Existence and non-existence results for some degenerate quasilinear problems with indefinite non-linearities

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Abstract. We study the existence of non-negative and non-trivial solutions to the bifurcation quasilinear problem

$$-\mathrm{div}\,(a(x)|\nabla u|^{p-2}\nabla u) + b(x)|u|^{p-2}u + h(x)|u|^{m-2}u = f(\lambda, x, u) \quad \text{in } \mathbf{R}^N,$$

under some degeneracy hypothesis on the function $a \not\equiv 0$, which is continuous and non-negative on $\mathbf{R}^N$. We assume throughout that $\lambda > 0$ is a real parameter, $\max\{2, p\} < m < p^* = Np/(N-p)$, $1 < p < N$, the functions b and h are positive while f is a subcritical non-linearity. We show that there exists $\lambda_0 > 0$ such that the above problem has at least a solution if $\lambda \geq \lambda_0$, but it has no solution if $\lambda \in (0, \lambda_0)$. This paper extends previous results obtained by Chabrowski [4].

1 Introduction and the main result

Consider the following bifurcation quasilinear problem

$$(\mathrm{P}_\lambda)\begin{cases} -\mathrm{div}\,(a(x)|\nabla u|^{p-2}\nabla u) + b(x)|u|^{p-2}u + h(x)|u|^{m-2}u = f(\lambda, x, u) & \text{in } \mathbf{R}^N, \\ u \geq 0, \quad u \not\equiv 0 & \text{in } \mathbf{R}^N, \end{cases}$$

where $\max\{2, p\} < m < pN/(N-p)$, $1 < p < N$ and $\lambda > 0$ is a real parameter.

Problems of this type arise in the study of physical phenomena related to equilibrium of anisotropic continuous media which possible are somewhere "perfect" insulators, cf. Dautray-Lions [7]. For instance, if $\vec{\tau}$ denotes the shear stress and $\nabla_p u$ is the velocity gradient then these quantities obey a relation of the form $\vec{\tau}(x) = a(x)\nabla_p u(x)$, where $\nabla_p u = |\nabla u|^{p-2}\nabla u$. The case $p = 2$ (respectively $p < 2$, $p > 2$) corresponds to a Newtonian (respectively pseudoplastic, dilatant) fluid. The resulting equations of motion then involve the quasilinear operator $\mathrm{div}\,(a\nabla_p u)$. We refer in this sense to Aronsson-Janfalk [2] for the mathematical treatment of the Hele-Shaw flow of "power-law fluids". The concept of Hele-Shaw flow refers to the flow between two closely-spaced parallel plates, close in the sense that the gap between the plates is small compared to the dimension of the plates. We also refer to the study of flow through porous media ($p = 3/2$, see Showalter-Walkington [18]) or glacial sliding

($p \in (1, 4/3]$, see Pélissier-Reynaud [17]). We mention the recent papers Cîrstea-Motreanu-Rădulescu [5], Cîrstea-Rădulescu [6], Drábek-Huang [9] and Drábek-Simader [10] for the mathematical treatment of bifurcation problems for several classes of quasilinear elliptic equations on unbounded domains and with respect to anisotropic spaces.

The purpose of this paper is to study the quasilinear problem (P_λ), under the following hypotheses

(B) $b \in L^\infty(\mathbf{R}^N)$ and there exists $b_0 > 0$ such that $b(x) \geq b_0$ a.e. in $\mathbf{R}^N$;

(H) $h : \mathbf{R}^N \to \mathbf{R}$ is a positive and continuous function satisfying

$$\int_{\mathbf{R}^N} \left(\frac{1}{h^q}\right)^{1/(m-q)} dx < \infty\,.$$

The integrability condition **(H)** of the ratio $1/h^q$ is inspired by assumption (1.4) in Alama-Tarantello [1].

Let $f(\lambda, x, t) : (0, \infty) \times \mathbf{R}^N \times \mathbf{R} \to \mathbf{R}$ be non-decreasing in λ, measurable in x, derivable in t satisfying

$(\mathbf{F_1})$ $f(\cdot, \cdot, 0) = 0, \quad f(\lambda, x, t) + f(\lambda, x, -t) \geq 0 \quad \forall \lambda > 0,$ a.e. $x \in \mathbf{R}^N,\ \forall t \in \mathbf{R}$;

$(\mathbf{F_2})$ $|f_t(\lambda, x, t)| \leq c_f \lambda |t|^{q-2}$ for some $m > q > \max\{2, p\},\ \forall \lambda > 0,$ a.e. $x \in \mathbf{R}^N,\ \forall t \in \mathbf{R}$;

$(\mathbf{F_3})$ $\displaystyle\lim_{t \to 0} \frac{f(\lambda, x, t)}{\lambda |t|^{q-2} t} = 1$ uniformly in x and in λ;

$(\mathbf{F_4})$ $|f(\lambda_1, x, t) - f(\lambda_2, x, t)| \leq C_f |\lambda_1 - \lambda_2| |t|^{q-1}, \quad \forall \lambda_1, \lambda_2 > 0,$ a.e. $x \in \mathbf{R}^N,\ \forall t \in \mathbf{R}$.

Remark 1 *It is easy to see that the non-linearity* $f(\lambda, x, t) = \lambda |t|^{q-2} t$, $\forall \lambda > 0$, *a.e.* $x \in \mathbf{R}^N$, $\forall t \in \mathbf{R}$ *satisfies the assumptions* $(\mathbf{F_1}) - (\mathbf{F_4})$ *provided that* $m > q > \max\{2, p\}$.

We are concerned here with the study of quasilinear elliptic problem (P_λ) in the *degenerate* case, that is the function a vanishes in at least one point in $\mathbf{R}^N$. More precisely, we assume

$(\mathbf{A_1})$ $a \in C(\mathbf{R}^N)$ and there exists $R_0 > 0$ such that

$$\{x : \ a(x) = 0\} \subset B(0, R_0) \quad \text{and} \quad 1/a \in L^s(B(0, R_0))$$

for some $s > Nq/(pN + pq - Nq)$;

$(\mathbf{A_2})$ $\inf_{\mathbf{R}^N \setminus B(0, R_0)} a(x) > 0$.

The degeneracy hypothesis $(\mathbf{A_1})$ is inspired by condition $(A-1)$ introduced in Murthy-Stampacchia [15]. In light of Proposition 1, assumption $(\mathbf{A_1})$ should be seen as a "subcritically" condition. Note that the study of degenerate elliptic boundary value problems was initiated in Mikhlin [13], [14] and many papers have been devoted in the past decades to the study of several questions related to these problems (see e.g. Murthy-Stampacchia [15], Stredulinsky [19], Passaseo [16] and the references therein).

For a positive continuous function l defined on $\mathbf{R}^N$ and $1 \leq m < \infty$, let $L^m(\mathbf{R}^N; l)$ be the space of all measurable functions u such that

$$\|u\|_{m,l} := \left(\int_{\mathbf{R}^N} |u(x)|^m l(x)\, dx\right)^{1/m} < \infty.$$

If $l \equiv 1$ on $\mathbf{R}^N$ then $L^m(\mathbf{R}^N;\, l)$ is the Lebesgue space $L^m(\mathbf{R}^N)$ and its norm will be denoted by $\|\cdot\|_m$.

Let $W^{1,p}_{a,b}(\mathbf{R}^N)$ be the Sobolev space defined as the completion of $C_0^\infty(\mathbf{R}^N)$ with respect to the norm

$$\|u\|_{a,b} = \left(\int_{\mathbf{R}^N} (a(x)|\nabla u|^p + b(x)|u|^p)\, dx\right)^{1/p}.$$

Consider the Banach space $E = W^{1,p}_{a,b}(\mathbf{R}^N) \cap L^m(\mathbf{R}^N;\, h)$ endowed with the norm

$$\|u\|_E^p := \|u\|_{a,b}^p + \left(\int_{\mathbf{R}^N} h(x)|u|^m\, dx\right)^{p/m}.$$

It is clear that the following embeddings

$$E \hookrightarrow W^{1,p}_{a,b}(\mathbf{R}^N) \quad \text{and} \quad E \hookrightarrow L^m(\mathbf{R}^N;\, h) \quad \text{are continuous.} \tag{1}$$

The energy functional corresponding to (P_λ) is given by $J_\lambda : E \to \mathbf{R}$,

$$J_\lambda(u) = \frac{1}{p}\int_{\mathbf{R}^N} (a(x)|\nabla u|^p + b(x)|u|^p)\, dx - \int_{\mathbf{R}^N} F(\lambda, x, u)\, dx + \frac{1}{m}\int_{\mathbf{R}^N} h(x)|u|^m\, dx,$$

where F is the primitive function of f with respect to the last variable, i.e. $F(\lambda, x, u) = \int_0^u f(\lambda, x, t)\, dt$. Solutions to problem (P_λ) will be found as non-negative and non-trivial critical points of J_λ. Thus, a function $u \in E$ is a solution of (P_λ) provided that $u \geq 0$, $u \not\equiv 0$ in $\mathbf{R}^N$ and for any $v \in E$,

$$\int_{\mathbf{R}^N} (a(x)|\nabla u|^{p-2}\nabla u \cdot \nabla v + b(x)|u|^{p-2}uv)\, dx + \int_{\mathbf{R}^N} h(x)|u|^{m-2}uv\, dx = \int_{\mathbf{R}^N} f(\lambda, x, u)v\, dx.$$

Since $f(\cdot,\cdot,0) = 0$, from hypothesis $(\mathbf{F_2})$ we deduce that

$$|f(\lambda, x, t)| \leq c\lambda|t|^{q-1}, \quad \forall\lambda > 0, \text{ a.e. } x \in \mathbf{R}^N, \ \forall t \in \mathbf{R}. \tag{2}$$

This inequality produces

$$|F(\lambda, x, t)| \leq C\lambda|t|^q, \quad \forall\lambda > 0, \text{ a.e. } x \in \mathbf{R}^N, \ \forall t \in \mathbf{R}, \tag{3}$$

where $c = c_f/(q-1)$ and $C = c/q$.

Our first result asserts that $W^{1,p}_{a,b}(\mathbf{R}^N)$ is continuously embedded in $L^q(\mathbf{R}^N)$. Using this fact and (3) we conclude that J_λ is well defined.

Proposition 1 *There exists a constant $C_q > 0$ such that, for any $u \in W^{1,p}_{a,b}(\mathbf{R}^N)$*

$$\left(\int_{\mathbf{R}^N} |u|^q\, dx\right)^{1/q} \leq C_q\left(\int_{\mathbf{R}^N} (a(x)|\nabla u|^p + b(x)|u|^p)\, dx\right)^{1/p}.$$

The main result of this paper is

Theorem 1 *There exists $\lambda_0 > 0$ such that the following hold:*
(i) Problem (P_λ) admits a solution, for any $\lambda \geq \lambda_0$;
(ii) Problem (P_λ) does not have any solution, provided that $0 < \lambda < \lambda_0$.

Remark 2 *Results of this type have been proven in Chabrowski [4], but in a particular framework. More precisely, Chabrowski considered the non-degenerate ($a \equiv 1$) problem (P_λ) for the non-linearity defined in Remark 1, $p = 2$ and $b \equiv 1$.*

2 Properties of J_λ

We have seen before that Proposition 1 plays an important role for proving the fact that J_λ is well defined on E. Therefore, we start with

Proof of Proposition 1. We follow the method used in Passaseo [16, Proposition 2.1] (see also Chabrowski [3, Proposition 1]). In view of hypotheses ($\mathbf{A_1}$) and ($\mathbf{A_2}$) we may assume, by taking R_0 large enough, that

$$\{x : a(x) = 0\} \subset B(0, R_0 - 1) \quad \text{and} \quad \inf_{\mathbf{R}^N - B(0,R_0-1)} a(x) > 0. \tag{4}$$

We define $r = ps/(s+1)$ with s appearing in ($\mathbf{A_1}$). We see that our hypothesis $s > Nq/(pN+pq-Nq)$ and the fact that $\max\{2, p\} < q < p^*$ imply $r < q < Nr/(N-r)$, where $1 < r < p < N$. Thus, by the Sobolev embedding theorem, $W_0^{1,r}(B(0, R_0))$ is continuously embedded in $L^q(B(0, R_0))$. From ($\mathbf{A_1}$) and Hölder's inequality we find

$$\begin{aligned}
\left(\int_{B(0,R_0)} |u|^q\,dx\right)^{1/q} &\le C_1 \left(\int_{B(0,R_0)} |\nabla u|^r\,dx\right)^{1/r} \\
&= C_1 \left(\int_{B(0,R_0)} \frac{1}{a(x)^{s/(s+1)}} |\nabla u|^r a(x)^{s/(s+1)}\,dx\right)^{1/r} \\
&\le C_1 \left(\int_{B(0,R_0)} \frac{1}{a(x)^s}\,dx\right)^{1/(ps)} \left(\int_{B(0,R_0)} a(x)|\nabla u|^p\,dx\right)^{1/p} \\
&\le C_2 \left(\int_{B(0,R_0)} (a(x)|\nabla u|^p + b(x)|u|^p\,dx\right)^{1/p}.
\end{aligned} \tag{5}$$

Let $\psi_{R_0-1} \in C^1(\mathbf{R}^N)$ be such that $\psi_{R_0-1} = 1$ in $\mathbf{R}^N \setminus B(0, R_0)$, $\psi_{R_0-1} = 0$ on $B(0, R_0 - 1)$ and $0 \le \psi_{R_0-1} \le 1$ in $\mathbf{R}^N$. Using the continuity of the embedding $W^{1,p}(\mathbf{R}^N) \hookrightarrow L^q(\mathbf{R}^N)$, (4) and our assumption ($\mathbf{B}$) we derive

$$\begin{aligned}
\left(\int_{\mathbf{R}^N\setminus B(0,R_0)} |u|^q\,dx\right)^{1/q} &= \left(\int_{\mathbf{R}^N\setminus B(0,R_0)} |u\psi_{R_0-1}|^q\,dx\right)^{1/q} \le \left(\int_{\mathbf{R}^N} |u\psi_{R_0-1}|^q\,dx\right)^{1/q} \\
&\le C_3 \left(\int_{\mathbf{R}^N} (|\nabla(u\psi_{R_0-1})|^p + |u\psi_{R_0-1}|^p)\,dx\right)^{1/p} \\
&\le C_4 \left(\int_{\mathbf{R}^N} (|u|^p|\nabla\psi_{R_0-1}|^p + |\psi_{R_0-1}|^p|\nabla u|^p + |u|^p|\psi_{R_0-1}|^p)\,dx\right)^{1/p} \\
&\le C_5 \left(\int_{\mathbf{R}^N\setminus B(R_0-1)} (|\nabla u|^p + |u|^p)\,dx\right)^{1/p} \\
&\le C_6 \left(\int_{\mathbf{R}^N\setminus B(0,R_0-1)} (a(x)|\nabla u|^p + b(x)|u|^p)\,dx\right)^{1/p},
\end{aligned} \tag{6}$$

where C_i with $i = 1, ..., 6$ denote some positive constants. From (5), (6) and the elementary inequality

$$(x+y)^{1/q} \le C(q)(x^{1/q} + y^{1/q}) \quad \text{for all } x, y > 0$$

we obtain

$$\left(\int_{\mathbf{R}^N} |u|^q\,dx\right)^{1/q} \le C(q)\left[\left(\int_{B(0,R_0)} |u|^q\,dx\right)^{1/q} + \left(\int_{\mathbf{R}^N\setminus B(0,R_0)} |u|^q\,dx\right)^{1/q}\right]$$
$$\le C_q\left(\int_{\mathbf{R}^N} (a(x)|\nabla u|^p + b(x)|u|^p)\,dx\right)^{1/p},$$

for some positive constants $C(q)$ and C_q depending only on q. This finishes the proof. ■

Remark 3 *The proof of Proposition 1 tells us, moreover, that the embedding $W^{1,p}_{a,b}(\mathbf{R}^N) \hookrightarrow L^j(\mathbf{R}^N)$ is continuous for all $q \ge j > \max\{2,p\}$.*

In this paper we denote by "$\rightharpoonup$" the weak convergence and by "$\to$" the strong convergence, in an arbitrary Banach space X. The duality pairing between X and X^* is denoted by $\langle\cdot,\cdot\rangle$.

Remark 4 *Let $\{u_n\}$ be a sequence that converges weakly to some u_0 in $W^{1,p}_{a,b}(\mathbf{R}^N)$. Then we can assume (passing eventually to subsequences) that*

$$u_n \to u_0 \ \text{in}\ L^k_{\mathrm{loc}}(\mathbf{R}^N)\ \ \forall p \le k \le q \quad \text{and} \quad u_n \to u_0 \ \text{a.e. in}\ \mathbf{R}^N \tag{7}$$

Indeed, since $\{u_n\}$ is bounded in $W^{1,p}_{a,b}(\mathbf{R}^N)$ we see that $\{u_n\}$ restricted to $\mathbf{R}^N \setminus B(0,R)$ is bounded in $W^{1,p}(\mathbf{R}^N \setminus B(0,R))$. It also follows from the proof of Proposition 1, that the sequence $\{u_n\}$ restricted to $B(0,R)$ is bounded in $W^{1,r}_0(B(0,r))$, $1<r<q<Nr/(N-r)$. Thus, (7) follows from the Sobolev embedding theorem.

We shall employ in what follows the following elementary inequality:

$$\alpha|u|^\mu - \beta|u|^\nu \le C_{\mu,\nu}\alpha\left(\frac{\alpha}{\beta}\right)^{\mu/(\nu-\mu)} \qquad \forall u\in\mathbf{R},\quad \forall\alpha,\beta\in(0,\infty),\quad \forall 0<\mu<\nu \tag{8}$$

where $C_{\mu,\nu}$ is a positive constant depending on μ and ν.

Lemma 1 *The functional J_λ is coercive for any $\lambda>0$.*

Proof. Using (3) we obtain the estimate

$$J_\lambda(u) = \frac{1}{p}\|u\|^p_{a,b} - \int_{\mathbf{R}^N} F(\lambda,x,u)\,dx + \frac{1}{m}\int_{\mathbf{R}^N} h|u|^m\,dx$$
$$\ge \frac{1}{p}\|u\|^p_{a,b} - \int_{\mathbf{R}^N}\left(C\lambda|u|^q - \frac{h}{2m}|u|^m\right)dx + \frac{1}{2m}\int_{\mathbf{R}^N} h|u|^m\,dx.$$

From (8) and hypothesis (**H**) we derive

$$\int_{\mathbf{R}^N}\left(C\lambda|u|^q - \frac{h}{2m}|u|^m\right)dx \le C'\lambda^{m/(m-q)}\int_{\mathbf{R}^N}\frac{dx}{h^{q/(m-q)}} \le C''.$$

It follows that

$$J_\lambda(u) \ge \frac{1}{p}\|u\|^p_{a,b} + \frac{1}{2m}\int_{\mathbf{R}^N} h|u|^m\,dx - C'',$$

which shows that J_λ is coercive. ■

Lemma 2 *Let $\lambda > 0$ be arbitrary and $\{u_n\}$ be a sequence in E such that $J_\lambda(u_n)$ is bounded. Then there exists a subsequence of $\{u_n\}$, denoted again by $\{u_n\}$, such that*

$$u_n \rightharpoonup u_0 \text{ in } E, \quad u_n \to u_0 \text{ a.e. in } \mathbf{R}^N \quad \text{and} \quad J_\lambda(u_0) \le \liminf_{n\to\infty} J_\lambda(u_n).$$

Proof. In virtue of Lemma 1, the boundedness of $J_\lambda(u_n)$ implies that $\{u_n\}$ is bounded in E. By (1) and Remark 4, we may assume (passing eventually to subsequences) that

$$u_n \rightharpoonup u_0 \text{ in } E, \quad u_n \to u_0 \text{ in } L^q_{\mathrm{loc}}(\mathbf{R}^N) \quad \text{and} \quad u_n \to u_0 \text{ a.e. in } \mathbf{R}^N.$$

Set

$$\Xi(x,u) = F(\lambda,x,u) - \frac{1}{m}h|u|^m \quad \text{and} \quad \xi(x,u) = \Xi_u(x,u).$$

From (8) and hypothesis $(\mathbf{F_2})$ we infer that

$$\xi_u(x,u) = f_u(\lambda,x,u) - (m-1)h|u|^{m-2} \le \lambda c_f|u|^{q-2} - (m-1)h|u|^{m-2} \le C_{m,q}\lambda\left(\frac{\lambda}{h}\right)^{(q-2)/(m-q)}.$$

It turns out that

$$\begin{aligned}\int_{\mathbf{R}^N}(\Xi(x,u_n)-\Xi(x,u_0))\,dx &= \int_{\mathbf{R}^N}\left(\int_0^1\int_0^z \xi_u(x,u_0+t(u_n-u_0))\,dt\,dz\right)(u_n-u_0)^2\,dx \\ &\le C'_{m,q}\int_{\mathbf{R}^N}\frac{(u_n-u_0)^2}{h^{(q-2)/(m-q)}}\,dx.\end{aligned}$$

Therefore, we obtain the following estimate for $J_\lambda(u_0) - J_\lambda(u_n)$

$$\begin{aligned}J_\lambda(u_0)-J_\lambda(u_n) &= \frac{1}{p}\left(\|u_0\|^p_{a,b} - \|u_n\|^p_{a,b}\right) + \int_{\mathbf{R}^N}(\Xi(x,u_n)-\Xi(x,u_0))\,dx \\ &\le \frac{1}{p}\left(\|u_0\|^p_{a,b} - \|u_n\|^p_{a,b}\right) + C'_{m,q}\int_{\mathbf{R}^N}\frac{(u_n-u_0)^2}{h^{(q-2)/(m-q)}}\,dx.\end{aligned} \tag{9}$$

Since assumption $(\mathbf{H})$ holds, for every $\varepsilon > 0$ we can choose $R = R(\varepsilon) > 0$ such that

$$\int_{|x|>R}\frac{dx}{h^{q/(m-q)}} < \varepsilon^{q/(q-2)}.$$

By Hölder's inequality we find the estimate

$$\begin{aligned}\int_{\mathbf{R}^N}\frac{(u_n-u_0)^2}{h^{(q-2)/(m-q)}}\,dx \le &\left(\int_{|x|\le R}\frac{dx}{h^{q/(m-q)}}\right)^{(q-2)/q}\left(\int_{|x|\le R}|u_n-u_0|^q\,dx\right)^{2/q} \\ &+\left(\int_{|x|>R}\frac{dx}{h^{q/(m-q)}}\right)^{(q-2)/q}\left(\int_{|x|>R}|u_n-u_0|^q\,dx\right)^{2/q}.\end{aligned}$$

From Proposition 1, $\{u_n\}$ is bounded in $L^q(\mathbf{R}^N)$. Since $u_n \to u_0$ in $L^q_{\mathrm{loc}}(\mathbf{R}^N)$ it follows that

$$\limsup_{n\to\infty}\int_{\mathbf{R}^N}\frac{(u_n-u_0)^2}{h^{(q-2)/(m-q)}}\,dx \le C_0\,\varepsilon$$

for some constant $C_0 > 0$ which is independent of n and ε. Since $\varepsilon > 0$ is arbitrary we obtain

$$\lim_{n\to\infty}\int_{\mathbf{R}^N}\frac{(u_n-u_0)^2}{h^{(q-2)/(m-q)}}\,dx = 0. \tag{10}$$

The lower semicontinuity of $\|\cdot\|_{a,b}$ with respect to the weak topology, (9) and (10) finish the proof. ■

3 Preliminary results

Proposition 2 *There exists $\lambda^* > 0$ such that for $0 < \lambda < \lambda^*$ problem (P_λ) has no solution.*

Proof. Let $\lambda > 0$ be chosen such that (P_λ) has at least a solution, say u_λ. Then using (2) we find

$$\|u_\lambda\|_{a,b}^p + \int_{\mathbf{R}^N} h(x)|u_\lambda|^m\,dx = \int_{\mathbf{R}^N} f(\lambda,x,u_\lambda)u_\lambda\,dx \le c\lambda\int_{\mathbf{R}^N}|u_\lambda|^q\,dx. \tag{11}$$

Now, the Young inequality implies

$$c\lambda\int_{\mathbf{R}^N}|u_\lambda|^q\,dx = \int_{\mathbf{R}^N}\frac{c\lambda}{h^{q/m}}h^{q/m}|u_\lambda|^q\,dx \le \frac{m-q}{m}(c\lambda)^{m/(m-q)}\int_{\mathbf{R}^N}\frac{dx}{h^{q/(m-q)}} + \frac{q}{m}\int_{\mathbf{R}^N}h|u_\lambda|^m\,dx.$$

Combining this inequality with (11) we obtain

$$\begin{aligned}\|u_\lambda\|_{a,b}^p &\le \frac{m-q}{m}(c\lambda)^{m/(m-q)}\int_{\mathbf{R}^N}\frac{dx}{h^{q/(m-q)}} + \frac{q-m}{m}\int_{\mathbf{R}^N}h|u_\lambda|^m\,dx\\ &\le \frac{m-q}{m}(c\lambda)^{m/(m-q)}\int_{\mathbf{R}^N}\frac{dx}{h^{q/(m-q)}}.\end{aligned}\tag{12}$$

By (11) and Proposition 1 we have

$$c_1\left(\int_{\mathbf{R}^N}|u_\lambda|^q\,dx\right)^{p/q} \le \|u_\lambda\|_{a,b}^p \le c\lambda\int_{\mathbf{R}^N}|u_\lambda|^q\,dx \tag{13}$$

where $c_1 > 0$ is a positive constant. It follows that

$$(c_1c^{-1}\lambda^{-1})^{q/(q-p)} \le \int_{\mathbf{R}^N}|u_\lambda|^q\,dx.$$

This combined with (13) gives

$$c_1(c_1c^{-1}\lambda^{-1})^{p/(q-p)} \le \|u_\lambda\|_{a,b}^p. \tag{14}$$

From (12) and (14) we derive

$$c_1(c_1c^{-1}\lambda^{-1})^{p/(q-p)} \le \frac{m-q}{m}(c\lambda)^{m/(m-q)}\int_{\mathbf{R}^N}\frac{dx}{h^{q/(m-q)}}, \tag{15}$$

which yields

$$(c\lambda)^{q(m-p)/(m-q)(q-p)} \ge \frac{m}{m-q}c_1^{q/(q-p)}\left(\int_{\mathbf{R}^N}\frac{dx}{h^{q/(m-q)}}\right)^{-1}.$$

We observe that our claim follows if we take

$$\lambda^* = c^{-1}\left[\frac{m}{m-q}c_1^{q/(q-p)}\left(\int_{\mathbf{R}^N}\frac{dx}{h^{q/(m-q)}}\right)^{-1}\right]^{(m-q)(q-p)/q(m-p)}.$$

Corollary 1 *Suppose $\lambda > 0$ such that (P_λ) has a solution u_λ. Then the following hold:*

i) $\|u_\lambda\|_{a,b}^p + \frac{m-q}{m}\int_{\mathbf{R}^N}h|u_\lambda|^m\,dx \le \frac{m-q}{m}(c\lambda)^{m/(m-q)}\int_{\mathbf{R}^N}\frac{dx}{h^{q/(m-q)}}$.

ii) There exists a positive constant $K > 0$ independent of u_λ such that

$$\|u_\lambda\|_{a,b} \ge K\lambda^{-1/(q-p)}.$$

Proof. The first assertion is already proved by estimate (12). The second part is implied by (14) which shows that the constant K can be chosen for example as $(c_1^{q/p}c^{-1})^{1/(q-p)}$. ■

Proposition 3 *Suppose* $\lambda_n \searrow \lambda_0 > 0$ *such that problem* (P_{λ_n}) *has a solution* u_n *for each* n. *Then* $\{u_n\}$ *converges weakly (up to a subsequence) in* E *to some* $u_0 \geq 0$ *which is a critical point of* J_{λ_0}.

Proof. By Corollary 1, $\{u_n\}$ is bounded in E. By (1) and Remark 4, we may assume (passing eventually to subsequences) that

$$u_n \rightharpoonup u_0 \text{ in } E,\ u_n \rightharpoonup u_0 \text{ in } L^m(\mathbf{R}^N;\, h),\ u_n \rightharpoonup u_0 \text{ in } W^{1,p}_{a,b}(\mathbf{R}^N),\ u_n \rightharpoonup u_0 \text{ in } L^p(\mathbf{R}^N;\, b). \tag{16}$$

$$u_n \to u_0 \text{ in } L^k_{\text{loc}}(\mathbf{R}^N),\ \forall p \leq k \leq q \quad \text{and} \quad u_n \to u_0 \text{ a.e. in } \mathbf{R}^N. \tag{17}$$

By (17) we derive that $u_0 \geq 0$ in $\mathbf{R}^N$. It remains to prove that u_0 is a critical point of J_{λ_0}.

Let $v \in E$ be arbitrary. Since u_n is a critical point of J_{λ_n} for each n, we have

$$\int_{\mathbf{R}^N} \left(a|\nabla u_n|^{p-2}\nabla u_n \cdot \nabla v + b|u_n|^{p-2}u_n v\right)\, dx + \int_{\mathbf{R}^N} h|u_n|^{m-2}u_n v\, dx = \int_{\mathbf{R}^N} f(\lambda_n, x, u_n)v\, dx\,. \tag{18}$$

By (16) we find that $\{|u_n|^{m-2}u_n\}$ is a bounded sequence in $L^{m/(m-1)}(\mathbf{R}^N;\, h)$, while by (17) we get $|u_n|^{m-2}u_n \to |u_0|^{m-2}u_0$ a.e. in $\mathbf{R}^N$. Combining these facts we obtain

$$|u_n|^{m-2}u_n \rightharpoonup |u_0|^{m-2}u_0 \text{ in } L^{m/(m-1)}(\mathbf{R}^N;\, h), \tag{19}$$

which implies

$$\lim_{n\to\infty} \int_{\mathbf{R}^N} h|u_n|^{m-2}u_n v\, dx = \int_{\mathbf{R}^N} h|u_0|^{m-2}u_0 v\, dx. \tag{20}$$

Similarly

$$\lim_{n\to\infty} \int_{\mathbf{R}^N} b|u_n|^{p-2}u_n v\, dx = \int_{\mathbf{R}^N} b|u_0|^{p-2}u_0 v\, dx. \tag{21}$$

Since $\{u_n\}$ is bounded in $L^q(\mathbf{R}^N)$ by using (2) and (17) we find

$$\{f(\lambda_0, x, u_n)\} \text{ is bounded in } L^{q/(q-1)}(\mathbf{R}^N) \quad \text{and} \quad f(\lambda_0, x, u_n) \to f(\lambda_0, x, u_0) \text{ a.e. in } \mathbf{R}^N.$$

It follows that

$$f(\lambda_0, x, u_n) \rightharpoonup f(\lambda_0, x, u_0) \text{ in } L^{q/(q-1)}(\mathbf{R}^N)$$

which produces

$$\lim_{n\to\infty} \int_{\mathbf{R}^N} f(\lambda_0, x, u_n)v\, dx = \int_{\mathbf{R}^N} f(\lambda_0, x, u_0)v\, dx. \tag{22}$$

On the other hand, from hypothesis $(\mathbf{F_4})$ and Hölder's inequality we have

$$\begin{aligned} \int_{\mathbf{R}^N} |(f(\lambda_n, x, u_n - f(\lambda_0, x, u_n))v|\, dx &\leq C_f|\lambda_n - \lambda_0| \int_{\mathbf{R}^N} |u_n|^{q-1}|v|\, dx \\ &\leq C_f|\lambda_n - \lambda_0| \left(\int_{\mathbf{R}^N} |u_n|^q\, dx\right)^{(q-1)/q} \left(\int_{\mathbf{R}^N} |v|^q\, dx\right)^{1/q}. \end{aligned} \tag{23}$$

Taking into account (22), (23) and the fact that

$$\left|\int_{\mathbf{R}^N} (f(\lambda_n,x,u_n)-f(\lambda_0,x,u_0))\,v\,dx\right| \le \left|\int_{\mathbf{R}^N} (f(\lambda_n,x,u_n)-f(\lambda_0,x,u_n))\,v\,dx\right| + \left|\int_{\mathbf{R}^N} (f(\lambda_0,x,u_n)-f(\lambda_0,x,u_0))\,v\,dx\right|$$

we have so proved that

$$\lim_{n\to\infty}\int_{\mathbf{R}^N} f(\lambda_n,x,u_n)v\,dx = \int_{\mathbf{R}^N} f(\lambda_0,x,u_0)v\,dx. \tag{24}$$

The next step is to show that

$$a^{1/p}\nabla u_n \to a^{1/p}\nabla u_0 \quad \text{a.e. in } \mathbf{R}^N. \tag{25}$$

To this end, we need only to show

$$a^{1/p}\nabla u_n \to a^{1/p}\nabla u_0 \quad \text{a.e. in } B(0,R) \text{ for any } R>0.$$

For this purpose we use the following inequalities (see Diaz [8, Lemma 4.10]) that hold for any $\xi,\zeta\in\mathbf{R}^N$

$$|\xi-\zeta|^p \le C(|\xi|^{p-2}\xi-|\zeta|^{p-2}\zeta)(\xi-\zeta), \qquad \text{for } p\ge 2\,; \tag{26}$$

$$|\xi-\zeta|^2 \le C(|\xi|^{p-2}\xi-|\zeta|^{p-2}\zeta)(\xi-\zeta)(|\xi|+|\zeta|)^{2-p}, \qquad \text{for } 1<p<2\,. \tag{27}$$

Therefore, it is sufficient to prove that

$$a(|\nabla u_n|^{p-2}\nabla u_n-|\nabla u_0|^{p-2}\nabla u_0)\cdot(\nabla u_n-\nabla u_0)\to 0 \quad \text{a.e. in } B(0,R) \text{ for any } R>0. \tag{28}$$

For a fixed $R>0$, choose $\vartheta\in C_0^\infty(\mathbf{R}^N)$ with $0\le\vartheta\le 1$ in $\mathbf{R}^N$, $\vartheta\equiv 1$ on $B(0,R)$ and $\vartheta\equiv 0$ on $\mathbf{R}^N\setminus B(0,2R)$. Then by (16) and (17) we have that

$$\vartheta u_n \rightharpoonup \vartheta u_0 \text{ in } W^{1,p}_{a,b}(\mathbf{R}^N) \quad \text{and} \quad \vartheta u_n \rightharpoonup \vartheta u_0 \text{ in } L^m(\mathbf{R}^N;\,h)$$

which yield

$$\int_{\mathbf{R}^N}\Big(a|\nabla u_0|^{p-2}\nabla u_0\cdot\nabla(\vartheta u_n-\vartheta u_0)+b\vartheta|u_0|^{p-2}u_0(u_n-u_0)\Big)\,dx\to 0 \tag{29}$$

and

$$I_{0,n}:=\int_{\mathbf{R}^N} h\vartheta|u_0|^{m-2}u_0(u_0-u_n)\,dx\to 0. \tag{30}$$

By Hölder's inequality we have the estimate

$$\left|\int_{\mathbf{R}^N} a(u_n-u_0)|\nabla u_0|^{p-2}\nabla u_0\cdot\nabla\vartheta\,dx\right| \le C_1\left(\int_{\operatorname{Supp}\vartheta} a|\nabla u_0|^p\,dx\right)^{(p-1)/p}\left(\int_{\operatorname{Supp}\vartheta}|u_n-u_0|^p\,dx\right)^{1/p}$$

which combined with (17) gives

$$\lim_{n\to\infty}\int_{\mathbf{R}^N} a(u_n-u_0)|\nabla u_0|^{p-2}\nabla u_0\cdot\nabla\vartheta\,dx=0.$$

Using this fact in (29) we obtain

$$\int_{\mathbf{R}^N} \Big(a\vartheta|\nabla u_0|^{p-2}\nabla u_0 \cdot \nabla(u_n - u_0) + b\vartheta|u_0|^{p-2}u_0(u_n - u_0)\Big)\, dx \to 0. \tag{31}$$

On the other hand, since $\langle J'_{\lambda_n}(u_n), \vartheta(u_n - u_0)\rangle = 0$ we have

$$\begin{aligned}&\int_{\mathbf{R}^N} \Big(a\vartheta|\nabla u_0|^{p-2}\nabla u_0 \cdot \nabla(u_n - u_0) + b\vartheta|u_0|^{p-2}u_0(u_n - u_0)\Big)\, dx\\ &\le \int_{\mathbf{R}^N} \Big(a\vartheta|\nabla u_n|^{p-2}\nabla u_n \cdot \nabla(u_n - u_0) + b\vartheta|u_n|^{p-2}u_n(u_n - u_0)\Big)\, dx := I_{1,n} + I_{2,n} + I_{3,n},\end{aligned} \tag{32}$$

where

$$\begin{aligned}I_{1,n} &:= \int_{\mathbf{R}^N} h\vartheta|u_n|^{m-2}u_n(u_0 - u_n)\, dx, \qquad I_{2,n} := \int_{\mathbf{R}^N} f(\lambda_n, x, u_n)\vartheta(u_n - u_0)\, dx\\ I_{3,n} &:= \int_{\mathbf{R}^N} a(u_0 - u_n)|\nabla u_n|^{p-2}\nabla u_n \cdot \nabla\vartheta\, dx.\end{aligned}$$

By Hölder's inequality, (2), (16) and (17) we derive

$$\begin{aligned}|I_{2,n}| &\le c\sup_{n\ge 1}\lambda_n \left(\int_{\mathrm{Supp}\,\vartheta} |u_n|^q\, dx\right)^{(q-1)/q} \left(\int_{\mathrm{Supp}\,\vartheta} |u_n - u_0|^q\, dx\right)^{1/q} \to 0\\ |I_{3,n}| &\le C_1 \left(\int_{\mathrm{Supp}\,\vartheta} a|\nabla u_n|^p\, dx\right)^{(p-1)/p} \left(\int_{\mathrm{Supp}\,\vartheta} |u_n - u_0|^p\, dx\right)^{1/p} \to 0.\end{aligned} \tag{33}$$

We now see that $I_{1,n} \le I_{0,n}$. Thus, by using relations (31), (33) and (30) in (32) we conclude that

$$\int_{\mathbf{R}^N} \Big(a\vartheta|\nabla u_n|^{p-2}\nabla u_n \cdot \nabla(u_n - u_0) + b\vartheta|u_n|^{p-2}u_n(u_n - u_0)\Big)\, dx \to 0. \tag{34}$$

Since

$$\begin{aligned}0 &\le \int_{\mathbf{R}^N} a\vartheta(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u_0|^{p-2}\nabla u_0)\cdot\nabla(u_n - u_0)\, dx\\ &\le \int_{\mathbf{R}^N} \Big(a\vartheta(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u_0|^{p-2}\nabla u_0)\cdot\nabla(u_n - u_0) + b\vartheta(|u_n|^{p-2}u_n - |u_0|^{p-2}u_0)(u_n - u_0)\Big)\, dx\end{aligned}$$

we deduce by (31) and (34) that

$$\lim_{n\to\infty} \int_{B(0,R)} a(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u_0|^{p-2}\nabla u_0)\cdot(\nabla u_n - \nabla u_0)\, dx = 0.$$

Hence (28) holds. Thus, the claim (25) is proved. It follows that

$$a^{(p-1)/p}|\nabla u_n|^{p-2}\frac{\partial u_n}{\partial x_i} \to a^{(p-1)/p}|\nabla u_0|^{p-2}\frac{\partial u_0}{\partial x_i} \quad \text{a.e. in } \mathbf{R}^N.$$

This and the fact that $\{a^{(p-1)/p}|\nabla u_n|^{p-2}\frac{\partial u_n}{\partial x_i}\}$ is bounded in $L^{p/(p-1)}(\mathbf{R}^N)$ imply

$$a^{(p-1)/p}|\nabla u_n|^{p-2}\frac{\partial u_n}{\partial x_i} \rightharpoonup a^{(p-1)/p}|\nabla u_0|^{p-2}\frac{\partial u_0}{\partial x_i} \quad \text{in } L^{p/(p-1)}(\mathbf{R}^N). \tag{35}$$

We define

$$\Phi(w)=\int_{\mathbf{R}^N} a^{1/p}\frac{\partial v}{\partial x_i} w\,dx \quad \text{for all } w\in L^{p/(p-1)}(\mathbf{R}^N).$$

Since $a^{1/p}\frac{\partial v}{\partial x_i}\in L^p(\mathbf{R}^N)$, Φ is well defined. Moreover, $\Phi\in\left(L^{p/(p-1)}(\mathbf{R}^N)\right)^*$. Therefore, in view of (35)

$$\Phi\left(a^{(p-1)/p}|\nabla u_n|^{p-2}\frac{\partial u_n}{\partial x_i}\right)\to\Phi\left(a^{(p-1)/p}|\nabla u_0|^{p-2}\frac{\partial u_0}{\partial x_i}\right)$$

or, in other words,

$$\lim_{n\to\infty}\int_{\mathbf{R}^N} a|\nabla u_n|^{p-2}\frac{\partial u_n}{\partial x_i}\frac{\partial v}{\partial x_i}\,dx=\int_{\mathbf{R}^N} a|\nabla u_0|^{p-2}\frac{\partial u_0}{\partial x_i}\frac{\partial v}{\partial x_i}\,dx.$$

This gives

$$\lim_{n\to\infty}\int_{\mathbf{R}^N} a|\nabla u_n|^{p-2}\nabla u_n\cdot\nabla v\,dx=\int_{\mathbf{R}^N} a|\nabla u_0|^{p-2}\nabla u_0\cdot\nabla v\,dx. \tag{36}$$

By (18), (20), (21), (24) and (36) we conclude that u_0 is a critical point of J_{λ_0}. ■

4 Proof of the main result

Let $\lambda>0$ be arbitrary. From Lemma 1 we see that $m_\lambda:=\inf_{u\in E} J_\lambda(u)$ is finite. Let $\{u_n\}$ be a sequence such that $\lim_{n\to\infty}J_\lambda(u_n)=m_\lambda$. According to Lemma 2, we can assume (up to a subsequence) that

$$u_n\rightharpoonup u_0 \text{ in } E \quad\text{and}\quad J_\lambda(u_0)\le\liminf_{n\to\infty}J_\lambda(u_n)=m_\lambda.$$

This shows that $\inf_{u\in E}J_\lambda(u)$ is attained in u_0. From $(\mathbf{F_1})$ we deduce that $F(\lambda,x,|u_0|)\ge F(\lambda,x,u_0)$ a.e. $x\in\mathbf{R}^N$. It follows that $J_\lambda(|u_0|)\le J_\lambda(u_0)$. Therefore, we may assume that $u_0\ge 0$ in $\mathbf{R}^N$. To ensure that $u_0\not\equiv 0$ we shall prove that m_λ is negative, provided that $\lambda>\tilde\lambda$ for some $\tilde\lambda>0$.

By hypothesis $(\mathbf{F_3})$ we deduce that there exists $\delta>0$ independent of x and λ such that

$$F(\lambda,x,u(x))\ge\frac{\lambda}{2q}|u(x)|^q \quad \text{a.e. } x\in\mathbf{R}^N,\ \forall u\in E \text{ with } \sup_{x\in\mathbf{R}^N}|u(x)|\le\delta. \tag{37}$$

Set $\zeta>0$ with the property that

$$Y=\left\{u\in E\setminus\{0\}:\ \sup_{x\in\mathbf{R}^N}|u(x)|\le\zeta\|u\|_q\right\}\neq\emptyset$$

and denote $\eta=\left(\frac{\delta}{\zeta}\right)^q$. Define

$$\tilde\lambda:=\inf\left\{\frac{2q}{\eta p}\|u\|_{a,b}^p+\frac{2q}{\eta m}\int_{\mathbf{R}^N}h|u|^m\,dx:\ u\in Z\right\},$$

where

$$Z=\left\{u\in E:\ \sup_{x\in\mathbf{R}^N}|u(x)|\le\delta,\ \int_{\mathbf{R}^N}|u|^q\,dx=\eta\right\}.$$

It is easy to check that $Z \neq \emptyset$. Indeed, if $y \in Y$ then $u = \dfrac{\eta^{1/q}}{\|y\|_q}\, y \in Z$.

We now claim that $\tilde{\lambda} > 0$. For this aim, we consider the constrained minimization problem

$$M := \inf \left\{ \|u\|^p_{a,b} :\ u \in W^{1,p}_{a,b}(\mathbf{R}^N), \int_{\mathbf{R}^N} |u|^q\, dx = \eta \right\}.$$

Since the embedding $W^{1,p}_{a,b}(\mathbf{R}^N) \hookrightarrow L^q(\mathbf{R}^N)$ is continuous, it follows that $M > 0$. Thus

$$\|u\|^p_{a,b} \geq M \qquad \text{for all } u \in E \text{ with } \int_{\mathbf{R}^N} |u|^q\, dx = \eta.$$

By applying the Hölder inequality we find

$$\int_{\mathbf{R}^N} |u|^q\, dx = \int_{\mathbf{R}^N} \frac{1}{h^{q/m}} h^{q/m} |u|^q\, dx \leq \left(\int_{\mathbf{R}^N} \frac{dx}{h^{q/(m-q)}} \right)^{(m-q)/m} \left(\int_{\mathbf{R}^N} h|u|^m\, dx \right)^{q/m}. \tag{38}$$

Therefore, we have the estimate

$$\frac{2q}{\eta p} \|u\|^p_{a,b} + \frac{2q}{\eta m} \int_{\mathbf{R}^N} h|u|^m\, dx \geq \frac{2qM}{\eta p} + \frac{2q}{\eta m} \eta^{m/q} \left(\int_{\mathbf{R}^N} \frac{dx}{h^{q/(m-q)}} \right)^{-(m-q)/q}$$

for all $u \in E$ with $\int_{\mathbf{R}^N} |u|^q\, dx = \eta$. It follows that

$$\tilde{\lambda} \geq \frac{2qM}{\eta p} + \frac{2q}{m} \eta^{(m-q)/q} \left(\int_{\mathbf{R}^N} \frac{dx}{h^{q/(m-q)}} \right)^{-(m-q)/q}$$

and our claim follows.

Let $\lambda > \tilde{\lambda}$. Then there exists a function $u_1 \in Z$ such that

$$\lambda > \frac{2q}{\eta p} \|u_1\|^p_{a,b} + \frac{2q}{\eta m} \int_{\mathbf{R}^N} h|u_1|^m\, dx.$$

This inequality and (37) imply

$$\begin{aligned} J_\lambda(u_1) &= \frac{1}{p} \|u_1\|^p_{a,b} + \frac{1}{m} \int_{\mathbf{R}^N} h|u_1|^m\, dx - \int_{\mathbf{R}^N} F(\lambda, x, u_1(x))\, dx \\ &\leq \frac{1}{p} \|u_1\|^p_{a,b} + \frac{1}{m} \int_{\mathbf{R}^N} h|u_1|^m\, dx - \frac{\lambda}{2q} \int_{\mathbf{R}^N} |u_1|^q\, dx < 0. \end{aligned}$$

Consequently, $\inf_{u \in E} J_\lambda(u) < 0$. Thus, the problem (P_λ) has a solution if $\lambda > \tilde{\lambda}$.

We define

$$\lambda_0 = \inf \{ \lambda > 0 : (P_\lambda) \text{ admits a solution} \}.$$

By Proposition 2, $\lambda_0 \geq \lambda^* > 0$.

We now show that for each $\lambda > \lambda_0$ problem (P_λ) admits a solution. Indeed, for every $\lambda > \lambda_0$ there exists $\rho \in (\lambda_0, \lambda)$ such that problem (P_ρ) has a solution u_ρ which is a subsolution of problem (P_λ). We consider the variational problem

$$\inf \{ J_\lambda(u) : u \in E \text{ and } u \geq u_\rho \}.$$

By Lemmas 1 and 2, this problem admits a solution $\bar{u}$. This minimizer $\bar{u}$ is a solution of problem (P_λ). It remains to show that problem (P_{λ_0}) has also a solution. Let $\lambda_n \to \lambda_0$ and $\lambda_n > \lambda_0$ for each n. Problem (P_{λ_n}) has a solution u_n for each n. Then, in virtue of Proposition 3, we may assume (up to a subsequence) that u_n converges weakly to some u_0 in E, which is a non-negative critical point of J_{λ_0}. Moreover,

$$u_n \rightharpoonup u_0 \text{ in } L^m(\mathbf{R}^N;h), \quad u_n \to u_0 \text{ in } L^q_{\text{loc}}(\mathbf{R}^N) \quad \text{and} \quad \{u_n\} \text{ is bounded in } L^q(\mathbf{R}^N) \tag{39}$$

To conclude that u_0 is a solution of problem (P_{λ_0}) it remains only to prove that $u_0 \not\equiv 0$.

Using $(\mathbf{F_4})$ we get the estimate

$$\left|\int_{\mathbf{R}^N} (f(\lambda_n,x,u_0) - f(\lambda_0,x,u_0))(u_n-u_0)\,dx\right| \le C_f(\lambda_n-\lambda_0)\int_{\mathbf{R}^N}|u_0|^{q-1}|u_n-u_0|\,dx.$$

Since $\{u_n\}$ is bounded in $L^q(\mathbf{R}^N)$, it follows from the Hölder inequality that

$$\begin{aligned}\int_{\mathbf{R}^N}|u_0|^{q-1}|u_n-u_0|\,dx &\le \int_{\mathbf{R}^N}|u_0|^{q-1}|u_n|\,dx + \int_{\mathbf{R}^N}|u_0|^q\,dx\\ &\le \left(\int_{\mathbf{R}^N}|u_0|^q\,dx\right)^{(q-1)/q}\left(\int_{\mathbf{R}^N}|u_n|^q\,dx\right)^{1/q} + \int_{\mathbf{R}^N}|u_0|^q\,dx < k\end{aligned}$$

for some constant $k>0$ which is independent of n. Therefore,

$$\lim_{n\to\infty}(\lambda_n-\lambda_0)\int_{\mathbf{R}^N}|u_0|^{q-1}|u_n-u_0|\,dx = 0,$$

which implies

$$\lim_{n\to\infty}\int_{\mathbf{R}^N}(f(\lambda_n,x,u_0)-f(\lambda_0,x,u_0))(u_n-u_0)\,dx = 0. \tag{40}$$

Let $\varepsilon>0$ be given. In view of $(\mathbf{H})$ we can take $R=R(\varepsilon)>0$ large enough such that

$$\int_{|x|>R}\frac{dx}{h^{q/(m-q)}} < \varepsilon^{m/(m-q)} \quad \text{and} \quad \int_{|x|>R}|u_0|^q\,dx < \varepsilon^q.$$

From (2) and the Hölder inequality we deduce

$$\begin{aligned}&\int_{\mathbf{R}^N}|(f(\lambda_n,x,u_n)-f(\lambda_n,x,u_0))(u_n-u_0)|\,dx \le c\lambda_n\int_{\mathbf{R}^N}(|u_n|^{q-1}+|u_0|^{q-1})|u_n-u_0|\,dx\\ &\le c\lambda_n\left[\int_{|x|\le R}|u_n|^{q-1}|u_n-u_0|\,dx + \int_{|x|\le R}|u_0|^{q-1}|u_n-u_0|\,dx + \int_{|x|>R}|u_0|^{q-1}|u_n-u_0|\,dx\right.\\ &\left.+\int_{|x|>R}|u_n|^q\,dx + \int_{|x|>R}|u_n|^{q-1}|u_0|dx\right] \le c\lambda_n\left\{\left(\int_{|x|>R}|u_0|^q\,dx\right)^{(q-1)/q}\left(\int_{|x|>R}|u_n-u_0|^q dx\right)^{1/q}\right.\\ &+\left(\int_{|x|>R}\frac{dx}{h^{q/(m-q)}}\right)^{(m-q)/m}\left(\int_{|x|>R}h|u_n|^m dx\right)^{q/m} + \left(\int_{|x|>R}|u_n|^q dx\right)^{(q-1)/q}\left(\int_{|x|>R}|u_0|^q dx\right)^{1/q}\\ &\left.+\left(\int_{|x|\le R}|u_n-u_0|^q\,dx\right)^{1/q}\left[\left(\int_{|x|\le R}|u_0|^q\,dx\right)^{(q-1)/q} + \left(\int_{|x|\le R}|u_n|^q\,dx\right)^{(q-1)/q}\right]\right\}.\end{aligned}$$

Thus, we find

$$\limsup_{n\to\infty}\int_{\mathbf{R}^N}|(f(\lambda_n,x,u_n)-f(\lambda_n,x,u_0))(u_n-u_0)|\,dx\le k_1\varepsilon$$

for some constant $k_1>0$ independent of n and ε. Since $\varepsilon>0$ is arbitrary it follows that

$$\lim_{n\to\infty}\int_{\mathbf{R}^N}(f(\lambda_n,x,u_n)-f(\lambda_n,x,u_0))(u_n-u_0)\,dx=0. \tag{41}$$

From (40) and (41) we obtain

$$\lim_{n\to\infty}\int_{\mathbf{R}^N}(f(\lambda_n,x,u_n)-f(\lambda_0,x,u_0))(u_n-u_0)\,dx=0.$$

In what follows, we denote

$$A_n=\int_{\mathbf{R}^N}\Big(a(|\nabla u_n|^{p-2}\nabla u_n-|\nabla u_0|^{p-2}\nabla u_0)\cdot(\nabla u_n-\nabla u_0)+b(|u_n|^{p-2}u_n-|u_0|^{p-2}u_0)(u_n-u_0)\Big)\,dx.$$

We see that $A_n\ge 0$ for all n. Since u_n and u_0 are critical points of J_{λ_n} and J_{λ_0}, respectively, we have

$$\begin{aligned} A_n &= \int_{\mathbf{R}^N}(f(\lambda_n,x,u_n)-f(\lambda_0,x,u_0))(u_n-u_0)\,dx-\int_{\mathbf{R}^N}h(|u_n|^{m-2}u_n-|u_0|^{m-2}u_0)(u_n-u_0)\,dx\\ &\le \int_{\mathbf{R}^N}(f(\lambda_n,x,u_n)-f(\lambda_0,x,u_0))(u_n-u_0)\,dx\to 0. \end{aligned}$$

It follows that $\lim_{n\to\infty}A_n=0$. Using this fact we show that $\|u_n-u_0\|_{a,b}\to 0$ as $n\to\infty$. We distinguish two cases which may occur

CASE 1: $p\ge 2$. Using (26) we obtain $\|u_n-u_0\|_{a,b}^p\le C_1A_n\to 0$ as $n\to\infty$.

CASE 2: $1<p<2$. We observe that it is enough to show that

$$\|u_n-u_0\|_{a,b}^2\le C_2A_n(\|u_n\|_{a,b}^{2-p}+\|u_0\|_{a,b}^{2-p}). \tag{42}$$

In order to prove (42) we recall the following result: for all $t>0$ there is a constant $C_t>0$ such that

$$(x+y)^t\le C_t(x^t+y^t)\qquad \text{for any } x,y\in(0,\infty). \tag{43}$$

Then we obtain

$$\begin{aligned} \|u_n-u_0\|_{a,b}^2= &\left(\int_{\mathbf{R}^N}(a(x)|\nabla u_n-\nabla u_0|^p+b(x)|u_n-u_0|^p)\,dx\right)^{2/p}\le\\ &C_p\left[\left(\int_{\mathbf{R}^N}a(x)|\nabla u_n-\nabla u_0|^p\,dx\right)^{2/p}+\left(\int_{\mathbf{R}^N}b(x)|u_n-u_0|^p\,dx\right)^{2/p}\right]. \end{aligned} \tag{44}$$

Using (27), (43) and the Hölder inequality we find

$$\begin{aligned} &\int_{\mathbf{R}^N}a|\nabla u_n-\nabla u_0|^p\,dx=\int_{\mathbf{R}^N}a(|\nabla u_n-\nabla u_0|^2)^{p/2}\,dx\\ &\le c_1\int_{\mathbf{R}^N}\Big(a(|\nabla u_n|^{p-2}\nabla u_n-|\nabla u_0|^{p-2}\nabla u_0)(\nabla u_n-Du_0)\Big)^{p/2}(a(|\nabla u_n|+|\nabla u_0|)^p)^{(2-p)/2}\,dx\\ &\le c_1\left(\int_{\mathbf{R}^N}a(|\nabla u_n|^{p-2}\nabla u_n-|\nabla u_0|^{p-2}\nabla u_0)(\nabla u_n-\nabla u_0)dx\right)^{p/2}\left(\int_{\mathbf{R}^N}a(|\nabla u_n|+|\nabla u_0|)^pdx\right)^{(2-p)/2}\\ &\le c_2\left(\int_{\mathbf{R}^N}a(|\nabla u_n|^{p-2}\nabla u_n-|\nabla u_0|^{p-2}\nabla u_0)(\nabla u_n-\nabla u_0)\,dx\right)^{p/2}\left(\|u_n\|_{a,b}^p+\|u_0\|_{a,b}^p\right)^{(2-p)/2}\\ &\le c_3\left(\int_{\mathbf{R}^N}a(|\nabla u_n|^{p-2}\nabla u_n-|\nabla u_0|^{p-2}\nabla u_0)(\nabla u_n-\nabla u_0)\,dx\right)^{p/2}\left(\|u_n\|_{a,b}^{(2-p)p/2}+\|u_0\|_{a,b}^{(2-p)p/2}\right). \end{aligned}$$

Using the last inequality and (43) we have the estimate

$$\left(\int_{\mathbf{R}^N} a|\nabla u_n - \nabla u_0|^p\,dx\right)^{2/p} \le c_p A_n(\|u_n\|_{a,b}^{2-p} + \|u_0\|_{a,b}^{2-p}). \tag{45}$$

In a similar way we can obtain the estimate

$$\left(\int_{\mathbf{R}^N} b|u_n - u_0|^p\,dx\right)^{2/p} \le c_p^{'} A_n(\|u_n\|_{a,b}^{2-p} + \|u_0\|_{a,b}^{2-p}). \tag{46}$$

We now observe that inequalities (44), (45) and (46) imply the estimate (42).

In both cases, we have obtained $\|u_n - u_0\|_{a,b} \to 0$ as $n \to \infty$. This shows that $\|u_n\|_{a,b} \to \|u_0\|_{a,b}$ as $n \to \infty$. By Corollary 1, we find $u_0 \not\equiv 0$. This concludes the proof of Theorem 1. ■

References

[1] S. Alama and G. Tarantello, Elliptic problems with nonlinearities indefinite in sign, *J. Funct. Anal.* **141** (1996), 159-215.

[2] G. Aronsson and U. Janfalk, On Hele-Shaw flow of power-law fluids, *European J. Appl. Math.* **3** (1992), 343-366.

[3] J. Chabrowski, Degenerate elliptic equation involving a subcritical Sobolev exponent, *Portugal. Math.* **53** (1996), 167-178.

[4] J. Chabrowski, Elliptic variational problems with indefinite nonlinearities, *Topol. Methods Nonlinear Anal.* **9** (1997), 221-231.

[5] F. Cîrstea, D. Motreanu and V. Rădulescu, Weak solutions of quasilinear problems with nonlinear boundary condition, *Nonlin. Anal., T.M.A.* **43** (2001), 623-636.

[6] F. Cîrstea and V. Rădulescu, Existence and nonexistence results for quasilinear problems with nonlinear boundary condition, *J. Math. Anal. Appl.* **244** (2000), 169-183.

[7] R. Dautray and J.-L. Lions, *Mathematical Analysis and Numerical Methods for Science and Technology, vol. 1: Physical Origins and Classical Methods*, Springer-Verlag, Berlin, 1985.

[8] J. I. Diaz, *Nonlinear partial differential equations and free boundaries. Elliptic equations.* Pitman Adv. Publ., Boston etc., 1986.

[9] P. Drábek and Y.X. Huang, Positive solutions and bifurcations problems for some quasilinear elliptic equations in $\mathbf{R}^N$, *Nonlin. Anal., T.M.A.* **30** (1997), 3671-3681.

[10] P. Drábek and C. Simader, Nonlinear eigenvalue problems for quasilinear equations on unbounded domains, *Math. Nachr.* **203** (1999), 5-30.

[11] D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, 2nd ed., Springer-Verlag, New York, 1983.

[12] O. A. Ladyzhenskaya and N. N. Ural'tseva, *Linear and Quasilinear Equations of Elliptic Type*, Nauka, Moscow, 1964.

[13] S.G. Mikhlin, On applicability of a variational method to certain degenerate elliptic equations, *Doklady Akad. Nauka SSSR (New Series)* **91** (1953), 723-726 (in Russian).

[14] S.G. Mikhlin, Degenerate elliptic equations, *Vestnik Leningrad Univ.* **9** (1954), 19-48 (in Russian).

[15] M.K.V. Murthy and G. Stampacchia, Boundary value problems for some degenerate elliptic operators, *Ann. Mat. Pura Appl.* **80** (1968), 1-122.

[16] D. Passaseo, Some concentration phenomena in degenerate semilinear elliptic problems, *Nonlin. Anal., T.M.A.* **24** (1995), 1011-1025.

[17] M.C. Pélissier and M.L. Reynaud, Étude d'un modèle mathématique d'écoulement de glacier, *C.R. Acad. Sci. Paris, Sér. I Math.* **279** (1974), 531-534.

[18] R.E. Showalter and N.J. Walkington, Diffusion of fluid in a fissured medium with microstructure, *SIAM J. Math. Anal.* **22** (1991), 1702-1722.

[19] E.W. Stredulinsky, *Weighted inequalities and degenerate elliptic partial differential equations*, Springer Verlag, Berlin, New York, 1984.

[20] M. Struwe, *Variational Methods*, Springer-Verlag, Berlin, 1990.