MONOTONICITY RESULTS FOR THE GAMMA FUNCTION

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ABSTRACT. The function $f(x) = \frac{[\Gamma(x+1)]^{1/x}}{x+1}$ is strictly decreasing on $[1,\infty)$, the function $g(x) = \frac{[\Gamma(x+1)]^{1/x}}{\sqrt{x}}$ is strictly increasing on $[2,\infty)$, and the function $h(x) = \frac{[\Gamma(x+1)]^{1/x}}{\sqrt{x+1}}$ is strictly increasing on $[1,\infty)$, respectively. From these, some inequalities, for example, the Minc-Sathre inequality, are deduced, and two open problems posed by the second author are partially solved.

1. Introduction

In [12], H. Minc and L. Sathre proved that, if r is a positive integer and $\phi(r) = (r!)^{1/r}$, then

$$1 < \frac{\phi(r+1)}{\phi(r)} < \frac{r+1}{r}. \tag{1}$$

In [1, 11], H. Alzer and J. S. Martins refined the right inequality in (1) and showed that, if n is a positive integer, then, for all positive real numbers r, we have

$$\frac{n}{n+1} < \left(\frac{1}{n}\sum_{i=1}^{n} i^r \Bigg/ \frac{1}{n+1}\sum_{i=1}^{n+1} i^r\right)^{1/r} < \frac{\sqrt[n]{n!}}{\sqrt[n+1]{(n+1)!}}. \tag{2}$$

Both bounds in (2) are the best possible.

Many extensions and generalizations of the inequalities in (2) have been found, please see [3, 4, 10, 13, 14, 20, 22, 27] and the references therein.

The inequalities in (1) were refined and generalized in [15, 21, 23, 24, 25] and the following inequalities were obtained:

$$\frac{n+k+1}{n+m+k+1} < \left(\prod_{i=k+1}^{n+k} i\right)^{1/n} \Bigg/ \left(\prod_{i=k+1}^{n+m+k} i\right)^{1/(n+m)} \le \sqrt{\frac{n+k}{n+m+k}}, \tag{3}$$

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where k is a nonnegative integer, n and m are natural numbers. For $n = m = 1$, the equality in (3) is valid.

In [16], the inequalities in (3) were generalized to obtain the following inequalities on the ratio for the geometric means of a positive arithmetic sequence with unit difference for any nonnegative integer k and natural numbers n and m:

$$\frac{n+k+1+\alpha}{n+m+k+1+\alpha} \le \frac{\left[\prod_{i=k+1}^{n+k}(i+\alpha)\right]^{1/n}}{\left[\prod_{i=k+1}^{n+m+k}(i+\alpha)\right]^{1/(n+m)}} \le \sqrt{\frac{n+k+\alpha}{n+m+k+\alpha}}\,, \tag{4}$$

where $\alpha \in [0, 1]$ is a constant.

Furthermore, for nonnegative integer k and natural numbers n and m, we have

$$\frac{a(n+k+1)+b}{a(n+m+k+1)+b} \le \frac{\left[\prod_{i=k+1}^{n+k}(ai+b)\right]^{\frac{1}{n}}}{\left[\prod_{i=k+1}^{n+m+k}(ai+b)\right]^{\frac{1}{n+m}}} \le \sqrt{\frac{a(n+k)+b}{a(n+m+k)+b}}\,, \tag{5}$$

where a and b are positive constants. See [8].

It is clear that the inequalities in (5) extend those in (4).

In [21], the following monotonicity result was obtained: The function

$$\frac{[\Gamma(x+y+1)/\Gamma(y+1)]^{1/x}}{x+y+1} \tag{6}$$

is decreasing in $x \ge 1$ for fixed $y \ge 0$. Then, for positive real numbers x and y, we have

$$\frac{x+y+1}{x+y+2} \le \frac{[\Gamma(x+y+1)/\Gamma(y+1)]^{1/x}}{[\Gamma(x+y+2)/\Gamma(y+1)]^{1/(x+1)}}. \tag{7}$$

Inequality (7) extends and generalizes inequality (3), since $\Gamma(n+1) = n!$.

In [8, 21], the second author, F. Qi, posed the following

Open Problem 1. *For positive real numbers x and y, we have*

$$\frac{[\Gamma(x+y+1)/\Gamma(y+1)]^{1/x}}{[\Gamma(x+y+2)/\Gamma(y+1)]^{1/(x+1)}} \le \sqrt{\frac{x+y}{x+y+1}}\,, \tag{8}$$

where Γ denotes the gamma function.

Open Problem 2. *For any positive real number z, define $z! = z(z-1)\cdots\{z\}$, where $\{z\} = z - [z-1]$, and $[z]$ denotes Gauss function whose value is the largest integer not more than z. Let $x > 0$ and $y \ge 0$ be real numbers, then*

$$\frac{x+1}{x+y+1} \le \frac{\sqrt[x]{x!}}{\sqrt[x+y]{(x+y)!}} \le \sqrt{\frac{x}{x+y}}\,. \tag{9}$$

It is well known that $\Gamma(z+1)=z!$, see [28, pp. 90–91]. Hence the inequalities in (8) and (9) are equivalent to the following monotonicity results in some sense for $x\ge 1$, which are the main results of this paper.

Theorem 1. *The function $f(x)=\frac{[\Gamma(x+1)]^{1/x}}{x+1}$ is strictly decreasing on $[1,\infty)$, the function $g(x)=\frac{[\Gamma(x+1)]^{1/x}}{\sqrt{x}}$ is strictly increasing on $[2,\infty)$, and the function $h(x)=\frac{[\Gamma(x+1)]^{1/x}}{\sqrt{x+1}}$ is strictly increasing on $[1,\infty)$, respectively.*

Remark 1. Note that the function $f(x)$ is a special case of the function (6). In this paper, we will give a new and simple proof for the monotonicity of $f(x)$. Theorem 1 partially solves the two open problems above.

Remark 2. In recent years, many monotonicity results and inequalities involving the gamma function and incomplete gamma functions have been established, please see [5, 6, 7, 17, 18, 19, 24, 26] and some of the references therein.

2. Proof of Theorem 1

For $x>1$, the following double inequalities are stated in [9, p. 431]:

$$0<\ln\Gamma(x)-\left[\left(x-\frac{1}{2}\right)\ln x-x+\frac{1}{2}\ln(2\pi)\right]<\frac{1}{x}, \tag{10}$$

$$\frac{1}{2x}<\ln x-\frac{\Gamma'(x)}{\Gamma(x)}<\frac{1}{x}, \tag{11}$$

$$\frac{1}{x}<\frac{\mathrm{d}^2}{\mathrm{d}x^2}\ln\Gamma(x)<\frac{1}{x-1}. \tag{12}$$

In [28, pp. 103–105], the following formula was given:

$$\frac{\Gamma'(z)}{\Gamma(z)}+\gamma=\int_0^\infty\frac{e^{-t}-e^{-zt}}{1-e^{-t}}\,\mathrm{d}t=\int_0^1\frac{1-t^{z-1}}{1-t}\,\mathrm{d}t, \tag{13}$$

where γ denotes Euler constant and $\gamma=0.57721566490153286060651\cdots$. See [28, p. 94]. Formula (13) can be used to calculate $\Gamma'(k)$ for $k\in\mathbb{N}$. We call $\psi(z)=\frac{\Gamma'(z)}{\Gamma(z)}$ the digamma or psi function. See [2, p. 71].

Taking the logarithm yields

$$\ln f(x)=\frac{1}{x}\ln\Gamma(x+1)-\ln(x+1). \tag{14}$$

Differentiating with respect to x on both sides of (14) and using the double inequalities (10) and (11) gives us

$$\begin{aligned} x^2\frac{f'(x)}{f(x)} &= -\ln\Gamma(x+1)+x\frac{\Gamma'(x+1)}{\Gamma(x+1)}-\frac{x^2}{x+1} \\ &< -\left[\left(x+\frac{1}{2}\right)\ln(x+1)-(x+1)+\frac{1}{2}\ln(2\pi)\right] \\ &\quad +x\left[\ln(x+1)-\frac{1}{2(x+1)}\right]-\frac{x^2}{x+1} \\ &= -\frac{1}{2}\ln(x+1)-\frac{1}{2(x+1)}+\frac{1}{2}[3-\ln(2\pi)] \\ &\triangleq \phi(x), \end{aligned} \tag{15}$$

By direct computation, we have

$$\phi'(x) = -\frac{x}{2(x+1)^2} < 0.$$

Thus, the function $\phi(x)$ is strictly decreasing, and then $\phi(x) \le \phi(1) = \frac{5}{4} - \frac{1}{2}\ln(4\pi) < 0$. Therefore $f'(x) < 0$ and $f(x)$ is strictly decreasing on $[1,\infty)$.

Straightforward calculation and utilizing the inequalities in (12) for $x > 1$ produces

$$\ln g(x) = \frac{1}{x}\ln\Gamma(x+1)-\frac{1}{2}\ln x, \tag{16}$$

$$x^2\frac{g'(x)}{g(x)} = -\ln\Gamma(x+1)+x\frac{\mathrm{d}}{\mathrm{d}x}\ln\Gamma(x+1)-\frac{1}{2}x \triangleq \varphi(x), \tag{17}$$

$$\begin{aligned} \varphi'(x) &= x\frac{\mathrm{d}^2}{\mathrm{d}x^2}\ln\Gamma(x+1)-\frac{1}{2} \\ &> \frac{x}{x+1}-\frac{1}{2} \\ &= \frac{x-1}{2(x+1)} \\ &> 0. \end{aligned} \tag{18}$$

Therefore, the function $\varphi(x)$ is strictly increasing, and $\varphi(x) \ge \varphi(2) = \Gamma'(3) - 1 - \ln 2 > 0$ by (13). Thus $g'(x) > 0$ and then $g(x)$ is strictly increasing on $[2,\infty)$.

Direct computation and using the inequalities in (12) for $x > 1$ produces

$$\ln h(x) = \frac{1}{x}\ln\Gamma(x+1)-\frac{1}{2}\ln(x+1), \tag{19}$$

$$x^2\frac{h'(x)}{h(x)} = -\ln\Gamma(x+1)+x\frac{\mathrm{d}}{\mathrm{d}x}\ln\Gamma(x+1)-\frac{x^2}{2(x+1)} \triangleq \tau(x), \tag{20}$$

$$\begin{aligned}\tau'(x) &= x\frac{\mathrm{d}^2}{\mathrm{d}x^2}\ln\Gamma(x+1)-\frac{x(2+x)}{2(1+x)^2}\\ &> \frac{x}{x+1}-\frac{x(2+x)}{2(1+x)^2}\\ &= \frac{x^2}{2(x+1)^2}\\ &> 0.\end{aligned} \tag{21}$$

Therefore, function $\tau(x)$ is strictly increasing, and $\tau(x) \geq \tau(1) = \Gamma'(2) - \frac{1}{4} > 0$. Thus $h'(x) > 0$ and then $h(x)$ is strictly increasing on $[1, \infty)$.

The proof is complete.

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