GEOMETRIC MEANS, INDEX MAPPINGS AND SUPERMULTIPLICATIVITY

By

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Abstract: Some maps are defined which derive naturally from weighted geometric means. They are shown to have convenient monotonicity and supermultiplicativity properties.

Key Words and Phrases: Geometric mean, supermultiplicativity, Jensen's inequality, convex functions.

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1 Introduction

In a companion article [1] in this volume we explored various subadditive and superadditive maps that arise naturally from weighted geometric means considered as functions of all their parameters. We now complement (and conclude) our study with an investigation of a more complicated but still fairly natural combination of weighted geometric means that exhibits supermultiplicativity properties.

We begin as before by setting

$$\begin{aligned}
\mathcal{P} &:= \{I | I \subset I\!N,\ 0 < |I| < \infty\},\\
\mathcal{J}^*(I) &:= \{x\, | p = (x_i)_{i \in I\!N},\ x_i > 0\ \forall i \in I\} \quad (I \in \mathcal{P}),\\
\mathcal{J}(I) &:= \{p\, | p = (p_i)_{i \in I\!N},\ p_i \geq 0\ \forall i \in I,\ P_I > 0\} \quad (I \in \mathcal{P}),
\end{aligned}$$

where $P_I := \sum_{i \in I} p_i$. For $I \in \mathcal{P}$, $p \in \mathcal{J}(I)$ and $x \in \mathcal{J}^*(I)$, we define the geometric mean of x with weights p to be

$$G_I(p, x) := \left(\prod_{i \in I} x_i^{p_i} \right)^{1/P_I}.$$

The following two lemmas will prove convenient for our analysis. The first (see [1]) expresses the basic superadditivity and monotonicity properties of the mapping $x \mapsto G_I(p, x)$. For $p, q \in \mathcal{J}(I)$, we write $p \geq q$ whenever $p_i \geq q_i$ for all $i \in I$.

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LEMMA A. *Suppose $I \in \mathcal{P}$ and $p \in \mathcal{J}(I)$. Then*
(i) for all $x, y \in \mathcal{J}^*(I)$,

$$G_I(p, x+y) \geq G_I(p,x) + G_I(p,y) \geq 0;$$

(ii) for all $x, y \in \mathcal{J}^*(I)$ with $x \geq y$,

$$G_I(p,x) \geq G_I(p,y) \geq 0.$$

The second lemma incorporates superadditivity and monotonicity properties associated with Jensen's inequality (see [3]).

LEMMA B. *Let $f : C \subseteq X \to I\!R$ be a convex function on the convex set C of a linear space X, and for $I \in \mathcal{P}$, $p \in \mathcal{J}$, $x \in C \cap \mathcal{J}^*(I)$ define*

$$\theta(I,p,f,x) := \sum_{i \in I} p_i f(x_i) - P_I f\left(\frac{1}{P_I} \sum_{i \in I} p_i x_i\right).$$

Then
(i) the mapping $\theta(I, \cdot, f, x)$ is superadditive, nonnegative and monotone nondecreasing;
(ii) the mapping $\theta(\cdot, p, f, x)$ is monotone nondecreasing and superadditive as an index set mapping.

For $I \in \mathcal{P}$, $p \in \mathcal{J}(I)$, $x, y \in \mathcal{J}^*(I)$ we may define

$$\psi(I,p,x,y) := \left[\frac{G_I(p,x+y)}{G_I(p,x) + G_I(p,y)}\right]^{P_I}.$$

It is immediate from Lemma A that

$$\psi(I,p,x,y) \geq 1.$$

The function ψ is the basic object of our study and is considered in the next section. In Section 3 we address an associated function of a real variable.

2 Supermultiplicativity

Our first result is as follows.

THEOREM 2.1. *Suppose $I \in \mathcal{P}$ and $x, y \in \mathcal{J}^*(I)$. Then $\psi(I, \cdot, x, y)$ is supermultiplicative and monotone nondecreasing.*

Proof. For $u \in \mathbf{R}$, define $f_0(u) := \ln(1 + e^u)$. Then

$$f_0''(u) = \frac{e^u}{(1+e^u)^2},$$

so that f_0 is convex. We have

$$\begin{aligned} \theta(I,p,f_0,x) &= \sum_{i \in I} p_i \ln(1 + \exp(x_i)) - P_I \ln\left(1 + \exp\left(\frac{1}{P_I}\sum_{i \in I} p_i x_i\right)\right) \\ &= \ln\left\{\frac{\prod_{i \in I}(1+\exp(x_i))^{p_i}}{\left[1 + \exp\left(\frac{1}{P_I}\sum_{i \in I} p_i x_i\right)\right]^{P_I}}\right\} \end{aligned}$$

for all $x_i \in \mathbf{R}$ $(i \in I)$.

For $x, y \in \mathcal{J}^*(I)$, define $\ln(x/y)$ by $(\ln(x/y))_i := \ln(x_i/y_i)$ $(i \in I)$. Then replacing x by $\ln(x/y)$ in the above displayed relation yields

$$\begin{aligned}
\theta(I, p, f_0, \ln(x/y)) &= \ln\left\{\frac{\prod_{i\in I}\left(1+\frac{x_i}{y_i}\right)^{p_i}}{\left[1+\exp\left(\frac{1}{P_I}\sum_{i\in I} p_i \ln\left(\frac{x_i}{y_i}\right)\right)\right]^{P_I}}\right\} \\
&= \ln\left\{\frac{\prod_{i\in I}(x_i+y_i)^{p_i} \Big/ \prod_{i\in I} y_i^{p_i}}{\left[1+\prod_{i\in I}\left(\frac{x_i}{y_i}\right)^{p_i/P_I}\right]^{P_I}}\right\} \\
&= \ln\left\{\frac{\prod_{i\in I}(x_i+y_i)^{p_i}}{\left[\left(\prod_{i\in I} x_i^{p_i}\right)^{1/P_I} + \left(\prod_{i\in I} y_i^{p_i}\right)^{1/P_I}\right]^{P_I}}\right\} \\
&= \ln\left[\frac{G_I(p, x+y)}{G_I(p,x)+G_I(p,y)}\right]^{P_I} \\
&= \ln\psi(I, p, x, y),
\end{aligned}$$

whence

$$\psi(I, p, x, y) = \exp[\theta(I, p, f_0, \ln(x/y))].$$

By Lemma B, $\theta(I, \cdot, f_0, \ln(x/y))$ is superadditive and monotone nondecreasing, since f_0 is convex. The desired result follows at once from the properties of $\exp(\cdot)$. □

The choice $I = \{1, 2, \cdots, n\}$ provides the following.

COROLLARY 2.2. *If $p_i \le 1$, $P_n > 0$ and $x_i, y_i > 0$ for $1 \le i \le n$, we have*

$$\left[\frac{\prod_{i=1}^n (x_i+y_i)^{\frac{1}{n}}}{\prod_{i=1}^n x_i^{\frac{1}{n}} + \prod_{i=1}^n y_i^{\frac{1}{n}}}\right]^n \ge \left[\frac{\prod_{i=1}^n (x_i+y_i)^{\frac{p_i}{P_n}}}{\prod_{i=1}^n x_i^{\frac{p_i}{P_n}} + \prod_{i=1}^n y_i^{\frac{1-p_i}{P_n}}}\right]^{P_n} \left[\frac{\prod_{i=1}^n (x_i+y_i)^{\frac{1-p_i}{n-P_n}}}{\prod_{i=1}^n x^{\frac{1-p_i}{n-P_n}} + \prod_{i=1}^n y_i^{\frac{1-p_i}{n-P_n}}}\right]^{n-P_n} \ge 1.$$

This leads to the following bound.

COROLLARY 2.3. *Set $I = \{1, 2, \ldots, n\}$ and suppose $x_i, y_i > 0$ for $i \in I$. Then*

$$\sup_{p\in\mathcal{J}(\mathcal{I})}\left[\frac{\prod_{i=1}^n (x_i+y_i)^{\frac{p_i}{P_n}}}{\prod_{i=1}^n x_i^{\frac{p_i}{P_n}} + \prod_{i=1}^n y_i^{\frac{p_i}{P_n}}}\right]^{P_n} = \left[\frac{\prod_{i=1}^n (x_i+y_i)^{\frac{1}{n}}}{\prod_{i=1}^n x_i^{\frac{1}{n}} + \prod_{i=1}^n y_i^{\frac{1}{n}}}\right]^n \ge 1.$$

In the following theorem we consider ψ as an index mapping.

THEOREM 2.4. *Suppose $I \in \mathcal{P}$, $p \in \mathcal{J}(I)$ and $x, y \in \mathcal{J}^*(I)$. Then $\psi(\cdot, p, x, y)$ is monotone nondecreasing and supermultiplicative as an index set map.*

Proof. From Lemma B the mapping $\theta(\cdot, p, f_0, \ln(x/y))$ is superadditive and monotone nondecreasing. The desired result follows from

$$\psi(I, p, f_0, \ln(x/y)) = \exp[\theta(I, p, f_0, \ln(x/y))].$$

□

COROLLARY 2.5. *Suppose $I = \{1, 2, \ldots, n\}$, $p, x, y \in \mathcal{J}^*(I)$. Then*

$$\sup_{J \subseteq I} \left[\frac{\prod_{i \in J} (x_i + y_i)^{\frac{p_i}{P_J}}}{\prod_{i \in J} x_i^{\frac{p_i}{P_J}} + \prod_{i \in J} y_i^{\frac{p_i}{P_J}}} \right]^{P_J} = \left[\frac{\prod_{i=1}^{n} (x_i + y_i)^{\frac{p_i}{P_n}}}{\prod_{i=1}^{n} x_i^{\frac{p_i}{P_n}} + \prod_{i=1}^{n} y_i^{\frac{p_i}{P_n}}} \right]^{P_n} \geq 1.$$

COROLLARY 2.6. *With the above assumptions*

$$\left[\frac{\prod_{i=1}^{n} (x_i + y_i)^{\frac{p_i}{P_n}}}{\prod_{i=1}^{n} x_i^{\frac{p_i}{P_n}} + \prod_{i=1}^{n} y_i^{\frac{p_i}{P_n}}} \right]^{P_n} \max_{1 \leq i < j \leq n} \left\{ \left[\frac{[(x_i + y_i)^{p_i} (x_j + y_j)^{p_j}]^{\frac{1}{p_i + p_j}}}{(x_i^{p_i} x_j^{p_j})^{\frac{1}{p_i + p_j}} + (y_i^{p_i} y_j^{p_j})^{\frac{1}{p_i + p_j}}} \right]^{p_i + p_j} \right\} \geq 1.$$

3 A function of a real variable

Suppose that $I \in \mathcal{P}$, $x, y \in \mathcal{J}^*(I)$, $p, q \in \mathcal{J}(I)$. We set

$$K(I, p, x, y) := \frac{G_I(p, x + y)}{G_I(p, x) + G_I(p, y)}$$

and for $t \geq 0$ define

$$\phi(t) := \left[\frac{K(I, p + tq, x, y)}{K(I, q, x, y)} \right]^{P_I + tQ_I}.$$

The properties of this mapping are embodied in the following theorem, wherein we set

$$H(I, p, q, x) := \frac{G_I(p, x)}{G_I(q, x)}.$$

THEOREM 3.1. *We have that on $[0, \infty)$*
(i) ϕ is monotone nondecreasing;
(ii) ϕ is logarithmically concave;
(iii) the inequality

$$\begin{aligned}
\left[\frac{K(I, p, x, y)}{K(I, q, x, y)} \right]^{P_I} &\leq \phi(t) \\
&\leq \left\{ \frac{[H(I, p, q, x + y)]^{G_I(q,x) + G_I(q,y))}}{[H(I, p, q, x)]^{G_I(q,x)} \, [H(I, p, q, y)]^{G_I(q,y)}} \right\}^{\frac{P_I}{G_I(q,x) + G_I(q,y)}}
\end{aligned}$$

is satisfied.

Proof. Let $t_2 > t_1 \geq 0$. Then by Theorem 2.1,

$$\begin{aligned}
\phi(t_2) &= \phi(t_1 + (t_2 - t_1)) \\
&= \left[\frac{K(I,(p+t_1q)+(t_2-t_1)q,x,y)}{K(I,q,x,y)}\right]^{P_I+t_2Q_I} \\
&= \frac{[K(I,(p+t_1q)+(t_2-t_1)q,x,y)]^{P_I+t_1Q_I+(t_2-t_1)Q_I}}{[K(I,q,x,y)]^{P_I+t_1Q_I}[K(I,q,x,y)]^{(t_2-t_1)Q_I}} \\
&\leq \frac{[K(I,p+t_1q,x,y)]^{P_I+t_1Q_I}[K(I,(t_2-t_1)q,x,y)]^{(t_2-t_1)Q_I}}{[K(I,q,x,y)]^{P_I+t_1Q_I}[K(I,q,x,y)]^{(t_2-t_1)Q_I}}.
\end{aligned}$$

Now $G_I(\alpha p, x) = G_I(p, x)$ for all $\alpha > 0$ and so

$$K(I,(t_2-t_1)q,x,y) = K(I,q,x,y) \quad \text{for all} \quad t_2 > t_1 \geq 0.$$

Hence

$$\phi(t_2) \leq \frac{[K(I,p+t_1q,x,y)]^{P_I+t_1Q_I}}{[K(I,q,x,y)]^{P_I+t_1Q_I}} = \phi(t_1)$$

and the monotonicity of ϕ is proved.

Let $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$ and $t_1, t_2 \geq 0$. Then by Theorem 2.1

$$\begin{aligned}
\phi(\alpha t_1 + \beta t_2) &= \left[\frac{K(I,\alpha(p+t_1q)+\beta(p+t_2q),x,y)}{K(I,q,x,y)}\right]^{\alpha(P_I+t_1Q_I)+\beta(P_I+t_2Q_I)} \\
&\geq \frac{[K(I,\alpha(p+t_1q),x,y)]^{\alpha(P_I+t_1Q_I)}}{[K(I,q,x,y)]^{\alpha(P_I+t_1Q_I)}} \cdot \frac{[K(I,\beta(p+t_2q),x,y)]^{\beta(P_I+t_2Q_I)}}{[K(I,q,x,y)]^{\beta(P_I+t_2Q_I)}} \\
&= \left\{\left[\frac{K(I,p+t_1q,x,y)}{K(I,q,x,y)}\right]^{P_I+t_1Q_I}\right\}^{\alpha} \left\{\left[\frac{K(I,p+t_2q,x,y)}{K(I,q,x,y)}\right]^{P_I+t_2Q_I}\right\}^{\beta} \\
&= [\phi(t_1)]^{\alpha}[\phi(t_2)]^{\beta},
\end{aligned}$$

which proves the logarithmic concavity of ϕ.

The first inequality in (iii) follows from the monotonicity of ϕ and

$$\phi(0) = \left[\frac{K(I,p,x,y)}{K(I,q,x,y)}\right]^{P_I}.$$

Observe that for $u > 0$,

$$\begin{aligned}
G_I\left(p + \frac{1}{u}\,q, x\right) &= \left[\prod_{i\in I} x_i^{\left(p_i+\frac{1}{u}\,q_i\right)}\right]^{1/\left(P_I+\frac{1}{u}\,Q_I\right)} \\
&= \left(\left[\prod_{i\in I} x_i^{(up_i+q_i)}\right]^{1/u}\right)^{u/(uP_I+Q_I)} \\
&= G_I(up+q,x).
\end{aligned}$$

Hence

$$\lim_{t\to\infty} K(I,p+tq,x,y) = \lim_{t\to\infty}\left[\frac{G_I(p+tq,x+y)}{G_I(p+tq,x)+G_I(p+tq,y)}\right]$$

$$
\begin{aligned}
&= \lim_{u\to 0+} \frac{G_I\left(p+\frac{1}{u}\,q, x+y\right)}{G_I\left(p+\frac{1}{u}\,q, x\right)+G_I\left(p+\frac{1}{u}\,q, y\right)} \\
&= \lim_{u\to 0+} \frac{G_I(up+q, x+y)}{G_I(up+q,x)+G_I(up+q,y)} \\
&= \frac{G_I(q,x+y)}{G_I(q,x)+G_I(p,y)} \\
&= K(I,q,x,y).
\end{aligned}
$$

Put

$$
p(t) := \frac{K(I,p+tq,x,y)-K(I,q,x,y)}{K(I,q,x,y)}.
$$

Then $\lim_{t\to\infty} p(t) = 0$ and thus

$$
\lim_{t\to\infty}\phi(t) = \lim_{t\to\infty}[(1+p(t))^{1/p(t)}]^{p(t)\cdot(P_I+tQ_I)} = \exp\left[\lim_{t\to\infty}[p(t)(P_I+tQ_I)]\right].
$$

To determine $\lim_{t\to\infty}\phi(t)$, we have to compute

$$
\begin{aligned}
\lim_{t\to\infty}[p(t)(P_I+tQ_I)] &= \lim_{u\to 0+}\left[\frac{K(I,up+q,x,y)-K(I,q,x,y)}{K(I,q,x,y)}\cdot\frac{uP_I+E_I}{u}\right] \\
&= \frac{Q_I}{K(I,q,x,y)}\cdot\lim_{u\to 0+}\left[\frac{K(I,up+q,x,y)-K(I,q,x,y)}{u}\right] \\
&= \frac{Q_I}{K(I,q,x,y)}\left.\frac{dg}{du}\right|_{u=0},
\end{aligned}
$$

where

$$
g(u) := K(I,up+q,x,y) = \frac{G_I(up+q,x+y)}{G_I(up+q,x)+G_I(up+q,y)}.
$$

The calculation is conveniently performed in terms of

$$
\begin{aligned}
h(u,z) &:= G_I(up+q,z) \\
&= \exp\left[\ln\left(\prod_{i\in I} z_i^{up_i+q_i}\right)^{1/(uP_I+E_I)}\right] \\
&= \exp\left[\frac{1}{uP_I+E_I}\sum_{i\in I}(up_i+e_i)\ln z_i\right]
\end{aligned}
$$

We have

$$
\begin{aligned}
\frac{\partial h(u,z)}{\partial u} &= \exp\left[\frac{1}{uP_I+Q_I}\sum_{i\in I}(up_i+q_i)\ln z_i\right]\frac{\partial}{\partial u}\left[\frac{1}{uP_I+Q_I}\sum_{i\in I}(up_i+q_i)\ln z_i\right] \\
&= G_I(up+q,z)\,\frac{\left(\sum_{i\in I}p_i\ln z_i\right)(uP_I+Q_I)-P_I\sum_{i\in I}(up_i+q_i)\ln z_i}{(uP_I+Q_I)^2} \\
&= G_I(up+q,z)\,\frac{\ln\left(\prod_{i\in I}z_i^{p_i}\right)^{uP_I+Q_I}-\ln\left(\prod_{i\in I}z_i^{up_i+q_i}\right)^{P_I}}{(uP_I+Q_I)^2}
\end{aligned}
$$

$$
= \quad G_I(up+q,z)\cdot\ln\left[\frac{\left(\prod_{i\in I} z_i^{p_i}\right)^{uP_I+Q_I}}{\left(\prod_{i\in I} z_i^{up_i+q_i}\right)^{P_I}}\right]^{1/(uP_I+Q_I)^2} \qquad .
$$

Hence

$$
\frac{\partial h(0,z)}{\partial u} = G_I(q,z)\cdot\ln\left[\frac{\left(\prod_{i\in I} z_i^{p_i}\right)^{1/Q_I}}{\left(\prod_{i\in I} z_i^{q_i}\right)^{P_I/Q_I^2}}\right] = G_I(q,z)\ \frac{P_I}{Q_I}\ln\left[H(I,p,q,z)\right].
$$

We now have

$$
\begin{aligned}
\frac{dg(0)}{du} &= \frac{\frac{\partial h(0,x+y)}{\partial u}\cdot[h(0,x)+h(0,y)] - h(0,x+y)\left[\frac{\partial h(0,x)}{\partial u}+\frac{\partial h(0,y)}{\partial u}\right]}{(h(0,x)+h(0,y))^2} \\
&= \frac{G_I(q,x+y)\ \frac{P_I}{Q_I}\ln\left[H(I,p,q,x+y)\right](G_I(q,x)+G_I(q,y)) - G_I(q,x+y)A}{(G_I(q,x)+G_I(q,y))^2},
\end{aligned}
$$

where

$$
\begin{aligned}
A &= G_I(q,x)\ \frac{P_I}{Q_I}\ln\left[H(I,p,q.x)\right] + G_I(q,y)\ \frac{P_I}{Q_I}\ln\left[H(I,p,q,y)\right] \\
&= \frac{P_I}{Q_I}\left[\ln\left\{[H(I,p,q,x)]^{G_I(q,x)}\,[H(I,p,q,y)]^{G_I(q,y)}\right\}\right].
\end{aligned}
$$

Hence we derive

$$
\frac{dg(0)}{du} = \frac{P_I}{Q_I}\cdot\frac{G_I(q,x+y)}{(G_I(q,x)+G_I(q,y))^2}\ln\left\{\frac{[H(I,p,q,x+y)]^{G_I(q,x)+G_I(q,y)}}{\left([H(I,p,q,x)]^{G_I(q,x)}\,[H(I,p,q,y)]^{G_I(q,y)}\right)}\right\}.
$$

Accordingly

$$
\begin{aligned}
&\lim_{t\to\infty}[p(t)(P_I+tQ_I)] \\
&\quad = \ln\left\{\frac{H(I,p,q,x+y)}{[H(I,p,q,x)]^{(G_I(q,x))/(G_I(q,x)+G_I(q,y))}\,[H(I,p,q,y)]^{(G_I(q,y))/(G_I(q,x)+G_I(q,y))}}\right\}^{P_I}
\end{aligned}
$$

Finally, we obtain

$$
\lim_{t\to\infty}\phi(t) = \left\{\frac{[H(I,p,q,x+y)]^{G_I(q,x)+G_I(q,y)}}{[H(I,p,q,x)]^{G_I(q,x)}\,[H(I,p,q,y)]^{G_I(q,y)}}\right\}^{P_I/(G_I(q,x)+G_I(q,y))}
$$

and the theorem is proved. $\square$

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