IMPROVEMENT OF LOWER BOUND IN WALLIS' INEQUALITY

CHAO-PING CHEN AND FENG QI

Abstract. For all natural number n, we have

$$\frac{1}{\sqrt{\pi(n+2/7)}} < \frac{(2n-1)!!}{(2n)!!} < \frac{1}{\sqrt{\pi(n+1/4)}}.$$

The constants $\frac{2}{7}$ and $\frac{1}{4}$ are the best possible. From this, the well-known Wallis' inequality is improved.

1. Introduction

Let

$$P_n = \frac{(2n-1)!!}{(2n)!!}, \tag{1}$$

then we have

$$\frac{1}{2\sqrt{n}} < \sqrt{\frac{2}{\pi}} \cdot \frac{1}{\sqrt{2n+1}} < P_n < \frac{2}{\sqrt{\pi}} \cdot \frac{1}{\sqrt{4n+1}} < \frac{1}{\sqrt{3n+1}} < \frac{1}{\sqrt{2n+1}} < \frac{1}{\sqrt{2n}} \tag{2}$$

for $n > 1$. The inequality (2) is called Wallis' inequality in [3, p. 103].

The lower and upper bounds of P_n in (2) are always cited and applied by mathematicians. The smallest upper bound $\frac{2}{\sqrt{\pi}} \cdot \frac{1}{\sqrt{4n+1}}$ and the largest lower bound $\sqrt{\frac{2}{\pi}} \cdot \frac{1}{\sqrt{2n+1}}$ in (2), that is, the following inequalities

$$\sqrt{\frac{2}{\pi}} \cdot \frac{1}{\sqrt{2n+1}} < P_n < \frac{2}{\sqrt{\pi}} \cdot \frac{1}{\sqrt{4n+1}} \tag{3}$$

are obtained by N. D. Kazarinoff. See [2, pp. 47–48 and pp. 65–67].

Date: July 16, 2002.

1991 *Mathematics Subject Classification.* Primary 26D15; Secondary 26A48.

The authors were supported in part by NNSF (#10001016) of China, SF for the Prominent Youth of Henan Province (#0112000200), SF of Henan Innovation Talents at Universities, NSF of Henan Province (#004051800), SF for Pure Research of Natural Science of the Education Department of Henan Province (#1999110004), Doctor Fund of Jiaozuo Institute of Technology, China.

This paper was typeset using $\mathcal{A}_\mathcal{M}\mathcal{S}$-LaTeX.

We can rewrite inequality (3) as

$$\frac{1}{\sqrt{\pi(n+\frac{1}{2})}} < P_n < \frac{1}{\sqrt{\pi(n+\frac{1}{4})}} \tag{4}$$

for $n \in \mathbb{N}$. See [1, p. 259].

It is well-known that factorials and their 'continuous' extension play an eminent role, for instance, in Combinatorics, Graph Theory, and Special Functions.

In this article, we will refine inequality (4). More precisely, we will ask for two best possible constants a and b such that the following double inequality

$$\frac{1}{\sqrt{\pi(n+a)}} < P_n < \frac{1}{\sqrt{\pi(n+b)}} \tag{5}$$

holds for all natural number n. In other words, the constants a and b can not be replaced by smaller or larger numbers in (5) respectively.

2. Lemma

Lemma 1. *Let* $f(x) = \frac{\Gamma(x+1)}{\Gamma(x+\frac{1}{2})\sqrt{x+c}}$, *where* c *is a constant. Then* $\lim_{x\to\infty} f(x) = 1$.

Proof. Taking logarithm yields

$$\ln f(x) = \ln\Gamma(x+1) - \frac{1}{2}\ln(x+c) - \ln\Gamma\Bigl(x+\frac{1}{2}\Bigr). \tag{6}$$

The following approximating expansion of $\Gamma(x)$ is given in [5, p. 107]:

$$\ln\Gamma(x) = \Bigl(x-\frac{1}{2}\Bigr)\ln x - x + \frac{1}{2}\ln(2\pi) + \sum_{r=1}^{n}\frac{(-1)^{r-1}B_r}{2r(2r-1)}x^{-2r+1} + O(x^{-2n-1}), \tag{7}$$

where B_r denotes the r-th Bernoulli number which is defined in [5, p. 1] by

$$t\left(\frac{1}{e^t-1}+\frac{1}{2}\right) = 1 + \sum_{n=1}^{\infty}(-1)^{n-1}\frac{t^{2n}}{(2n)!}B_n. \tag{8}$$

Utilizing (7) in (6) and simplifying can give the following

$$\ln f(x) = x\ln\frac{x+1}{x+\frac{1}{2}} + \frac{1}{2}\ln\frac{x+1}{x+c} - \frac{1}{2} + O\left(\frac{1}{x}\right) \to 0 \quad (x\to\infty), \tag{9}$$

and then we have $\lim_{x\to\infty} f(x) = 1$. The proof is complete. ∎

Corollary 1. *For any constant* c, *we have*

$$\lim_{n\to\infty}\frac{\Gamma(n+1)}{\Gamma(n+\frac{1}{2})\sqrt{n+c}} = 1. \tag{10}$$

Proof. This is a special case of Lemma 1. ∎

3. Main results

Now we will give the main results of this paper.

Theorem 1. *For all natural number n, we have*

$$\frac{1}{\sqrt{\pi(n+\frac{2}{7})}} < \frac{(2n-1)!!}{(2n)!!} < \frac{1}{\sqrt{\pi(n+\frac{1}{4})}}. \tag{11}$$

The constants $\frac{2}{7}$ and $\frac{1}{4}$ are the best possible, which means that they cannot be replaced by smaller number and larger number, respectively.

Proof. Let

$$\Omega_n = \frac{\Gamma(n+1)}{\Gamma(n+\frac{1}{2})\sqrt{n+c}} \tag{12}$$

for $n \in \mathbb{N}$. Since

$$\Gamma(n+1) = n!, \quad \Gamma\Big(n+\frac{1}{2}\Big) = \frac{(2n-1)!!}{2^n}\sqrt{\pi}, \quad 2^n n! = (2n)!!,$$

then we have

$$\Omega_n = \frac{1}{\sqrt{\pi(n+c)}} \cdot \frac{1}{P_n}, \tag{13}$$

furthermore,

$$\frac{\Omega_{n+1}}{\Omega_n} = \frac{2(n+1)}{2n+1}\sqrt{\frac{n+c}{n+1+c}}. \tag{14}$$

It is easy to see that the sequence $\{\Omega_n\}_{n=1}^{\infty}$ decreases strictly if and only if

$$\frac{2(n+1)}{2n+1}\sqrt{\frac{n+c}{n+1+c}} < 1$$

holds for all natural number $n \in \mathbb{N}$, which is equivalent to $-1 < c \leq \frac{1}{4}$.

Since $\lim_{n\to\infty} \Omega_n = 1$ from (10), thus we have

$$\Omega_n = \frac{1}{\sqrt{\pi(n+c)}} \cdot \frac{1}{P_n} > 1,$$

which is equivalent to

$$P_n < \frac{1}{\sqrt{\pi(n+c)}} \tag{15}$$

for $n \in \mathbb{N}$ and $-1 < c \leq \frac{1}{4}$.

From (14), it follows that the sequence $\{\Omega_n\}_{n=1}^{\infty}$ increases strictly if and only if

$$\frac{2(n+1)}{2n+1}\sqrt{\frac{n+c}{n+1+c}} > 1 \tag{16}$$

holds for all natural number $n \in \mathbb{N}$, which is equivalent to $c > \frac{2}{7}$. Therefore, if $c > \frac{2}{7}$, the following inequality

$$P_n > \frac{1}{\sqrt{\pi(n+c)}} \tag{17}$$

holds for all $n \in \mathbb{N}$.

For $c = \frac{2}{7}$, inequality (16) holds for $n \geq 2$. This means that the sequence

$$\left\{ \frac{1}{P_n \sqrt{\pi(n+\frac{2}{7})}} \right\}_{n=2}^{\infty}$$

is strictly increasing and tends to 1 as $n \to \infty$. Therefore

$$P_n > \frac{1}{\sqrt{\pi(n+\frac{2}{7})}} \tag{18}$$

holds for all $n \geq 2$. Direct computing reveals that inequality (18) is valid for $n = 1$. Hence, inequality (17) holds for all natural number n and $c \geq \frac{2}{7}$.

In conclusion, the following double inequality

$$\frac{1}{\sqrt{\pi(n+a)}} < P_n < \frac{1}{\sqrt{\pi(n+b)}} \tag{19}$$

holds for all $n \in \mathbb{N}$ and for all $a \geq \frac{2}{7}$ and $b \leq \frac{1}{4}$. The proof is complete. ∎

References

[1] P. S. Bullen, *A Dictionary of Inequalities*, Pitman Monographs and Surveys in Pure and Applied Mathematics **97**, Addison Wesley Longman Limited, 1998.

[2] N. D. Kazarrinoff, *Analytic Inequalities*, Holt, Rhinehart and Winston, New York, 1961.

[3] J.-Ch. Kuang, *Chángyòng Bùděngshì* (*Applied Inequalities*), 2nd edition, Hunan Education Press, Changsha, China, 1993. (Chinese)

[4] F. Qi, L.-H. Cui, and S.-L. Xu, *Some inequalities constructed by Tchebysheff's integral inequality, Math. Inequal. Appl. 2 (1999), no. 4, 517–528.*

[5] *Zh.-X. Wang and D.-R. Guo, Tèshū Hánshù Gàilùn (Introduction to Special Function), The Series of Advanced Physics of Peking University, Peking University Press, Beijing, China, 2000. (Chinese)*

Department of Applied Mathematics and Informatics, Jiaozuo Institute of Technology, #142, Mid-Jiefang Road, Jiaozuo City, Henan 454000, CHINA

E-mail address: `qifeng@jzit.edu.cn`

URL: `http://rgmia.vu.edu.au/qi.html`