A COMPANION OF THE GRÜSS INEQUALITY AND APPLICATIONS

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Abstract. A companion of the Grüss inequality in the general setting of measurable spaces and abstract Lebesgue integrals is proven. Some particular inequalities are mentioned as well. An application for the moments of guessing mapping is also provided.

1. Introduction

Let $(\Omega, \mathcal{A}, \mu)$ be a measurable space consisting of a set Ω, a $\sigma-$algebra of parts of Ω and a countably additive and positive measure μ on $\mathcal{A}$ with values in $\mathbb{R} \cup \{\infty\}$.

For a $\mu-$measurable function $w : \Omega \to \mathbb{R}$, with $w(x) \geq 0$ for $\mu-$a.e. $x \in \Omega$, assume $\int_\Omega w(x) \, d\mu(x) > 0$. Consider the Lebesgue space $L_w(\Omega, \mu) := \{f : \Omega \to \mathbb{R},\ f \text{ is } \mu-\text{measurable and } \int_\Omega w(x) |f(x)| \, d\mu(x) < \infty\}$.

If $f, g : \Omega \to \mathbb{R}$ are $\mu-$measurable functions and $f, g, fg \in L_w(\Omega, \mu)$, then we may consider the Čebyšev functional

$$\begin{aligned} T_w(f,g) &:= \frac{1}{\int_\Omega w(x)\, d\mu(x)} \int_\Omega w(x) f(x) g(x) \, d\mu(x) \\ &\quad - \frac{1}{\int_\Omega w(x)\, d\mu(x)} \int_\Omega w(x) f(x) \, d\mu(x) \cdot \frac{1}{\int_\Omega w(x)\, d\mu(x)} \int_\Omega w(x) g(x) \, d\mu(x). \end{aligned} \tag{1.1}$$

The following result is known in the literature as the Grüss inequality

$$|T_w(f,g)| \leq \frac{1}{4} (\Gamma - \gamma)(\Delta - \delta), \tag{1.2}$$

provided

$$-\infty < \gamma \leq f(x) \leq \Gamma < \infty, \quad -\infty < \delta \leq g(x) \leq \Delta < \infty \tag{1.3}$$

for $\mu-$a.e. $x \in \Omega$.

The constant $\frac{1}{4}$ is sharp in the sense that it cannot be replaced by a smaller constant.

Note that if $\Omega = \{1, \dots, n\}$ and μ is the discrete measure on Ω, then we obtain the discrete Grüss inequality

$$\left| \frac{1}{W_n} \sum_{i=1}^n w_i x_i y_i - \frac{1}{W_n} \sum_{i=1}^n w_i x_i \cdot \frac{1}{W_n} \sum_{i=1}^n w_i y_i \right| \leq \frac{1}{4} (\Gamma - \gamma)(\Delta - \delta), \tag{1.4}$$

provided $\gamma \leq x_i \leq \Gamma$, $\delta \leq y_i \leq \Delta$ for each $i \in \{1, \dots, n\}$ and $w_i \geq 0$ with $W_n := \sum_{i=1}^n w_i > 0$. The constant $\frac{1}{4}$ is best in this inequality as well.

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For some Grüss type inequalities see [1]-[10].

2. A Companion of the Grüss Inequality

We start to the following lemma that is interesting in itself as well

Lemma 1. *Assume that* $\rho, h, l : \Omega \to \mathbb{R}$ *are Lebesgue* μ*-measurable on* Ω *with* $\rho, \rho h, \rho l, \rho h l \in L(\Omega, \mu)$. *If* $\rho(x) \geq 0$ *for* μ*-a.e.* $x \in \Omega$ *with* $\int_\Omega \rho(x)\, d\mu(x) > 0$ *and there exists the real constants* a, A, b, B *with the property that*

$$-\infty < a \leq h(x) \leq A < \infty, -\infty < b \leq l(x) \leq B < \infty \text{ for } \mu\text{-a.e. } x \in \Omega \tag{2.1}$$

and

$$\int_\Omega \rho(x)\, l(x)\, d\mu(x) = \int_\Omega \rho(x)\, h(x)\, d\mu(x) = 0; \tag{2.2}$$

then we have the inequality

$$\left| \frac{1}{\int_\Omega \rho(x)\, d\mu(x)} \int_\Omega \rho(x)\, h(x)\, l(x)\, d\mu(x) + \frac{a+A}{2} \cdot \frac{b+B}{2} \right| \leq \frac{1}{4}(A-a)(B-b). \tag{2.3}$$

The constant $\frac{1}{4}$ *is best possible in the sense that it can not be replaced by a smaller one.*

Proof. Firstly, let observe, by the assumption (2.2), that

$$\begin{aligned} & \int_\Omega \rho(x) \left(h(x) - \frac{a+A}{2} \right) \left(l(x) - \frac{b+B}{2} \right) d\mu(x) \\ = & \int_\Omega \rho(x)\, h(x)\, l(x)\, d\mu(x) + \frac{a+A}{2} \cdot \frac{b+B}{2} \int_\Omega \rho(x)\, d\mu(x). \end{aligned} \tag{2.4}$$

On the other hand, by (2.1), we have

$$\left| h(x) - \frac{a+A}{2} \right| \leq \frac{A-a}{2}, \left| l(x) - \frac{b+B}{2} \right| \leq \frac{B-b}{2} \text{ for } \mu\text{-a.e. } x \in \Omega$$

and thus

$$\begin{aligned} & \left| \int_\Omega \rho(x) \left(h(x) - \frac{a+A}{2} \right) \left(l(x) - \frac{b+B}{2} \right) d\mu(x) \right| \\ \leq & \int_\Omega \rho(x) \left| h(x) - \frac{a+A}{2} \right| \left| l(x) - \frac{b+B}{2} \right| d\mu(x) \\ \leq & \frac{A-a}{2} \cdot \frac{B-b}{2} \int_\Omega \rho(x)\, d\mu(x). \end{aligned} \tag{2.5}$$

Now, using the equality (2.4), the inequality (2.5) and dividing by $\int_\Omega \rho(x)\, d\mu(x) > 0$, we deduce the desired inequality (2.3).

To prove the sharpness of the constant $\frac{1}{4}$, we assume that (2.3) holds with a constant $C > 0$, i.e.,

$$\left| \frac{1}{\int_\Omega \rho(x)\, d\mu(x)} \int_\Omega \rho(x)\, h(x)\, l(x)\, d\mu(x) + \frac{a+A}{2} \cdot \frac{b+B}{2} \right| \leq C(A-a)(B-b). \tag{2.6}$$

If we consider the functions $h, l : [\alpha, \beta] \subset \mathbb{R} \to \mathbb{R}$, $l = h$ and

$$h(x) = \begin{cases} -1 \text{ if } x \in \left[\alpha, \frac{\alpha+\beta}{2}\right] \\ 1 \text{ if } x \in \left(\frac{\alpha+\beta}{2}, \beta\right], \end{cases}$$

then for $\rho = 1$, we have

$$\begin{aligned} a &= b = -1, A = B = 1 \\ \int_\alpha^\beta h(x)\,dx &= \int_\alpha^\beta l(x)\,dx = 0 \\ \int_\alpha^\beta h(x)\,l(x)\,dx &= 1 \end{aligned}$$

and thus, by (2.3), we deduce $C \geq \frac{1}{2}$. ∎

The above lemma gives us the opportunity to state the following companion of Grüss inequality.

Theorem 1. *Assume that $\rho, f, g : \Omega \to \mathbb{R}$ are Lebesgue μ-measurable on Ω with $\rho, \rho f, \rho g, \rho f g \in L(\Omega, \mu)$. If $\rho(x) \geq 0$ for μ-a.e. $x \in \Omega$ with $\int_\Omega \rho(x)\,d\mu(x) > 0$ and there exists the real constants m, M, n, N with the property that*

$$-\infty < m \leq f(x) \leq M < \infty, -\infty < n \leq g(x) \leq N < \infty \text{ for } \mu\text{-a.e. } x \in \Omega \tag{2.7}$$

then we have the inequality

$$\begin{aligned} &\Bigg| \frac{1}{\int_\Omega \rho(x)\,d\mu(x)} \int_\Omega \rho(x) f(x) g(x)\,d\mu(x) \\ &-\frac{m+M}{2} \cdot \frac{1}{\int_\Omega \rho(x)\,d\mu(x)} \int_\Omega \rho(x) g(x)\,d\mu(x) \\ &-\frac{n+N}{2} \cdot \frac{1}{\int_\Omega \rho(x)\,d\mu(x)} \int_\Omega \rho(x) f(x)\,d\mu(x) + \frac{m+M}{2} \cdot \frac{n+N}{2} \Bigg| \\ \leq\ & \frac{1}{4}(M-m)(N-n). \end{aligned} \tag{2.8}$$

The constant $\frac{1}{4}$ is best possible.

Proof. If we choose in Lemma 1

$$\begin{aligned} h(x) &= f(x) - \frac{1}{\int_\Omega \rho(y)\,d\mu(y)} \int_\Omega \rho(y) f(y)\,d\mu(y), \\ l(x) &= g(x) - \frac{1}{\int_\Omega \rho(y)\,d\mu(y)} \int_\Omega \rho(y) g(y)\,d\mu(y); \end{aligned}$$

where $x \in \Omega$, and

$$\begin{aligned} a &= m - \frac{1}{\int_\Omega \rho(y)\,d\mu(y)} \int_\Omega \rho(y) f(y)\,d\mu(y), \\ A &= M - \frac{1}{\int_\Omega \rho(y)\,d\mu(y)} \int_\Omega \rho(y) f(y)\,d\mu(y), \\ b &= n - \frac{1}{\int_\Omega \rho(y)\,d\mu(y)} \int_\Omega \rho(y) g(y)\,d\mu(y); \\ B &= N - \frac{1}{\int_\Omega \rho(y)\,d\mu(y)} \int_\Omega \rho(y) g(y)\,d\mu(y); \end{aligned}$$

then a simple calculation will reveal that

$$
\begin{aligned}
& \frac{1}{\int_\Omega \rho(x)\, d\mu(x)} \int_\Omega \rho(x)\, h(x)\, l(x)\, d\mu(x) \\
= & \frac{1}{\int_\Omega \rho(x)\, d\mu(x)} \int_\Omega \rho(x)\, f(x)\, g(x)\, d\mu(x) \\
& -\frac{1}{\int_\Omega \rho(x)\, d\mu(x)} \int_\Omega \rho(x)\, f(x)\, d\mu(x) \cdot \frac{1}{\int_\Omega \rho(x)\, d\mu(x)} \int_\Omega \rho(x)\, g(x)\, d\mu(x)
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{a+A}{2} &= \frac{m+M}{2} - \frac{1}{\int_\Omega \rho(y)\, d\mu(y)} \int_\Omega \rho(y)\, f(y)\, d\mu(y) \\
\frac{b+B}{2} &= \frac{n+N}{2} - \frac{1}{\int_\Omega \rho(y)\, d\mu(y)} \int_\Omega \rho(y)\, g(y)\, d\mu(y)
\end{aligned}
$$

and since

$$A-a = M-m, B-b = N-n$$

then by (2.3) we deduce the desired inequality (2.8).

The sharpness of the constant follows from Lemma 1 and we omit the details. ∎

The following corollary is a natural consequence of this theorem

Corollary 1. *Assume that $\rho, f : \Omega \to \mathbb{R}$ are Lebesgue μ-measurable on Ω with $\rho, \rho f, \rho f^2 \in L(\Omega, \mu)$. If $\rho(x) \geq 0$ for μ-a.e. $x \in \Omega$ with $\int_\Omega \rho(y)\, d\mu(y) > 0$ and there exists the real constants m, M with the property that*

$$-\infty < m \leq f(x) \leq M < \infty, \text{ for } \mu\text{-a.e. } x \in \Omega; \tag{2.9}$$

then we have the inequality

$$
\begin{aligned}
0 \leq & \frac{1}{\int_\Omega \rho(x)\, d\mu(x)} \int_\Omega \rho(x)\, f^2(x)\, d\mu(x) \\
& -\left(\frac{1}{\int_\Omega \rho(x)\, d\mu(x)} \int_\Omega \rho(x)\, f(x)\, d\mu(x)\right)^2 \qquad (2.10)\\
\leq & \frac{1}{4}(M-m)^2 - \left(\frac{1}{\int_\Omega \rho(x)\, d\mu(x)} \int_\Omega \rho(x)\, f(x)\, d\mu(x) - \frac{m+M}{2}\right)^2 \\
& \left(\leq \frac{1}{4}(M-m)^2\right).
\end{aligned}
$$

The constant $\frac{1}{4}$ is best possible in (2.10).

The proof is obvious by the above theorem on choosing $f = g$.

3. Some Particular Inequalities

The following particular inequalities are of interest.

1. Let $f, g : [a,b] \to \mathbb{R}$ be Lebesgue measurable functions. If $f, g, fg \in L[a,b]$, where $L[a,b]$ is the usual Lebsgue space, and

$$-\infty < m \leq f(x) \leq M < \infty, -\infty < n \leq g(x) \leq N < \infty \text{ for } \mu-\text{a.e. } x \in [a,b], \tag{3.1}$$

then we have the inequalities

$$\begin{aligned} &\left| \frac{1}{b-a} \int_a^b f(x) g(x) \, dx - \frac{n+N}{2} \cdot \frac{1}{b-a} \int_a^b f(x) \, dx \right. \\ &\left. - \frac{m+M}{2} \cdot \frac{1}{b-a} \int_a^b g(x) \, dx + \frac{m+M}{2} \cdot \frac{n+N}{2} \right| \\ \leq \; & \frac{1}{4} (M-m)(N-n) . \end{aligned} \tag{3.2}$$

The constant $\frac{1}{4}$ is sharp in the inequality (3.2).

2. Let $\bar{\mathbf{a}} = (a_1, \dots, a_n)$, $\bar{\mathbf{b}} = (b_1, \dots, b_n)$, $\bar{\mathbf{p}} = (p_1, \dots, p_n)$ be $n-$tuples of real numbers with $p_i \geq 0$ $(i \in \{1, \dots, n\})$ and $\sum_{i=1}^n p_i = 1$. If

$$a \leq a_i \leq A, b \leq b_i \leq B, \quad i \in \{1, \dots, n\}, \tag{3.3}$$

then one has the inequality

$$\begin{aligned} &\left| \sum_{i=1}^n p_i a_i b_i - \frac{b+B}{2} \cdot \sum_{i=1}^n p_i a_i - \frac{a+A}{2} \cdot \sum_{i=1}^n p_i b_i + \frac{b+B}{2} \cdot \frac{a+A}{2} \right| \\ &\leq \frac{1}{4} (A-a)(B-b) . \end{aligned} \tag{3.4}$$

The constant $\frac{1}{4}$ is sharp in the inequality (3.4).

4. Applications for Moments of Guessing Mapping

In 1994, J.L. Massey [11] considered the problem of guessing the value taken on by a discrete random variable X in one trial of a random experiment by asking questions of the form

$$\text{“Did } X \text{ take on its } i^{\text{th}} \text{ possible value?”} \tag{4.1}$$

until the answer is

$$\text{“Yes!”}. \tag{4.2}$$

This problem arises for instance when a cryptologist must try out possible secret keys one at a time after *minimising the possibilities by some cryptoanalysis.*

Consider a random variable X with finite range $X = \{x_1, \dots, x_n\}$ and distribution $P_X(x_k) = p_k$ for $k = 1, 2, \dots, n$.

A one-to-one function $G : \chi \to \{1, \dots, n\}$ is a guessing function for X. Thus

$$E(G^m) := \sum_{k=1}^n k^m p_k \tag{4.3}$$

is the m^{th} moment of this function, provided we renumber the x_i such that x_k is always the k^{th} guess.

In [11], Massey observed that, $E(G)$, the average number of guesses, *is minimised by a guessing strategy that guesses the possible values of X in decreasing order of probability.*

In the same paper [11], Massey proved that

$$E(G) \geq \frac{1}{4} 2^{H(X)} + 1 \;\; \text{provided } H(X) \geq 2 \text{ bits}, \tag{4.4}$$

for an optimal guessing strategy, where $H(X)$ is the Shannon entropy

$$H(X) = -\sum_{i=1}^{n} p_i \log_2 (p_i). \tag{4.5}$$

He also has shown that $E(G)$ may be arbitrarily large when $H(X)$ is an arbitrarily small positive number such that there is no interesting upper bound on $E(G)$ in terms of $H(X)$.

In 1996, Arikan [12] has proved that any guessing algorithm for X obeys the lower bound

$$E(G^\rho) \geq \frac{\left[\sum_{k=1}^{n} p_k^{\frac{1}{1+\rho}}\right]^{1+\rho}}{[1+\ln n]^\rho}, \quad \rho \geq 0 \tag{4.6}$$

where an optimal guessing algorithm for X satisfies

$$E(G^\rho) \leq \left[\sum_{k=1}^{n} p_k^{\frac{1}{1+\rho}}\right]^{1+\rho}, \quad \rho \geq 0. \tag{4.7}$$

In 1997, Boztaş [13] proved that for $m \geq 1$, integer

$$\begin{aligned} E(G^m) &\leq \frac{1}{m+1}\left[\sum_{k=1}^{n} p_k^{\frac{1}{1+m}}\right]^{1+m} \\ &+ \frac{1}{m+1}\left\{\binom{m+1}{2} E\left(G^{m-1}\right) - \binom{m+1}{3} E\left(G^{m-2}\right) + \cdots + (-1)^{m+1}\right\} \end{aligned} \tag{4.8}$$

provided the guessing strategy satisfies:

$$p_{k+1}^{\frac{1}{1+m}} \leq \frac{1}{k}\left(p_1^{\frac{1}{1+m}} + \cdots + p_k^{\frac{1}{1+m}}\right), \quad k = 1, \ldots, n-1. \tag{4.9}$$

In 1997, Dragomir and Boztaş [14] obtained for any guessing sequence:

$$\left|E(G) - \frac{n+1}{2}\right| \leq \frac{(n-1)(n+1)}{6} \max_{1 \leq i < j \leq n} |p_i - p_j|, \tag{4.10}$$

$$\left|E(G) - \frac{n+1}{2}\right| \leq \sqrt{\frac{(n-1)(n+1)\left(n\|p\|_2^2 - 1\right)}{12}}, \tag{4.11}$$

where $\|p\|_2^2 = \sum_{i=1}^{n} p_i^2$ and

$$\left|E(G) - \frac{n+1}{2}\right| \leq \left[\frac{n+1}{2}\right]\left(n - \left[\frac{n+1}{2}\right]\right) \max_{1 \leq k \leq n} \left|p_k - \frac{1}{n}\right|, \tag{4.12}$$

where $[x]$ is the integer part of x.

For other results on $E(G^p)$, $p > 0$ see also [15]. We mention only, by making use of Grüss inequality, one has for $p, q > 0$ that

$$\left|E\left(G^{p+q}\right) - E(G^p) E(G^q)\right| \leq \frac{1}{4}(n^q - 1)(n^p - 1). \tag{4.13}$$

By the use of the inequality (3.4) the above result may be complemented in the following way.

Theorem 2. *With the above assumptions, we have the inequality*

$$\left|E\left(G^{p+q}\right)-\frac{1+n^{q}}{2}E\left(G^{p}\right)-\frac{1+n^{p}}{2}E\left(G^{q}\right)+\frac{1+n^{q}}{2}\cdot\frac{1+n^{p}}{2}\right| \leq \frac{1}{4}\left(n^{q}-1\right)\left(n^{p}-1\right). \tag{4.14}$$

for any $p,q>0$.

Applications for different particular instances of $p,q>0$ may be provided, but we omit the details.

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