A FEW NEW INEQUALITIES FOR THE MEAN

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1. INTRODUCTION AND LEMMAS

let a, b be positive real numbers.Let us denote the harmonic mean of a and b by $H = \frac{1}{\frac{1/a+1/b}{2}} = \frac{2ab}{a+b}$, thegeometric mean by $G = \sqrt{ab}$, the arithmetic mean by $A = \frac{a+b}{2}$ and the quadratic mean by $K = \sqrt{\frac{a^2+b^2}{2}}$.

The following inequalities

$$\frac{2ab}{a+b} \leq \sqrt{ab} \leq \frac{a+b}{2} \leq \sqrt{\frac{a^2+b^2}{2}}$$

that is

$$H \leq G \leq A \leq K$$

are valid,and equalities occur for $a = b$.

Two new inequalities for the mean was given by Z.F.Starc in [1],these two new inequalities can be stated as follows

$$AK + AG \geq 2G^2 \tag{1}$$

and

$$AG + HK \geq 2A^2 \tag{2}$$

In the inqualities (1) and (2),equality occurs if and only if $a = b$.

In this paper we will give also a few new inequalities similar to (1) and (2)

We shall need the following well known results

LEMMA[2] **1** If $x > 0$, $y > 0$ and $\frac{1}{p} + \frac{1}{q}, p > 1$ then

$$x^{\frac{1}{p}} y^{\frac{1}{q}} \leq \frac{x}{p} + \frac{y}{q} \tag{3}$$

equality holds if and only if $x = y$

LEMMA[3] **2** If $a_1 \leq \cdots \leq a_n$ and $b_1 \leq \cdots \leq b_n$ or $a_1 \geq \cdots \geq a_n$ and $b_1 \geq \cdots \geq b_n$ then

$$(\frac{1}{n} \sum_{i=1}^{n} a_i)(\frac{1}{n} \sum_{i=1}^{n} b_i) \leq \frac{1}{n} \sum_{i=1}^{n} a_i b_i$$

equality holds if and only if $a_1 = a_2 = \cdots = a_n$ or $b_1 = b_2 = \cdot = b_n$.

2 **MAIN RESULTS**

Our main results are given in following

RESULT 1

$$G^2 + A^2 - AG \leq \frac{1}{2}(AK + HK) \tag{4}$$

equality holds if and only if $a = b$.

From (1) and (2), we can get inequality (4).

RESULT 2

$$\frac{(1+G)(1+A)}{(1+H)(1+K)} \geq (\frac{1+\sqrt{GA}}{1+\sqrt{HA}})^2 \tag{5}$$

equality holds it and only if $a = b$.

Proof Because $x + y \geq 2x^{\frac{1}{2}} y^{\frac{1}{2}}$, where $x \geq 0, y \geq 0$,equality holds and only if $x = y$.

We have $1 + x + y + xy \geq 1 + 2x^{\frac{1}{2}} y^{\frac{1}{2}} + xy$.

Thus $(1 + x)(1 + y) \geq (1 + x^{\frac{1}{2}} y^{\frac{1}{2}})^2$.

Then from

$$(1 + G)(1 + A) \geq (1 + G^{\frac{1}{2}} A^{\frac{1}{2}})^2,$$

and

$$(1 + H)(1 + K) \geq (1 + H^{\frac{1}{2}} K^{\frac{1}{2}})^2,$$

we can get inequality (5).

RESULT 3

$$GAKH \leq (\frac{G^p}{p} + \frac{A^q}{q})(\frac{K^p}{p} + \frac{H^Q}{q}) \tag{6}$$

where $\frac{1}{p} + \frac{1}{q} = 1$ and $p > 1$, and equality holds if and only if $a^p = b^q$.

In Lemma 1,let x^p take the pace of x , y^q take the pace of y, we have

$$xy \leq \frac{x^p}{p} + \frac{y^q}{q},$$

holds if and only if $x^p = y^q$.

Thus

$$GA \leq \frac{G^p}{p} + \frac{A^q}{q}, \tag{7}$$

$$KH \leq \frac{K^p}{p} + \frac{H^q}{q}. \tag{8}$$

From (7) and (8), we easy get (6).

RESULT 4

$$H + G + A + K \leq 2\sqrt{H^2 + G^2 + A^2 + K^2} \tag{9}$$

equality holds if and only if $a = b$.

In Lemma 2 ,let $a_i = b_i$ and $n = 4$,then

$$\frac{1}{4}(\sum_{i=1}^{4} a_i)^2 \leq \sum_{i=1}^{4} a_i^2 \tag{10}$$

where $a_1 \leq a_2 \leq a_3 \leq a_4$ or $a_1 \geq a_2 \geq a_3 \geq a_4$.

On the orther hand

$$H \leq G \leq A \leq K$$

Hence (9) is valid.

References

[1] Z.F.Starc. Two inequalities for the mean . Function 23(1999):153-154.

[2] Chang-jian Zhao and Mihaly Bencze, About Hilbert's Integral Inequalities, Octogon Mathematical Magazine, 10(2002):9-12.

[3] D.S.Mitrinovic. Analytic inequalities,Springer-Verlag,1970.