phi(G) \;<\; \Phi(G^*)\,. \tag{42}$$

By straightforward calculations, we find that

$$\Phi(G) = \alpha^2(1-\alpha) + \beta^2(1-\beta)$$

and

$$\Phi(G^*) = \frac{1-D^2}{4} = \frac{1-(\beta^2-\alpha^2)^2}{4}. \tag{43}$$

Hence (42) is equivalent to

$$1 - 4\alpha^2 - 4\beta^2 + 4\alpha^3 + 4\beta^3 - \alpha^4 - \beta^4 + 2\alpha^2\beta^2 > 0. \tag{44}$$

Now we recall that the numbers α and β have to satisfy some side conditions for being admissible. From their definition, it is clear that they are non-negative. Since their squares are equal to the integrals $\int_{-1}^{1} G_-(t)\mathrm{d}t$ and $\int_{-1}^{1} G_+(t)\mathrm{d}t$, respectively, we readily conclude that α and β are bounded by 1. Since D was supposed to be non-negative, we have $\beta \geq \alpha$, and since G shall have an interval of zeros, the inequality $-1+2\alpha < 1-2\beta$ must hold. Altogether, these side conditions on α and β may be expressed as

$$\alpha,\, \beta \in [0,1], \qquad \alpha \leq \beta, \qquad \alpha + \beta < 1. \tag{45}$$

Next we mention that, on the left-hand side of (44), the positive term $1-\alpha-\beta$ (which is half the length of the interval of zeros of G) can be factored out. Carrying out that division (or using a computer algebra system) and grouping the resulting terms appropriately, we find that (44) is equivalent to

$$\alpha(1-\alpha)^2 + \beta(1-\beta)^2 + \alpha(\beta-\alpha) + (1-\beta^2) + (1-\alpha-\beta)\alpha\beta > 0.$$

But this inequality is *definitely* true since, under the restrictions (45), the terms on the left-hand side are all non-negative and $1-\beta^2$ is even positive. This completes the proof of (42).

Thus we have shown that $\Omega := (1-D^2)/4$ is the maximum value of the functional Φ and that this maximum is attained for the function G^*. Combining (36)–(40), we readily obtain (19). Functions f for which equality is attained are easily deduced from G^*. ▮

Proof (Proof of Theorem 2.6). Using the definition of $K_n(t)$, we start with

$$\int_a^b K_n(t) f^{(n)}(t)\,\mathrm{d}t = \int_a^x (t-a)^n f^{(n)}(t)\,\mathrm{d}t + \int_x^{a+b-x} Q_n(t) f^{(n)}(t)\,\mathrm{d}t$$

$$+\int_{a+b-x}^{b}(t-b)^n f^{(n)}(t)\,\mathrm{d}t\,.$$

Performing $n-1$ successive integrations by part on the right-hand side, we obtain

$$\frac{1}{n!}\int_a^b K_n(t)f^{(n)}(t)\,\mathrm{d}t \;=\; A+B+C+(-1)^{n-1}\int_a^b K^*(t)f'(t)\,\mathrm{d}t, \tag{46}$$

where

$$A \;=\; \sum_{j=0}^{n-2}(-1)^j\frac{(x-a)^{n-j}}{(n-j)!}\,f^{(n-j-1)}(x),$$

$$B \;=\; \sum_{j=0}^{n-2}(-1)^j\left[\frac{Q_n^{(j)}(a+b-x)}{n!}\,f^{(n-j-1)}(a+b-x)-\frac{Q_n^{(j)}(x)}{n!}\,f^{(n-j-1)}(x)\right],$$

$$C \;=\; \sum_{j=0}^{n-2}(-1)^{j-1}\frac{(a-x)^{n-j}}{(n-j)!}\,f^{(n-j-1)}(a+b-x)\,,$$

and

$$K^*(t) \;:=\; \begin{cases} t-a & \text{for} \quad a\le t\le x \\ \dfrac{Q_n^{(n-1)}(t)}{n!} & \text{for} \quad x<t\le a+b-x \\ t-b & \text{for} \quad a+b-x<t\le b\,. \end{cases}$$

Changing the index of summation, we easily find that

$$A+C \;=\; (-1)^{n-1}\sum_{\nu=1}^{n-1}\frac{(x-a)^{\nu+1}}{(\nu+1)!}\left(f^{(\nu)}(a+b-x)+(-1)^{\nu}f^{(\nu)}(x)\right). \tag{47}$$

The hypotheses on Q_n imply that

$$Q_n^{(j)}(t)=(-1)^{n-j}Q_n^{(j)}(a+b-t) \qquad \text{and} \qquad \frac{Q_n^{(n-1)}(t)}{n!} \;=\; t-\frac{a+b}{2}\,.$$

Hence

$$B \;=\; (-1)^n\sum_{\nu=1}^{n-1}\frac{Q_n^{(n-1-\nu)}(x)}{n!}\left(f^{(\nu)}(a+b-x)+(-1)^{\nu}f^{(\nu)}(x)\right), \tag{48}$$

and

$$K^*(t) \;=\; -\,K(t)\,, \tag{49}$$

where K is the function defined in (33). Now, combining (46)–(49) and applying Lemma 3.2, we obtain the desired result at once. ▮

Proof (Proof of Corollary 2.2). Let us first suppose that $f^{(n-1)}$ is piecewise continuously differentiable and $\left|f^{(n)}(t)\right| \le M$ at all points t where $f^{(n)}$ exists. Then Theorem 2.6 is applicable and

$$Q_n(t) \;:=\; \frac{1}{2^{2n}}\,\bigl(2x-a-b\bigr)^n\, U_n\!\left(\frac{2t-a-b}{2x-a-b}\right) \tag{50}$$

is an admissible polynomial. Differentiating it k times at the point $t=x$, we obtain

$$Q_n^{(k)}(x) \;=\; \frac{1}{2^{2n}}\,\bigl(2x-a-b\bigr)^{n-k}\, U_n^{(k)}(1).$$

From formulae (4.7.2), (4.7.3), and (4.7.14) in [9, pp. 80–81], it follows that

$$U_n^{(k)}(1) \;=\; \frac{2^k k!\,(n+k+1)!}{(n-k)!\,(2k+1)!}\,,$$

and so we can calculate $Q_n^{(k)}(x)$ explicitly. Replacing k by $n-\nu-1$, we find after some manipulations that

$$\frac{Q_n^{(n-\nu-1)}(x)}{n!} \;=\; \frac{1}{(\nu+1)!}\left(x-\frac{a+b}{2}\right)^{n-k}\prod_{j=0}^{\nu}\frac{1}{2}\left(1+\frac{n-\nu}{n-j}\right).$$

With this, it is easily seen that, if $\phi_{n\nu}(x)$ is as in the corollary, then

$$\left.\begin{aligned}&\left|E(f;x)-\sum_{\nu=1}^{n-1}\phi_{n\nu}(x)\,\frac{f^{(\nu)}(a+b-x)-(-1)^\nu f^{(\nu)}(x)}{(\nu+1)!\,(b-a)}\right|\\ &\qquad=\;\frac{1}{n!\,(b-a)}\left|\int_a^b K_n(t)f^{(n)}(t)\,\mathrm{d}t\right|.\end{aligned}\right\} \tag{51}$$

Now the right-hand side may be estimated as follows:

$$\left|\int_a^b K_n(t)f^{(n)}(t)\,\mathrm{d}t\right| \;\le\; M\int_a^b |K_n(t)|\,\mathrm{d}t \tag{52}$$

$$
\begin{aligned}
&= M\left[\frac{2(x-a)^{n+1}}{n+1} + \int_x^{a+b-x} |Q_n(t)|\ \mathrm{d}t\right] \\
&= M\left[\frac{2(x-a)^{n+1}}{n+1} + \frac{(a+b-2x)^{n+1}}{2^{2n+1}} \int_{-1}^{1} |U_n(\xi)|\ \mathrm{d}\xi\right].
\end{aligned}
$$

Since $\int_{-1}^{1} |U_n(\xi)|\,\mathrm{d}\xi = 2$, we obtain the bound of the corollary immediately.

In equation (51), we can avoid the appearance of $f^{(n)}$. In fact, an integration by parts shows, that $\int_a^b K_n(t) f^{(n)}(t)\,\mathrm{d}t$ can be replaced by

$$
\begin{aligned}
&\big((x-a)^n - Q_n(x)\big)\left(f^{(n-1)}(x) + (-1)^n f^{(n-1)}(a+b-x)\right) \\
&\qquad - \int_a^b K_n'(t) f^{(n-1)}(t)\,\mathrm{d}t\,,
\end{aligned}
$$

where K_n' is the piecewise existing derivative of K_n. Modifying (51) this way, we obtain a version of (51) which holds for all functions f satisfying the hypotheses of Corollary 2.2. But now we may employ Lemma 3.1 with $f^{(n-1)}$ taking the role of f. Given any $\varepsilon > 0$, we can choose the partition (30) such that

$$
\widetilde{f}(x) \;=\; f^{(n-1)}(x), \qquad \widetilde{f}(a+b-x) \;=\; f^{(n-1)}(a+b-x),
$$

and

$$
\left| f^{(n-1)}(t) \;-\; \widetilde{f}(t)\right| \;<\; \varepsilon \qquad (a \le t \le b),
$$

where $\widetilde{f}$ is piecewise continuously differentiable with $\left|\widetilde{f}'(t)\right| \le M$ for $t \in [a,b] \setminus \{t_0, \ldots, t_N\}$. Now the first part of the proof implies that the desired inequaltity holds with M replaced by $M + \varepsilon$, and so the result follows by letting $\varepsilon \to 0$.

Finally we note that the function f_*, defined in Corollary 2.2, is $n-1$ times continuously differentiable as a consequence of Lemma 3.3. Moreover, it has a piecewise existing n-th derivative which assumes only the values ± 1 such that

$$
K_n(t) f_*^{(n)}(t) \;=\; |K_n(t)| \qquad (a \le t \le b).
$$

Hence if $p(t)$ is any polynomial of degree at most $n-1$ and $f(t)$ is taken as $\pm M f_*(t) + p(t)$, then equality occurs in (52). This completes the proof. ∎

Proof (Proof of Corollary 2.3). Under the hypotheses of Corollary 2.3, Theorem 2.6 applies and gives

$$|E(f;x)| = \frac{1}{n!\,(b-a)} \left| \int_a^b K_n(t) f^{(n)}(t)\,\mathrm{d}t \right| . \tag{53}$$

The right-hand side may be estimated with the help of Hölder's inequality. For $p \in [1,\infty]$ and q defined by $p^{-1} + q^{-1} = 1$ (interpreting ∞^{-1} as 0 and vice versa), we obtain

$$|E(f;x)| \le \frac{1}{n!\,(b-a)} \|f^{(n)}\|_p \cdot \|K_n\|_q \,.$$

Thus, when $\|f^{(n)}\|_p \neq 0$,

$$\frac{n!\;|E(f;x)|}{\|f^{(n)}\|_p} \le \frac{\|K_n\|_q}{b-a} \,. \tag{54}$$

Hence the supremum of the left-hand side, taken over all admissible functions f with $\|f^{(n)}\|_p \neq 0$, is finite and gives the number $R(n,p,x)$ we are looking for. In particular, in view of (53) we have

$$\frac{\left| \int_a^b K_n(t) f^{(n)}(t)\,\mathrm{d}t \right|}{(b-a)\|f^{(n)}\|_p} \le R(n,p,x) \le \frac{\|K_n\|_q}{b-a} \tag{55}$$

for all admissible functions f with $\|f^{(n)}\|_p \neq 0$.

Now we want to determine $R(n,p,x)$ for $p = 1$, 2, and ∞. For this we try to find a kernel K_n (by an appropiate choice of the polynomial Q_n) so that there exist admissible functions f with $\|f^{(n)}\|_p \neq 0$ for which the two sides of (55) are either equal or the left-hand side can be made arbitrarily close to the right-hand side. Then, of course,

$$R(n,p,x) = \frac{\|K_n\|_q}{b-a} \,. \tag{56}$$

The case $p = \infty$. We take the polynomial Q_n in (50) so that K_n is as in the proof of Corollary 2.2. Then, by (52),

$$\|K_n\|_1 = \frac{2(x-a)^{n+1}}{n+1} + \frac{(a+b-2x)^{n+1}}{4^n} \,. \tag{57}$$

We know already that the function $f = c_0 + c_1 f_*$, with f_* as defined in Corollary 2.2, furnishes equality in the estimate of Corollary 2.2 when $M := \|f^{(n)}\|_\infty = |c_1|$. As a fortunate incidence, we observe that, for this particular function f,

$$f^{(\nu)}(x) \;=\; f^{(\nu)}(a+b-x) \;=\; 0 \qquad (\nu = 1, \ldots, n-1),$$

as a consequence of Lemma 3.3. Hence that function f with $c_1 \neq 0$ satisfies all the hypotheses of Corollary 2.3 and yields equality in (55) with $p = \infty$, $q = 1$ and $\|K_n\|_1$ given by (57). This settles the case where $p = \infty$.

The case $p = 2$. This time, we define K_n by taking

$$Q_n(t) \;:=\; \frac{(2x-a-b)^n}{\binom{2n}{n}}\, P_n\!\left(\frac{2t-a-b}{2x-a-b}\right),$$

where P_n is the n-th Legendre polynomial. Referring to [9] for properties of P_n, we find by a straightforward calculation, that

$$\|K_n\|_2^2 \;=\; \frac{2(x-a)^{2n+1}}{2n+1} \;+\; \frac{(a+b-2x)^{2n+1}}{(2n+1)\binom{2n}{n}^2}.$$

Now let $f := c_0 + c_1 f_*$, where f_* is the function defined in Corollary 2.3 and $c_1 \neq 0$. It is easily seen that this function f satisfies the hypotheses of Corollary 2.3. Moreover, taking care of the Rodrigues formula for the Legendre polynomials (see [9, § 4.3]), we find that

$$f^{(n)}(t) \;=\; \frac{c_1}{n!}\, K_n(t) \qquad (a \le t \le b),$$

and so

$$\left| \int_a^b K_n(t) f^{(n)}(t)\, \mathrm{d}t \right| \;=\; \|f^{(n)}\|_2 \cdot \|K_n\|_2 \,.$$

Hence, for our present choice of K_n, f, and p, the two sides of (55) are again equal, so that (56) holds. This settles the case where $p = 2$.

The case $p = 1$. Denoting by T_n the n-th Chebyshev polynomial of the first kind and setting

$$Q_n(t) \;:=\; \frac{(2x-a-b)^n}{2^{2n-1}}\, T_n\!\left(\frac{2t-a-b}{2x-a-b}\right),$$

we find under the condition (24) that

$$\|K_n\|_\infty \;=\; \max_{x \le t \le a+b-x} |Q_n(t)| \;=\; \frac{(a+b-2x)^n}{2^{2n-1}}\,. \tag{58}$$

Now we want to construct an admissible function f so that the two sides of (55) are "nearly equal" when $p = 1$ and $q = \infty$. First we introduce

$$g(t) := \begin{cases} 0 & \text{for} \quad t \in [a, b] \setminus [x, a + b - x] \\ \operatorname{sgn} U_{n-1}\left(\dfrac{2t - a - b}{a + b - 2x}\right) & \text{for} \quad t \in [x, a + b - 2x]. \end{cases}$$

Since $U_{n-1} = T_n'/n$, we note that g has its discontinuities exactly at the points where $|Q_n(t)| = \|K_n\|_\infty$. Denoting these points by t_ν $(\nu = 0, \ldots, n)$, we may number them such that

$$x = t_0 < t_1 < \cdots < t_n = a + b - x\,.$$

Next, choosing ε as a sufficiently small positive number, we want to redefine g on the intervals $I_0 := [t_0,\, t_0 + \varepsilon/2]$, $I_n := [t_n - \varepsilon/2,\, t_n]$, and

$$I_\nu := [t_\nu - \varepsilon/2,\, t_\nu + \varepsilon/2] \qquad (\nu = 1, \ldots, n - 1),$$

so that the modified function g_ε, say, is continuous and piecewise continuously differentiable. For this, we set

$$g_\varepsilon(t) := \begin{cases} g(t) & \text{for} \quad t \in [a, b] \setminus \big(I_0 \cup \cdots \cup I_n\big) \\ (-1)^{n-\nu}\dfrac{2(t - t_\nu)}{\varepsilon} & \text{for} \quad t \in I_\nu \ \ (\nu = 0, \ldots, n). \end{cases}$$

Then g_ε has the desired properties; moreover,

$$\|g_\varepsilon\|_\infty = \|g - g_\varepsilon\|_\infty = 1$$

and

$$\|g_\varepsilon'\|_1 = \sum_{\nu=0}^{n} \int_{I_\nu} |g_\varepsilon'(t)|\, \mathrm{d}t = 2n.$$

After these preparations, we first consider the case where $n \geq 2$ and define

$$\widetilde{f}_\varepsilon(t) := \int_x^t \frac{(t - \xi)^{n-2}}{(n - 2)!}\, g_\varepsilon(t)\, dt \qquad (a \leq t \leq b).$$

Then $\widetilde{f}_\varepsilon^{(n-1)} = g_\varepsilon$, and so $\widetilde{f}_\varepsilon^{(n-1)}(a + b - x) = 0$; furthermore, by the definition of $\widetilde{f}_\varepsilon$,

$$\widetilde{f}_\varepsilon^{(\nu)}(x) = 0 \qquad (\nu = 1, \ldots, n - 1),$$

while, as a consequence of Lemma 3.3,

$$\widetilde{f}_\varepsilon^{(\nu)}(a+b-x) \;=\; \int_x^{a+b-x} \frac{(a+b-x-\xi)^{n-2-\nu}}{(n-2-\nu)!}\,\big(g_\varepsilon(\xi)-g(\xi)\big)\,\mathrm{d}\xi$$

for $\nu = 1, \ldots, n-2$. The last equation implies that

$$\left|\widetilde{f}_\varepsilon^{(\nu)}(a+b-x)\right| \;\le\; \frac{(a+b-2x)^{n-2-\nu}}{(n-2-\nu)!}\,n\varepsilon \;\le\; c_1\varepsilon \qquad (\nu = 1, \ldots, n-2), \tag{59}$$

where c_1 and subsequently c_2, c_3, ... shall denote appropriate positive numbers which may depend on a, b, x, and n, but which do not depend on ε.

Now we modify $\widetilde{f}_\varepsilon$ by subtracting a polynomial

$$P_\varepsilon(t) \;:=\; a_1 t + \cdots + a_{2n-2}t^{2n-2}$$

such that

$$f_\varepsilon(t) \;:=\; \widetilde{f}_\varepsilon(t) - P_\varepsilon(t)$$

satisfies the hypotheses of Corollary 2.3. For this, we only need that

$$\left.\begin{array}{l} P_\varepsilon^{(\nu)}(x) = 0 \quad (\nu = 1, \ldots, n-1), \quad P_\varepsilon^{(n-1)}(a+b-x) = 0 \\ P_\varepsilon^{(\nu)}(a+b-x) = \widetilde{f}_\varepsilon^{(\nu)}(a+b-x) \quad (\nu = 1, \ldots, n-2). \end{array}\right\} \tag{60}$$

These conditions constitute a system of $2n-2$ linear equations for the coefficients of P_ε. We claim that this system has a unique solution. If not, then the corresponding homogeneous system would have a non-trivial solution. Thus there would exist a non-constant polynomial of degree at most $2n-2$ whose derivative would have zeros of multiplicity $n-1$ at x and at $a+b-x$; a contradiction.

Thinking of Cramer's rule for solving the linear system (60), we observe that

$$|a_\nu| \;\le\; c_2\varepsilon \qquad (\nu = 1, \ldots, 2n-2).$$

Hence

$$\||f_\varepsilon^{(n)}\|_1 \;\le\; \|\widetilde{f}_\varepsilon^{(n)}\|_1 + \|P_\varepsilon^{(n)}\|_1 \;=\; \|g_\varepsilon'\|_1 + \|P_\varepsilon^{(n)}\|_1 \;\le\; 2n + c_3\varepsilon\,.$$

Finally,

$$\begin{aligned}\left|\int_a^b K_n(t) f_\varepsilon^{(n)}(t)\,\mathrm{d}t\right| &= \left|\int_a^b K_n(t)\left(g'_\varepsilon(t) - P_\varepsilon^{(n)}(t)\right)\mathrm{d}t\right| \\ &= \left|\sum_{\nu=0}^{n}\int_{I_\nu} K_n(t) g'_\varepsilon(t)\,\mathrm{d}t - \int_a^b K_n(t) P_\varepsilon^{(n)}(t)\,\mathrm{d}t\right| \\ &\geq \left(\|K_n\|_\infty - c_4\varepsilon\right) 2n \;-\; c_5\varepsilon \\ &\geq \left(\|K_n\|_\infty - c_4\varepsilon\right)\left(\|f_\varepsilon^{(n)}\|_1 - c_3\varepsilon\right) \;-\; c_5\varepsilon\,.\end{aligned}$$

This show that, for $n \geq 2$, $p = 1$, and $f := f_\varepsilon$, the left-hand side of (55) approaches the right-hand side as $\varepsilon \to 0_+$. For $n = 1$, the same result is obtained by setting $f := g_\varepsilon$. Hence (56) holds, and (58) leads to (25). ▮

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