# SOME INEQUALITIES FOR THE INTEGRAL MEAN OF HÖLDER CONTINUOUS FUNCTIONS DEFINED ON DISKS IN A PLANE

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Abstract. Some bounds for the derivation of the integral mean of a function defined on a compact disk from the value at the central point and related results are presented. A version of Ostrowski's inequality for functions defined on the unit disk is also presented.

## 1. Introduction

In a recent paper [2], S.S. Dragomir proved the following Hermite-Hadamard type inequality for convex functions on the compact disk $D(C,R)$.

**Theorem 1.** *If the mapping $f : D(C,R) \to \mathbb{R}$ is convex on the disk $D(C,R)$ centered at the point $C$ and having the radius $R > 0$, then,*

$$f(C) \leq \frac{1}{\pi R^2} \iint_{D(C,R)} f(x,y)\,dx\,dy \leq \frac{1}{2\pi R} \int_{\sigma(C,R)} f(\gamma)\,dl(\gamma), \tag{1.1}$$

*where $\sigma(C,R)$ is the circle centered at the point $C$ with radius $R$.*
*The above inequalities are sharp.*

In fact, the author provided an upper bound for the integral mean,

$$\frac{1}{\pi R^2} \iint_{D(C,R)} f(x,y)\,dx\,dy,$$

i.e., the inequality (see [2], inequality (2.4))

$$\frac{1}{\pi R^2} \iint_{D(C,R)} f(x,y)\,dx\,dy \leq \frac{2}{3} \cdot \frac{1}{2\pi R} \int_{\sigma(C,R)} f(\gamma)\,dl(\gamma) + \frac{1}{3} f(C). \tag{1.2}$$

Applications for different functionals associated with (1.1) were also provided.

For other Hermite-Hadamard type inequalities in several dimensions see [4]-[7] and the monograph on line [3].

In this paper we drop the assumption of convexity and compare the integral mean

$$\frac{1}{\pi R^2} \iint_{D(C,R)} f(x,y)\,dx\,dy$$

with different quantities including $f(C)$ and $\frac{1}{2\pi R}\int_{\sigma(C,R)} f(\gamma)\,dl(\gamma)$, under the assumption of certain Hölder type conditions for $f$.

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## 2. Some Inequalities for Hölder Type Functions

The following result holds.

**Theorem 2.** *Let $f : D(C,R) \to \mathbb{R}$ $\left(C=(a,b)\in\mathbb{R}^2\right)$ be a function satisfying the Hölder type condition:*

$$|f(a,b)-f(x,y)| \le L_1 |x-a|^{\alpha_1} + L_2 |y-b|^{\alpha_2}, \tag{2.1}$$

*where $L_1, L_2 > 0$ and $\alpha_1, \alpha_2 \in (-1,\infty)$, $(x,y)\in D(C,R)\backslash C$, then we have,*

$$\left| f(C) - \frac{1}{\pi R^2}\iint_{D(C,R)} f(x,y)\,dx\,dy \right| \tag{2.2}$$
$$\le \frac{2}{\sqrt{\pi}}\left[\frac{R^{\alpha_1}}{\alpha_1+2}\cdot\frac{\Gamma\left(\frac{\alpha_1+1}{2}\right)}{\Gamma\left(\frac{\alpha_1+2}{2}\right)}\cdot L_1 + \frac{R^{\alpha_2}}{\alpha_2+2}\cdot\frac{\Gamma\left(\frac{\alpha_2+1}{2}\right)}{\Gamma\left(\frac{\alpha_2+2}{2}\right)}\cdot L_2\right].$$

*For $\alpha_1, \alpha_2 > 0$, the constant $\frac{2}{\sqrt{\pi}}$ is sharp.*

*Proof.* Integrating (2.1) on $D(C,R)$, we get

$$\left| f(C) - \frac{1}{\pi R^2}\iint_{D(C,R)} f(x,y)\,dx\,dy \right| \tag{2.3}$$
$$\le \frac{1}{\pi R^2}\iint_{D(C,R)} |f(a,b)-f(x,y)|\,dx\,dy$$
$$\le \frac{1}{\pi R^2}\left[L_1\iint_{D(C,R)}|x-a|^{\alpha_1}\,dx\,dy + L_2\iint_{D(C,R)}|y-b|^{\alpha_2}\,dx\,dy\right].$$

Denote

$$A_1 := \iint_{D(C,R)}|x-a|^{\alpha_1}\,dx\,dy \quad \text{and} \quad A_2 := \iint_{D(C,R)}|y-b|^{\alpha_2}\,dx\,dy.$$

If we use the change of coordinates

$$\begin{cases} x = a + r\cos\theta, \\ y = b + r\sin\theta, \end{cases} \quad r\in[0,R],\ \ \theta\in[0,2\pi],$$

then

$$\frac{\partial(x,y)}{\partial(r,\theta)} = r$$

and

$$A_1 = \int_0^R\int_0^{2\pi} r^{\alpha_1}|\cos\theta|^{\alpha_1}\cdot r\,dr d\theta = \int_0^R r^{\alpha_1+1}dr\cdot\int_0^{2\pi}|\cos\theta|^{\alpha_1}\,d\theta.$$

Since

$$\int_0^R r^{\alpha_1+1}dr = \frac{R^{\alpha_1+2}}{\alpha_1+2}$$

and

$$\int_0^{2\pi}|\cos\theta|^{\alpha_1}\,d\theta = 4\cdot\int_0^{\frac{\pi}{2}}(\cos\theta)^{\alpha_1}\,d\theta = 4\cdot\frac{\sqrt{\pi}}{2}\cdot\frac{\Gamma\left(\frac{\alpha_1+1}{2}\right)}{\Gamma\left(\frac{\alpha_1+2}{2}\right)},$$

then

$$A_1 = \frac{2\sqrt{\pi}}{\alpha_1+2}R^{\alpha_1+2}\cdot\frac{\Gamma\left(\frac{\alpha_1+1}{2}\right)}{\Gamma\left(\frac{\alpha_1+2}{2}\right)}.$$

Also,

$$A_2 = 4 \cdot \frac{R^{\alpha_2+2}}{\alpha_2+2} \cdot \int_0^{\frac{\pi}{2}} (\sin\theta)^{\alpha_2}\, d\theta$$

and since

$$\int_0^{\frac{\pi}{2}} (\sin\theta)^{\alpha_2}\, d\theta = \frac{\sqrt{\pi}}{2} \cdot \frac{\Gamma\left(\frac{\alpha_2+1}{2}\right)}{\Gamma\left(\frac{\alpha_2+2}{2}\right)}$$

we deduce

$$A_2 = \frac{2\sqrt{\pi}}{\alpha_2+2} R^{\alpha_2+2} \cdot \frac{\Gamma\left(\frac{\alpha_2+1}{2}\right)}{\Gamma\left(\frac{\alpha_2+2}{2}\right)}.$$

Now using (2.3) we deduce the desired inequality (2.2).

To prove the sharpness of the constant $\frac{2}{\sqrt{\pi}}$, we observe that if we choose (for $\alpha_1, \alpha_2 > 0$)

$$f : D(C,R) \to \mathbb{R},\ f(x,y) = L_1 |x-a|^{\alpha_1} + L_2 |y-b|^{\alpha_2},$$

then

$$\begin{aligned} &\left| f(C) - \frac{1}{\pi R^2} \iint_{D(C,R)} f(x,y)\, dx\, dy \right| \\ = &\ \frac{1}{\pi R^2} \iint_{D(C,R)} |f(a,b) - f(x,y)|\, dx\, dy \\ = &\ \frac{1}{\pi R^2} \left[ L_1 \iint_{D(C,R)} |x-a|^{\alpha_1}\, dx\, dy + L_2 \iint_{D(C,R)} |y-b|^{\alpha_2}\, dx\, dy \right]. \end{aligned}$$

The theorem is now completely proved. ∎

The following corollary for Lipschitzian functions holds.

**Corollary 1.** *If $f : D(C,R) \to \mathbb{R}$ satisfies the Lipschitzian condition*

$$|f(a,b) - f(x,y)| \le M_1 |x-a| + M_2 |y-b|,\ (x,y) \in D(C,R), \tag{2.4}$$

*where $M_1, M_2 > 0$, then we have the inequality,*

$$\left| f(C) - \frac{1}{\pi R^2} \iint_{D(C,R)} f(x,y)\, dx\, dy \right| \le \frac{4}{3\pi} R (M_1 + M_2). \tag{2.5}$$

*The constant in (2.5), $\frac{4}{3\pi}$, is sharp.*

The proof follows by the above theorem with $\alpha_1 = \alpha_2 = 1$.

In practical applications, the following corollary may be useful.

**Corollary 2.** *If $f : D(C,R) \to \mathbb{R}$ has continuous partial derivatives on $D(C,R)$ and*

$$\begin{aligned} \left\| \frac{\partial f}{\partial x} \right\|_{D(C,R),\infty} &:= \sup_{(x,y)\in D(C,R)} \left| \frac{\partial f(x,y)}{\partial x} \right| < \infty, \\ \left\| \frac{\partial f}{\partial y} \right\|_{D(C,R),\infty} &:= \sup_{(x,y)\in D(C,R)} \left| \frac{\partial f(x,y)}{\partial y} \right| < \infty; \end{aligned}$$

*then*

$$\left|f(C)-\frac{1}{\pi R^2}\iint_{D(C,R)}f(x,y)\,dx\,dy\right| \leq \frac{4}{3\pi}R\left[\left\|\frac{\partial f}{\partial x}\right\|_{D(C,R),\infty}+\left\|\frac{\partial f}{\partial y}\right\|_{D(C,R),\infty}\right]. \tag{2.6}$$

*The constant* $\frac{4}{3\pi}$ *is sharp.*

The following theorem also holds.

**Theorem 3.** *Assume that the function* $f:D(C,R)\to\mathbb{R}$ *satisfies the condition*

$$\begin{aligned}&|f(r\cos\theta+a,r\sin\theta+b)-f(R\cos\theta+a,R\sin\theta+b)|\\ \leq\;& L_1(R-r)^{\alpha_1}|\cos\theta|^{\alpha_2}+L_2(R-r)^{\alpha_3}|\sin\theta|^{\alpha_4}\end{aligned} \tag{2.7}$$

*for all* $r\in[0,R]$, $\theta\in[0,2\pi]$, *where* $\alpha_i\in(-1,\infty)$ $\left(i=\overline{1,4}\right)$, *then,*

$$\begin{aligned}&\left|\frac{1}{\pi R^2}\iint_{D(C,R)}f(x,y)\,dx\,dy-\frac{1}{2\pi R}\int_{\sigma(C,R)}f(\gamma)\,dl(\gamma)\right|\\ \leq\;&\frac{2}{\sqrt{\pi}}\left[\frac{R^{\alpha_1}}{(\alpha_1+1)(\alpha_1+2)}\cdot\frac{\Gamma\left(\frac{\alpha_2+1}{2}\right)}{\Gamma\left(\frac{\alpha_2+2}{2}\right)}\cdot L_1\right.\\ &\left.+\frac{R^{\alpha_3}}{(\alpha_3+1)(\alpha_3+2)}\cdot\frac{\Gamma\left(\frac{\alpha_4+1}{2}\right)}{\Gamma\left(\frac{\alpha_4+2}{2}\right)}\cdot L_2\right].\end{aligned} \tag{2.8}$$

*Proof.* If we multiply the condition (2.7) by $r>0$, we get,

$$\begin{aligned}&|f(r\cos\theta+a,r\sin\theta+b)r-f(R\cos\theta+a,R\sin\theta+b)r|\\ \leq\;& L_1r(R-r)^{\alpha_1}|\cos\theta|^{\alpha_2}+L_2r(R-r)^{\alpha_3}|\sin\theta|^{\alpha_4}.\end{aligned} \tag{2.9}$$

Integrating (2.9) over $[0,R]\times[0,2\pi]$ and using the change of variable

$$\begin{cases}x=r\cos\theta+a,\\ y=r\sin\theta+b,\end{cases}\quad r\in[0,R],\quad\theta\in[0,2\pi],$$

for which we have

$$\frac{\partial(x,y)}{\partial(r,\theta)}=r,$$

we obtain

$$\begin{aligned}&\left|\int_0^R\int_0^{2\pi}f(r\cos\theta+a,r\sin\theta+b)\,r\,drd\theta\right.\\ &\left.\quad-\frac{R^2}{2}\int_0^{2\pi}f(R\cos\theta+a,R\sin\theta+b)\,d\theta\right|\\ \leq\;& L_1\int_0^R r(R-r)^{\alpha_1}dr\cdot\int_0^{2\pi}|\cos\theta|^{\alpha_2}d\theta\\ &+L_2\int_0^R r(R-r)^{\alpha_3}dr\cdot\int_0^{2\pi}|\sin\theta|^{\alpha_4}d\theta.\end{aligned} \tag{2.10}$$

Since

$$\int_0^R \int_0^{2\pi} f(r\cos\theta + a, r\sin\theta + b)\, r\, dr d\theta = \iint_{D(C,R)} f(x,y)\, dx\, dy,$$

$$\int_0^{2\pi} f(R\cos\theta + a, R\sin\theta + b)\, d\theta = \int_{\sigma(C,R)} f(\gamma)\, dl(\gamma)$$

$$\left( = \int_0^{2\pi} f(x(\theta), y(\theta)) \left[ (\dot{x}(\theta))^2 + (\dot{y}(\theta))^2 \right]^{\frac{1}{2}} d\theta \right),$$

$$\int_0^R r (R-r)^{\alpha_1}\, dr = \frac{R^{\alpha_1+2}}{(\alpha_1+1)(\alpha_1+2)},$$

$$\int_0^R r (R-r)^{\alpha_3}\, dr = \frac{R^{\alpha_3+2}}{(\alpha_3+1)(\alpha_3+2)},$$

$$\int_0^{2\pi} |\cos\theta|^{\alpha_2}\, d\theta = 4\int_0^{\frac{\pi}{2}} [\cos\theta]^{\alpha_2}\, d\theta = 2\sqrt{\pi}\frac{\Gamma\left(\frac{\alpha_2+1}{2}\right)}{\Gamma\left(\frac{\alpha_2+2}{2}\right)},$$

$$\int_0^{2\pi} |\sin\theta|^{\alpha_4}\, d\theta = 4\int_0^{\frac{\pi}{2}} [\sin\theta]^{\alpha_4}\, d\theta = 2\sqrt{\pi}\frac{\Gamma\left(\frac{\alpha_4+1}{2}\right)}{\Gamma\left(\frac{\alpha_4+2}{2}\right)},$$

then, by (2.10), on dividing by $\pi R^2$, we deduce the desired result (2.8). ∎

The following corollary for Lipschitzian functions holds.

**Corollary 3.** *Assume that the function* $f : D(C,R) \to \mathbb{R}$ *is Lipschitzian on* $D(C,R)$ *with the constants* $K_1, K_2 > 0$*, i.e.,*

$$|f(x,y) - f(u,v)| \le K_1 |x-u| + K_2 |y-v| \;\; \textit{for any} \;\; (x,y) \in D(C,R), \tag{2.11}$$

*then,*

$$\left| \frac{1}{\pi R^2} \iint_{D(C,R)} f(x,y)\, dx\, dy - \frac{1}{2\pi R}\int_{\sigma(C,R)} f(\gamma)\, dl(\gamma) \right| \le \frac{2R}{3\pi}(K_1 + K_2). \tag{2.12}$$

A more practical result is embodied in the following corollary.

**Corollary 4.** *Assume that* $f : D(C,R) \to \mathbb{R}$ *has partial derivatives continuous on* $D(C,R)$*, then,*

$$\left| \frac{1}{\pi R^2} \iint_{D(C,R)} f(x,y)\, dx\, dy - \frac{1}{2\pi R}\int_{\sigma(C,R)} f(\gamma)\, dl(\gamma) \right| \le \frac{2R}{3\pi}\left[ \left\| \frac{\partial f}{\partial x} \right\|_{D(C,R),\infty} + \left\| \frac{\partial f}{\partial y} \right\|_{D(C,R),\infty} \right]. \tag{2.13}$$

It is also quite natural to compare the value of the function in $C$ with the integral mean taken on the boundary.

**Theorem 4.** *Assume that the function* $f : D(C,R) \to \mathbb{R}$ *satisfies the condition:*

$$|f(a,b) - f(R\cos\theta + a, R\sin\theta + b)| \le L_1 R^{\alpha_1} |\cos\theta|^{\alpha_2} + L_2 R^{\alpha_3} |\sin\theta|^{\alpha_4}, \;\; \theta \in [0, 2\pi], \tag{2.14}$$

*where $L_1, L_2 > 0$ and $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in (-1, \infty)$, then we have,*

$$\left| f(C) - \frac{1}{2\pi R} \int_{\sigma(C,R)} f(\gamma)\, dl(\gamma) \right| \leq \frac{1}{\sqrt{\pi}} \left[ L_1 R^{\alpha_1} \cdot \frac{\Gamma\left(\frac{\alpha_2+1}{2}\right)}{\Gamma\left(\frac{\alpha_2+2}{2}\right)} + L_2 R^{\alpha_3} \cdot \frac{\Gamma\left(\frac{\alpha_4+1}{2}\right)}{\Gamma\left(\frac{\alpha_4+2}{2}\right)} \right]. \tag{2.15}$$

*The constant $\frac{1}{\sqrt{\pi}}$ is sharp for $\alpha_1 = \alpha_2$ and $\alpha_3 = \alpha_4$.*

*Proof.* Integrating the condition (2.14) and $\theta \in [0, 2\pi]$, we have

$$\begin{aligned} & \left| 2\pi f(a,b) - \int_0^{2\pi} f(R\cos\theta + a, R\sin\theta + b)\, d\theta \right| \\ \leq\ & L_1 R^{\alpha_1} \int_0^{2\pi} |\cos\theta|^{\alpha_2}\, d\theta + L_2 R^{\alpha_3} \int_0^{2\pi} |\sin\theta|^{\alpha_4}\, d\theta \\ =\ & L_1 R^{\alpha_1} \cdot 2\sqrt{\pi} \cdot \frac{\Gamma\left(\frac{\alpha_2+1}{2}\right)}{\Gamma\left(\frac{\alpha_2+2}{2}\right)} + L_2 R^{\alpha_3} \cdot 2\sqrt{\pi} \cdot \frac{\Gamma\left(\frac{\alpha_4+1}{2}\right)}{\Gamma\left(\frac{\alpha_4+2}{2}\right)} \end{aligned}$$

and dividing by $2\pi$, we deduce (2.15).

The equality in (2.15) holds for $f(x,y) = L_1 |x-a|^{\alpha_1} + L_2 |y-b|^{\alpha_3}$. ∎

**Corollary 5.** *If the function $f : D(C,R) \to \mathbb{R}$ satisfies the Lipschitzian condition (2.4), then we have the inequality:*

$$\left| f(C) - \frac{1}{2\pi R} \int_{\sigma(C,R)} f(\gamma)\, dl(\gamma) \right| \leq \frac{2R}{\pi} [M_1 + M_2]. \tag{2.16}$$

*The constant $\frac{2}{\pi}$ is sharp.*

**Corollary 6.** *If $f : D(C,R) \to \mathbb{R}$ has continuous partial derivatives on $D(C,R)$, then*

$$\left| f(C) - \frac{1}{2\pi R} \int_{\sigma(C,R)} f(\gamma)\, dl(\gamma) \right| \leq \frac{2R}{\pi} \left[ \left\| \frac{\partial f}{\partial x} \right\|_{D(C,R),\infty} + \left\| \frac{\partial f}{\partial y} \right\|_{D(C,R),\infty} \right]. \tag{2.17}$$

*The constant $\frac{2}{\pi}$ is sharp.*

## 3. An Ostrowski Type Inequality on the Disk

Using the notation

$${}_pF_q \left[ \begin{array}{c} a_1, \ldots, a_p \\ \\ b_1, \ldots, b_q \end{array} ;\quad x \right] := \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \cdot \frac{x^n}{n!}, \tag{3.1}$$

where

$$(a)_0 := 1,\ \ (a)_n := a(a+1)\cdots(a+n-1),\ \ n \geq 1,\ n \in \mathbb{N}$$

we have Euler's representation formula

$${}_2F_1 \left[ \begin{array}{c} a, b \\ c \end{array} ;\ y \right] = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 (1-yt)^{-a}\, t^{b-1} (1-t)^{c-b-1}\, dt, \tag{3.2}$$

where $|y| < 1$, $\operatorname{Re} c > \operatorname{Re} b > 0$ (see for example [1]).

The following technical lemma holds.

**Lemma 1.** *For $x \in [-1,1]$ and $\alpha > -1$, consider the integral*

$$I(\alpha, x) := \int_{-1}^{1} |x-u|^{\alpha} \sqrt{1-u^2}\, du. \tag{3.3}$$

*We then have the representation,*

$$I(\alpha,x) = \begin{cases} \frac{\sqrt{2\pi}}{2} \cdot \frac{\Gamma(\alpha+1)}{\Gamma\left(\alpha+\frac{5}{2}\right)} \left\{ (x+1)^{\alpha+\frac{3}{2}} \, {}_2F_1\left[ \begin{matrix} -\frac{1}{2}, \frac{3}{2} \\ \alpha+\frac{5}{2} \end{matrix} ; \frac{x+1}{2} \right] \right. \\ \left. \quad + (1-x)^{\alpha+\frac{3}{2}} \, {}_2F_1\left[ \begin{matrix} -\frac{1}{2}, \frac{3}{2} \\ \alpha+\frac{5}{2} \end{matrix} ; \frac{1-x}{2} \right] \right\}, & \text{if} \quad x \in (-1,1) \\ 2^{\alpha+2} B\left(\frac{3}{2}, \alpha+\frac{3}{2}\right), \quad \text{if} \quad x \in \{-1,1\} \end{cases} \tag{3.4}$$

*Proof.* Let $x \in (-1,1)$. First, observe that

$$\begin{aligned} I(\alpha,x) &= \int_{-1}^{x} (x-u)^{\alpha} (1-u)^{\frac{1}{2}} (u+1)^{\frac{1}{2}} du \\ &\quad + \int_{x}^{1} (u-x)^{\alpha} (1-u)^{\frac{1}{2}} (1+u)^{\frac{1}{2}} du. \end{aligned}$$

Applying the change of variable,

$$u = (1-\lambda)(-1) + \lambda x, \quad \lambda \in [0,1]$$

in the first integral, we obtain,

$$\begin{aligned} I_1(\alpha,x) &= \int_{-1}^{x} (x-u)^{\alpha} (1-u)^{\frac{1}{2}} (u+1)^{\frac{1}{2}} du \\ &= (x+1)^{\alpha+\frac{3}{2}} \int_0^1 (1-\lambda)^{\alpha} \lambda^{\frac{1}{2}} \left[2-(1+x)\lambda\right]^{\frac{1}{2}} d\lambda \\ &= \sqrt{2}\,(x+1)^{\alpha+\frac{3}{2}} \int_0^1 (1-\lambda)^{\alpha+\frac{5}{2}-\frac{3}{2}-1} \lambda^{\frac{3}{2}-1} \left(1-\left(\frac{x+1}{2}\right)\lambda\right)^{-\left(-\frac{1}{2}\right)} d\lambda. \end{aligned}$$

Using (3.2) for $a = -\frac{1}{2}$, $b = \frac{3}{2}$, $c = \alpha + \frac{5}{2}$, $y = \frac{x+1}{2}$, we observe that $|y| < 1$, $\operatorname{Re} c > \operatorname{Re} b > 0$ and then

$$\begin{aligned} &\int_0^1 (1-\lambda)^{\alpha+\frac{5}{2}-\frac{3}{2}-1} \lambda^{\frac{3}{2}-1} \left(1-\left(\frac{x+1}{2}\right)\lambda\right)^{-\left(-\frac{1}{2}\right)} d\lambda \\ &= \frac{\Gamma\left(\frac{3}{2}\right)\Gamma(\alpha+1)}{\Gamma\left(\alpha+\frac{5}{2}\right)} \, {}_2F_1\left[ \begin{matrix} -\frac{1}{2}, \frac{3}{2} \\ \alpha+\frac{5}{2} \end{matrix} ; \frac{x+1}{2} \right] \end{aligned}$$

giving

$$I_1(\alpha,x) = \frac{\sqrt{2\pi}}{2} \cdot \frac{\Gamma(\alpha+1)}{\Gamma\left(\alpha+\frac{5}{2}\right)} (x+1)^{\alpha+\frac{3}{2}} \, {}_2F_1\left[ \begin{matrix} -\frac{1}{2}, \frac{3}{2} \\ \alpha+\frac{5}{2} \end{matrix} ; \frac{x+1}{2} \right]. \tag{3.5}$$

If we consider the change of variable

$$u = (1-\lambda)x + \lambda, \quad \lambda \in [0,1]$$

in the second integral, we obtain:

$$\begin{aligned} I_2(\alpha,x) &= \int_x^1 (u-x)^\alpha (1-u)^{\frac{1}{2}} (1+u)^{\frac{1}{2}}\, du \\ &= (1-x)^{\alpha+\frac{3}{2}} \int_0^1 \lambda^\alpha (1-\lambda)^{\frac{1}{2}} [1+x+\lambda(1-x)]^{\frac{1}{2}}\, d\lambda \\ &= (1-x)^{\alpha+\frac{3}{2}} \int_0^1 (1-\mu)^\alpha \mu^{\frac{1}{2}} [2-(1-x)\mu]^{\frac{1}{2}}\, d\mu \\ &= \sqrt{2}(1-x)^{\alpha+\frac{3}{2}} \int_0^1 (1-\mu)^\alpha \mu^{\frac{1}{2}} \left[1-\left(\frac{1-x}{2}\right)\mu\right]^{\frac{1}{2}} d\mu \\ &= \sqrt{2}(1-x)^{\alpha+\frac{3}{2}} \int_0^1 (1-\mu)^{\alpha+\frac{5}{2}-\frac{3}{2}-1} \mu^{\frac{3}{2}-1} \left(1-\left(\frac{1-x}{2}\right)\mu\right)^{-\left(-\frac{1}{2}\right)} d\mu. \end{aligned}$$

Using (3.2) for $a = \frac{-1}{2}$, $b = \frac{3}{2}$, $c = \alpha + \frac{5}{2}$, $y = \frac{1-x}{2}$, we observe that $|y| < 1$, $\operatorname{Re} c > \operatorname{Re} b > 0$ and thus

$$\begin{aligned} &\int_0^1 (1-\mu)^{\alpha+\frac{5}{2}-\frac{3}{2}-1} \mu^{\frac{3}{2}-1} \left(1-\left(\frac{1-x}{2}\right)\mu\right)^{-\left(-\frac{1}{2}\right)} d\mu \\ &= \frac{\Gamma\left(\frac{3}{2}\right)\Gamma(\alpha+1)}{\Gamma\left(\alpha+\frac{5}{2}\right)} {}_2F_1\left[\begin{matrix} -\frac{1}{2}, \frac{3}{2} \\ \alpha+\frac{5}{2} \end{matrix} ; \frac{1-x}{2}\right] \end{aligned}$$

giving

$$I_2(\alpha,x) = \frac{\sqrt{2\pi}}{2} \cdot \frac{\Gamma(\alpha+1)}{\Gamma\left(\alpha+\frac{5}{2}\right)} (1-x)^{\alpha+\frac{3}{2}} {}_2F_1\left[\begin{matrix} -\frac{1}{2}, \frac{3}{2} \\ \alpha+\frac{5}{2} \end{matrix} ; \frac{1-x}{2}\right].$$

For $x = -1$ or $x = 1$ the identity (3.4) is obvious and we omit the details. ∎

We may now state the following Ostrowski type inequality for functions of two independent variables defined on the compact unity disk.

**Theorem 5.** *Assume that the function* $f : D(0,1) \subset \mathbb{R}^2 \to \mathbb{R}$ *where* $D(0,1) = \left\{(x,y) \in \mathbb{R}^2 | x^2+y^2 \le 1\right\}$ *satisfy the condition*

$$|f(x,y) - f(u,v)| \le L_1 |x-u|^{\alpha_1} + L_2 |y-v|^{\alpha_2}, \quad (\alpha_1, \alpha_2 > -1) \tag{3.6}$$

*for any* $(x,y)$, $(u,v) \in D(0,1)$, $(u,v) \ne (x,y)$. *We have the inequality,*

$$\left| f(x,y) - \frac{1}{\pi} \iint_{D(0,1)} f(u,v)\, du\, dv \right| \le \frac{2}{\pi} \left[L_1 I(\alpha_1, x) + L_2 I(\alpha_2, y)\right] \tag{3.7}$$

*for any* $(x,y) \in D(0,1)$, *where the functions* $I(\cdot,\cdot)$ *are defined by (3.3).*

*Proof.* By (3.6) we have, for $(x,y) \in D(0,1)$,

$$\begin{aligned} & \left| f(x,y) - \frac{1}{\pi} \iint_{D(0,1)} f(u,v)\, du\, dv \right| \\ \leq & \frac{1}{\pi} \iint_{D(0,1)} |f(x,y) - f(u,v)|\, du\, dv \\ \leq & \frac{1}{\pi} \iint_{D(0,1)} \left[ L_1 |x-u|^{\alpha_1} + L_2 |y-v|^{\alpha_2} \right] du\, dv \\ = & \frac{2}{\pi} \left[ L_1 I(\alpha_1, x) + L_2 I(\alpha_2, y) \right] \end{aligned}$$

and the inequality (3.7) is proved. ∎

**Corollary 7.** *Assume that the function* $f : D(0,1) \subset \mathbb{R}^2 \to \mathbb{R}$ *satisfy the Lipschitzian condition,*

$$|f(x,y) - f(u,v)| \leq M_1 |x-u| + M_2 |y-v| \tag{3.8}$$

*for any* $(x,y)$, $(u,v) \in D(0,1)$, *then we have the inequality,*

$$\begin{aligned} & \left| f(x,y) - \frac{1}{\pi} \iint_{D(0,1)} f(u,v)\, du\, dv \right| \\ \leq & \frac{2}{\pi} \left[ M_1 \left( x \arcsin x + \frac{1}{3} \sqrt{1-x^2} \left(2+x^2\right) \right) \right. \\ & \left. + M_2 \left( y \arcsin y + \frac{1}{3} \sqrt{1-y^2} \left(2+y^2\right) \right) \right]. \end{aligned} \tag{3.9}$$

*for any* $(x,y) \in D(0,1)$.

**Remark 1.** *Finally, we observe that if* $f$ *has bounded partial derivatives on* $D(0,1)$, *then the inequality (3.9) becomes*

$$\begin{aligned} & \left| f(x,y) - \frac{1}{\pi} \iint_{D(0,1)} f(u,v)\, du\, dv \right| \\ \leq & \frac{2}{\pi} \left[ \left\| \frac{\partial f}{\partial x} \right\|_{D(0,1),\infty} \left( x \arcsin x + \frac{1}{3} \sqrt{1-x^2} \left(2+x^2\right) \right) \right. \\ & \left. + \left\| \frac{\partial f}{\partial y} \right\|_{D(0,1),\infty} \left( y \arcsin y + \frac{1}{3} \sqrt{1-y^2} \left(2+y^2\right) \right) \right]. \end{aligned} \tag{3.10}$$

*for any* $(x,y) \in D(0,1)$.

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