A GENERALISED TRAPEZOID TYPE INEQUALITY FOR CONVEX FUNCTIONS

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ABSTRACT. A generalised trapezoid inequality for convex functions and applications for quadrature rules are given. A refinement and a counterpart result for the Hermite-Hadamard inequalities are obtained and some inequalities for pdf's and (HH) −divergence measure are also mentioned.

1. INTRODUCTION

The following integral inequality for the generalised trapezoid formula was obtained in [2] (see also [1, p. 68]):

Theorem 1. *Let $f : [a, b] \to \mathbb{R}$ be a function of bounded variation. We have the inequality*

$$\left|\int_a^b f(t)\,dt - [(x-a) f(a) + (b-x) f(b)]\right| \leq \left[\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right|\right] \bigvee_a^b (f), \tag{1.1}$$

holding for all $x \in [a, b]$, where $\bigvee_a^b (f)$ denotes the total variation of f on the interval $[a, b]$.
The constant $\frac{1}{2}$ is the best possible one.

This result may be improved if one assumes the monotonicity of f as follows (see [1, p. 76])

Theorem 2. *Let $f : [a, b] \to \mathbb{R}$ be a monotonic nondecreasing function on $[a, b]$. Then we have the inequality:*

$$\begin{aligned} &\left|\int_a^b f(t)\,dt - [(x-a) f(a) + (b-x) f(b)]\right| \\ \leq\ & (b-x) f(b) - (x-a) f(a) + \int_a^b sgn(x-t) f(t)\,dt \\ \leq\ & (x-a)[f(x) - f(a)] + (b-x)[f(b) - f(x)] \\ \leq\ & \left[\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right|\right][f(b) - f(a)] \end{aligned} \tag{1.2}$$

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for all $x \in [a,b]$.
The above inequalities are sharp.

If the mapping is Lipschitzian, then the following result holds as well [3] (see also [1, p. 82]).

Theorem 3. *Let* $f:[a,b]\to\mathbb{R}$ *be an* $L-$*Lipschitzian function on* $[a,b]$, *i.e.., f satisfies the condition:*

$$\text{(L)}\qquad |f(s)-f(t)| \le L|s-t| \quad \text{for any} \quad s,t\in[a,b] \quad (L>0 \text{ is given}).$$

Then we have the inequality:

$$\left|\int_a^b f(t)\,dt - [(x-a)f(a)+(b-x)f(b)]\right| \le \left[\frac{1}{4}(b-a)^2+\left(x-\frac{a+b}{2}\right)^2\right]L \tag{1.3}$$

for any $x\in[a,b]$.
The constant $\frac{1}{4}$ *is best in (1.3).*

If we would assume absolute continuity for the function f, then the following estimates in terms of the Lebesgue norms of the derivative f' hold [1, p. 93].

Theorem 4. *Let* $f:[a,b]\to\mathbb{R}$ *be an absolutely continuous function on* $[a,b]$. *Then for any* $x\in[a,b]$, *we have*

$$\left|\int_a^b f(t)\,dt - [(x-a)f(a)+(b-x)f(b)]\right| \le \begin{cases} \left[\frac{1}{4}(b-a)^2+\left(x-\frac{a+b}{2}\right)^2\right]\|f'\|_\infty & \text{if } f'\in L_\infty[a,b]; \\ \frac{1}{(q+1)^{\frac{1}{q}}}\left[(x-a)^{q+1}+(b-x)^{q+1}\right]^{\frac{1}{q}}\|f'\|_p & \text{if } f'\in L_p[a,b], \\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]\|f'\|_1, & \end{cases} \tag{1.4}$$

where $\|\cdot\|_p$ $(p\in[1,\infty])$ *are the Lebesgue norms, i.e.,*

$$\|f'\|_\infty = ess\sup_{s\in[a,b]}|f'(s)|$$

and

$$\|f'\|_p := \left(\int_a^b |f'(s)|\,ds\right)^{\frac{1}{p}},\quad p\ge 1.$$

In this paper we point out some similar results for convex functions. Applications for quadrature formulae, for probability density functions and $HH-$Divergences in Information Theory are also considered.

2. The Results

The following theorem providing a lower bound for the difference

$$(x-a)\,f(a)+(b-x)\,f(b)-\int_a^b f(t)\,dt$$

holds.

Theorem 5. *Let $f:[a,b]\to\mathbb{R}$ be a convex function on $[a,b]$. Then for any $x\in(a,b)$ we have the inequality*

$$\begin{aligned}\frac{1}{2}\left[(b-x)^2 f'_+(x)-(x-a)^2 f'_-(x)\right] \\ \leq\quad (x-a)\,f(a)+(b-x)\,f(b)-\int_a^b f(t)\,dt.\end{aligned} \tag{2.1}$$

The constant $\frac{1}{2}$ in the left hand side of (2.1) is sharp in the sense that it cannot be replaced by a larger constant.

Proof. It is easy to see that for any locally absolutely continuous function $f:(a,b)\to\mathbb{R}$, we have the identity

$$(x-a)\,f(a)+(b-x)\,f(b)-\int_a^b f(t)\,dt=\int_a^b (t-x)\,f'(t)\,dt \tag{2.2}$$

for any $x\in(a,b)$, where f' is the derivative of f which exists a.e. on $[a,b]$.

Since f is convex, then it is locally Lipschitzian and thus (2.2) holds. Moreover, for any $x\in(a,b)$, we have the inequalities:

$$f'(t)\leq f'_-(x)\quad\text{for a.e.}\quad t\in[a,x] \tag{2.3}$$

and

$$f'(t)\geq f'_+(x)\quad\text{for a.e.}\quad t\in[x,b]. \tag{2.4}$$

If we multiply (2.3) by $x-t\geq 0$, $t\in[a,x]$ and integrate on $[a,x]$, we get

$$\int_a^x (x-t)\,f'(t)\,dt\leq\frac{1}{2}(x-a)^2 f'_-(x) \tag{2.5}$$

and if we multiply (2.4) by $t-x\geq 0$, $t\in[x,b]$ and integrate on $[x,b]$, we also have

$$\int_x^b (t-x)\,f'(t)\,dt\geq\frac{1}{2}(b-x)^2 f'_+(x). \tag{2.6}$$

Finally, if we subtract (2.5) from (2.6) and use the representation (2.2), we deduce the desired inequality (2.1).

Now, assume that (2.1) holds with a constant $C>0$ instead of $\frac{1}{2}$, i.e.,

$$\begin{aligned}C\left[(b-x)^2 f'_+(x)-(x-a)^2 f'_-(x)\right] \\ \leq\quad (x-a)\,f(a)+(b-x)\,f(b)-\int_a^b f(t)\,dt.\end{aligned} \tag{2.7}$$

Consider the convex function $f_0(t) := k\left|t - \frac{a+b}{2}\right|$, $k > 0$, $t \in [a,b]$. Then

$$\begin{aligned} f'_{0+}\left(\frac{a+b}{2}\right) &= k, \quad f'_{0-}\left(\frac{a+b}{2}\right) = -k, \\ f_0(a) &= \frac{k(b-a)}{2} = f_0(b), \quad \int_a^b f_0(t)\,dt = \frac{1}{4}k(b-a)^2. \end{aligned}$$

If in (2.7) we choose f_0 as above and $x = \frac{a+b}{2}$, then we get

$$C\left[\frac{1}{4}(b-a)^2 k + \frac{1}{4}(b-a)^2 k\right] \le \frac{1}{4}k(b-a)^2$$

giving $C \le \frac{1}{2}$, and the sharpness of the constant is proved. ∎

Now, recall that the following inequality which is well known in the literature as the *Hermite-Hadamard inequality* for convex functions holds

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(t)\,dt \le \frac{f(a)+f(b)}{2}. \tag{H-H}$$

The following corollary gives a sharp lower bound for the difference

$$\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)\,dt.$$

Corollary 1. *Let $f : [a,b] \to \mathbb{R}$ be a convex function on $[a,b]$. Then*

$$\begin{aligned} 0 &\le \frac{1}{8}\left[f'_+\left(\frac{a+b}{2}\right) - f'_-\left(\frac{a+b}{2}\right)\right](b-a) \\ &\le \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)\,dt. \end{aligned} \tag{2.8}$$

The constant $\frac{1}{8}$ is sharp.

The proof is obvious by the above theorem. The sharpness of the constant is obtained for $f_0(t) = k\left|t - \frac{a+b}{2}\right|$, $t \in [a,b]$, $k > 0$.

When x is a point of differentiability, we may state the following corollary as well.

Corollary 2. *Let f be as in Theorem 5. If $x \in (a,b)$ is a point of differentiability for f, then*

$$(b-a)\left(\frac{a+b}{2} - x\right)f'(x) \le (x-a)f(a) + (b-x)f(b) - \int_a^b f(t)\,dt. \tag{2.9}$$

Remark 1. *If $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is convex on I and if we choose $x \in \mathring{I}$ ($\mathring{I}$ is the interior of I), $b = x + \frac{h}{2}$, $a = x - \frac{h}{2}$, $h > 0$ is such that $a, b \in I$, then from (2.1) we may write*

$$0 \le \frac{1}{8}h^2\left[f'_+(x) - f'_-(x)\right] \le \frac{f(a)+f(b)}{2}\cdot h - \int_{x-\frac{h}{2}}^{x+\frac{h}{2}} f(t)\,dt \tag{2.10}$$

and the constant $\frac{1}{8}$ is sharp in (2.10).

The following result providing an upper bound for the difference

$$(x-a)f(a) + (b-x)f(b) - \int_a^b f(t)\,dt$$

also holds.

Theorem 6. *Let $f : [a,b] \to \mathbb{R}$ be a convex function on $[a,b]$. Then for any $x \in [a,b]$, we have the inequality:*

$$(x-a) f(a) + (b-x) f(b) - \int_a^b f(t)\,dt \leq \frac{1}{2}\left[(b-x)^2 f'_-(b) - (x-a)^2 f'_+(a)\right]. \tag{2.11}$$

The constant $\frac{1}{2}$ is sharp in the sense that it cannot be replaced by a smaller constant.

Proof. If either $f'_+(a) = -\infty$ or $f'_-(b) = +\infty$, then the inequality (2.11) evidently holds true.

Assume that $f'_+(a)$ and $f'_-(b)$ are finite.

Since f is convex on $[a,b]$, we have

$$f'(t) \geq f'_+(a) \quad \text{for a.e.} \quad t \in [a,x] \tag{2.12}$$

and

$$f'(t) \leq f'_-(b) \quad \text{for a.e.} \quad t \in [x,b]. \tag{2.13}$$

If we multiply (2.12) by $(x-t) \geq 0$, $t \in [a,x]$ and integrate on $[a,x]$, then we deduce

$$\int_a^x (x-t) f'(t)\,dt \geq \frac{1}{2}(x-a)^2 f'_+(a) \tag{2.14}$$

and if we multiply (2.13) by $t - x \geq 0$, $t \in [x,b]$ and integrate on $[x,b]$, then we also have

$$\int_x^b (t-x) f'(t)\,dt \leq \frac{1}{2}(b-x)^2 f'_-(b). \tag{2.15}$$

Finally, if we subtract (2.14) from (2.15) and use the representation (2.2), we deduce the desired inequality (2.11).

Now, assume that (2.11) holds with a constant $D > 0$ instead of $\frac{1}{2}$, i.e.,

$$(x-a) f(a) + (b-x) f(b) - \int_a^b f(t)\,dt \leq D\left[(b-x)^2 f'_-(b) - (x-a)^2 f'_+(a)\right]. \tag{2.16}$$

If we consider the convex function $f_0 : [a,b] \to \mathbb{R}$, $f_0(t) = k\left|t - \frac{a+b}{2}\right|$, then we have $f'_-(b) = k$, $f'_+(a) = -k$ and by (2.16) we deduce for $x = \frac{a+b}{2}$ that

$$\frac{1}{4}k(b-a)^2 \leq D\left[\frac{1}{4}k(b-a)^2 + \frac{1}{4}k(b-a)^2\right]$$

giving $D \geq \frac{1}{2}$, and the sharpness of the constant is proved. ∎

The following corollary related to the Hermite-Hadamard inequality is interesting as well.

Corollary 3. *Let $f : [a,b] \to \mathbb{R}$ be convex on $[a,b]$. Then*

$$0 \leq \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)\,dt \leq \frac{1}{8}\left[f'_-(b) - f'_+(a)\right](b-a) \tag{2.17}$$

and the constant $\frac{1}{8}$ is sharp.

Remark 2. *Denote* $B := f'_{-}(b)$, $A := f'_{+}(a)$ *and assume that* $B \neq A$, *i.e.,* f *is not constant on* (a,b). *Then*

$$(b-x)^2 B - (x-a)^2 A = (B-A)\left[x - \left(\frac{bB-aA}{B-A}\right)\right]^2 - \frac{AB}{B-A}(b-a)^2$$

and by (2.11) we get

$$\begin{aligned} &(x-a) f(a) + (b-x) f(b) - \int_a^b f(t)\,dt \\ \leq\ & (B-A)\left[x - \left(\frac{bB-aA}{B-A}\right)\right]^2 - \frac{AB}{(B-A)^2}(b-a)^2 \end{aligned} \tag{2.18}$$

for any $x \in [a,b]$.
If $A \geq 0$, *then* $x_0 = \frac{bB-aA}{B-A} \in [a,b]$, *and by (2.18) for* $x = \frac{bB-aA}{B-A}$ *we get that*

$$0 \leq \frac{1}{2} \cdot \frac{AB}{B-A}(b-a) \leq \frac{Bf(a) - Af(b)}{B-A} - \frac{1}{b-a}\int_a^b f(t)\,dt \tag{2.19}$$

which is an interesting inequality in itself as well.

3. The Composite Case

Consider the division $I_n : a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$ and denote $h_i := x_{i+1} - x_i$ $\left(i = \overline{0, n-1}\right)$. If $\xi_i \in [x_i, x_{i+1}]$ $\left(i = \overline{0, n-1}\right)$ are intermediate points, then we will denote by

$$G_n(f; I_n, \boldsymbol{\xi}) := \sum_{i=0}^{n-1} \left[(\xi_i - x_i) f(x_i) + (x_{i+1} - \xi_i) f(x_{i+1})\right] \tag{3.1}$$

the generalised trapezoid rule associated to f, I_n and $\boldsymbol{\xi}$.

The following theorem providing upper and lower bounds for the remainder in approximating the integral $\int_a^b f(t)\,dt$ of a convex function f in terms of the generalised trapezoid rule holds.

Theorem 7. *Let* $f : [a,b] \to \mathbb{R}$ *be a convex function and* I_n *and* $\boldsymbol{\xi}$ *be as above. Then we have:*

$$\int_a^b f(t)\,dt = G_n(f; I_n, \boldsymbol{\xi}) - S_n(f; I_n, \boldsymbol{\xi}), \tag{3.2}$$

where $G_n(f; I_n, \boldsymbol{\xi})$ *is the generalised Trapezoid Rule defined by (3.1) and the remainder* $S_n(f; I_n, \boldsymbol{\xi})$ *satisfies the estimate:*

$$\begin{aligned} &\frac{1}{2}\left[\sum_{i=0}^{n-1} (x_{i+1} - \xi_i)^2 f'_{+}(\xi_i) - \sum_{i=0}^{n-1} (\xi_i - x_i)^2 f'_{-}(\xi_i)\right] \\ \leq\ & S_n(f; I_n, \boldsymbol{\xi}) \\ \leq\ & \frac{1}{2}\Bigg[\left(b - \xi_{n-1}\right)^2 f'_{-}(b) + \sum_{i=1}^{n-1}\left[\left(x_i - \xi_{i-1}\right)^2 f'_{-}(x_i) - (\xi_i - x_i)^2 f'_{+}(x_i)\right] \\ & - (\xi_0 - a)^2 f'_{+}(a)\Bigg]. \end{aligned} \tag{3.3}$$

Proof. If we write the inequalities (2.1) and (2.11) on the interval $[x_i, x_{i+1}]$ and for the intermediate points $\xi_i \in [x_i, x_{i+1}]$, then we have

$$\begin{aligned} & \frac{1}{2}\left[(x_{i+1}-\xi_i)^2 f'_+(x_i) - (\xi_i - x_i)^2 f'_-(\xi_i)\right] \\ \leq\ & (\xi_i - x_i) f(x_i) + (x_{i+1}-\xi_i) f(x_{i+1}) - \int_{x_i}^{x_{i+1}} f(t)\,dt \\ \leq\ & \frac{1}{2}\left[(x_{i+1}-\xi_i)^2 f'_-(x_{i+1}) - (\xi_i - x_i)^2 f'_+(x_i)\right]. \end{aligned}$$

Summing the above inequalities over i from 0 to $n-1$, we deduce

$$\begin{aligned} & \frac{1}{2}\sum_{i=0}^{n-1}\left[(x_{i+1}-\xi_i)^2 f'_+(\xi_i) - (\xi_i - x_i)^2 f'_-(\xi_i)\right] \\ \leq\ & G_n(f; I_n, \boldsymbol{\xi}) - \int_a^b f(t)\,dt \\ \leq\ & \frac{1}{2}\left[\sum_{i=0}^{n-1}(x_{i+1}-\xi_i)^2 f'_-(x_{i+1}) - \sum_{i=0}^{n-1}(\xi_i - x_i)^2 f'_+(x_i)\right]. \end{aligned} \tag{3.4}$$

However,

$$\begin{aligned} \sum_{i=0}^{n-1}(x_{i+1}-\xi_i)^2 f'_-(x_{i+1}) &= \left(b-\xi_{n-1}\right)^2 f'_-(b) + \sum_{i=0}^{n-2}\left[(x_{i+1}-\xi_i)^2 f'_-(x_{i+1})\right] \\ &= \left(b-\xi_{n-1}\right)^2 f'_-(b) + \sum_{i=1}^{n-1}\left(x_i-\xi_{i-1}\right)^2 f'_-(x_i) \end{aligned}$$

and

$$\sum_{i=0}^{n-1}(\xi_i - x_i)^2 f'_+(x_i) = \sum_{i=1}^{n-1}(\xi_i - x_i)^2 f'_+(x_i) + (\xi_0 - a)^2 f'_+(a)$$

and then, by (3.4), we deduce the desired estimate (3.3). ∎

The following corollary may be useful in practical applications.

Corollary 4. *Let $f : [a,b] \to \mathbb{R}$ be a differentiable convex function on $[a,b]$. Then we have the representation (3.2) and $S_n(f; I_n, \boldsymbol{\xi})$ satisfies the estimate:*

$$\begin{aligned} & \sum_{i=0}^{n-1}\left(\frac{x_i + x_{i+1}}{2} - \xi_i\right) h_i f'(\xi_i) \\ \leq\ & S_n(f; I_n, \boldsymbol{\xi}) \\ \leq\ & \frac{1}{2}\Bigg[\left(b-\xi_{n-1}\right)^2 f'_-(b) - (\xi_0 - a)^2 f'_+(a) \\ & + \sum_{i=1}^{n-1}\left[\left(x_i - \frac{\xi_i + \xi_{i-1}}{2}\right)\left(\xi_i - \xi_{i-1}\right) f'(x_i)\right]\Bigg]. \end{aligned} \tag{3.5}$$

We may also consider the trapezoid quadrature rule:

$$T_n(f; I_n) := \sum_{i=0}^{n-1}\frac{f(x_i) + f(x_{i+1})}{2}\cdot h_i. \tag{3.6}$$

Using the above results, we may state the following corollary.

Corollary 5. *Assume that $f:[a,b]\to\mathbb{R}$ is a convex function on $[a,b]$ and I_n is a division as above. Then we have the representation*

$$\int_a^b f(t)\,dt = T_n(f;I_n) - Q_n(f;I_n) \tag{3.7}$$

where $T_n(f;I_n)$ is the mid-point quadrature formula given in (3.6) and the remainder $Q_n(f;I_n)$ satisfies the estimates

$$\begin{aligned} 0 &\le \frac{1}{8}\sum_{i=0}^{n-1}\left[f'_+\left(\frac{x_i+x_{i+1}}{2}\right) - f'_-\left(\frac{x_i+x_{i+1}}{2}\right)\right]h_i^2 \\ &\le Q_n(f;I_n) \le \frac{1}{8}\sum_{i=0}^{n-1}\left[f'_+(x_{i+1}) - f'_-(x_i)\right]h_i^2. \end{aligned} \tag{3.8}$$

The constant $\frac{1}{8}$ is sharp in both inequalities.

4. Applications for P.D.F.s

Let X be a random variable with the *probability density function* $f:[a,b]\subset\mathbb{R}\to[0,\infty)$ and with *cumulative distribution function* $F(x)=\Pr(X\le x)$.

The following theorem holds.

Theorem 8. *If $f:[a,b]\subset\mathbb{R}\to\mathbb{R}_+$ is monotonically increasing on $[a,b]$, then we have the inequality:*

$$\begin{aligned} &\frac{1}{2}\left[(b-x)^2 f_+(x) - (x-a)^2 f_-(x)\right] + x \\ \le\ & E(X) \\ \le\ & \frac{1}{2}\left[(b-x)^2 f_+(b) - (x-a)^2 f_-(a)\right] + x \end{aligned} \tag{4.1}$$

for any $x\in(a,b)$, where $f_\pm(\alpha)$ represent respectively the right and left limits of f in α and $E(X)$ is the expectation of X.
The constant $\frac{1}{2}$ is sharp in both inequalities.
The second inequality also holds for $x=a$ or $x=b$.

Proof. Follows by Theorem 5 and 6 applied for the convex cdf function $F(x)=\int_a^x f(t)\,dt$, $x\in[a,b]$ and taking into account that

$$\int_a^b F(x)\,dx = b - E(X).$$

∎

Finally, we may state the following corollary in estimating the expectation of X.

Corollary 6. *With the above assumptions, we have*

$$\begin{aligned} &\frac{1}{8}\left[f_+\left(\frac{a+b}{2}\right) - f_-\left(\frac{a+b}{2}\right)\right](b-a)^2 + \frac{a+b}{2} \\ \le\ & E(X) \le \frac{1}{8}\left[f_+(b) - f_-(a)\right](b-a)^2 + \frac{a+b}{2}. \end{aligned} \tag{4.2}$$

5. Applications for $HH-$Divergence

Assume that a set χ and the $\sigma-$finite measure μ are given. Consider the set of all probability densities on μ to be

$$\Omega := \left\{ p | p : \Omega \to \mathbb{R},\ p(x) \geq 0,\ \int_{\chi} p(x)\, d\mu(x) = 1 \right\}. \tag{5.1}$$

Csiszár's $f-$divergence is defined as follows [4]

$$D_f(p, q) := \int_{\chi} p(x)\, f\left[\frac{q(x)}{p(x)} \right] d\mu(x),\ p, q \in \Omega, \tag{5.2}$$

where f is convex on $(0, \infty)$. It is assumed that $f(u)$ is zero and strictly convex at $u = 1$. By appropriately defining this convex function, various divergences are derived.

In [5], Shioya and Da-te introduced the generalised Lin-Wong $f-$divergence $D_f\left(p, \frac{1}{2}p + \frac{1}{2}q\right)$ and the Hermite-Hadamard (HH) divergence

$$D_{HH}^{f}(p, q) := \int_{\chi} \frac{p^2(x)}{q(x) - p(x)} \left(\int_1^{\frac{q(x)}{p(x)}} f(t)\, dt \right) d\mu(x),\ p, q \in \Omega, \tag{5.3}$$

and, by the use of the Hermite-Hadamard inequality for convex functions, proved the following basic inequality

$$D_f\left(p, \frac{1}{2}p + \frac{1}{2}q\right) \leq D_{HH}^{f}(p, q) \leq \frac{1}{2} D_f(p, q), \tag{5.4}$$

provided that f is convex and normalised, i.e., $f(1) = 0$.

The following result in estimating the difference

$$\frac{1}{2} D_f(p, q) - D_{HH}^{f}(p, q)$$

holds.

Theorem 9. *Let $f : [0, \infty) \to \mathbb{R}$ be a normalised convex function and $p, q \in \Omega$. Then we have the inequality:*

$$\begin{aligned} 0 &\leq \frac{1}{8} \left[D_{f'_+ \cdot \left| \frac{\cdot + 1}{2} \right|}(p, q) - D_{f'_- \cdot \left| \frac{\cdot + 1}{2} \right|}(p, q) \right] \\ &\leq \frac{1}{2} D_f(p, q) - D_{HH}^{f}(p, q) \\ &\leq \frac{1}{8} D_{f'_- \cdot (\cdot - 1)}(p, q). \end{aligned} \tag{5.5}$$

Proof. Using the double inequality

$$\begin{aligned} 0 &\leq \frac{1}{8} \left[f'_+\left(\frac{a+b}{2} \right) - f'_-\left(\frac{a+b}{2} \right) \right] |b - a| \\ &\leq \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(t)\, dt \\ &\leq \frac{1}{8} \left[f_-(b) - f'_+(a) \right] (b - a) \end{aligned}$$

for the choices $a = 1$, $b = \frac{q(x)}{p(x)}$, $x \in \chi$, multiplying with $p(x) \geq 0$ and integrating over x on χ we get

$$\begin{aligned} 0 &\leq \frac{1}{8}\int_{\chi}\left[f'_{+}\left(\frac{p(x)+q(x)}{2p(x)}\right) - f'_{-}\left(\frac{p(x)+q(x)}{2p(x)}\right)\right]|q(x)-p(x)|\,d\mu(x) \\ &\leq \frac{1}{2}D_f(p,q) - D_{HH}^{f}(p,q) \\ &\leq \frac{1}{8}\int_{\chi}\left[f'_{-}\left(\frac{q(x)}{p(x)}\right) - f'_{+}(1)\right](q(x)-p(x))\,d\mu(x), \end{aligned}$$

which is clearly equivalent to (5.5). ∎

Corollary 7. *With the above assumptions and if f is differentiable on $(0,\infty)$, then*

$$0 \leq \frac{1}{2}D_f(p,q) - D_{HH}^{f}(p,q) \leq \frac{1}{8}D_{f'\cdot(\cdot-1)}(p,q). \tag{5.6}$$

References

[1] P. CERONE and S.S. DRAGOMIR, Trapezoidal-type rules from an inequalities point of view, in *Handbook of Analytic-Computational Methods in Applied Mathematics*, G.A. Anastassiou (Ed), Chapman & Hall/CRC Press, New York, 2000, 65-134.

[2] P. CERONE, S.S. DRAGOMIR and C.E.M. PEARCE, A generalised trapezoid inequality for functions of bounded variation, *Turkish J. of Math.,* **24**(2) (2000), 147-163.

[3] S.S. DRAGOMIR, On the trapezoid quadrature formula for Lipschitzian mappings and applications, *Tamkang J. of Math.,* **30**(2) (1999), 133-138.

[4] I. CSISZÁR, Information-type measures of difference of probability distributions and indirect observations, *Studia Math. Hungarica,* **2** (1967), 299-318.

[5] H. SHIOYA and T. DA-TE, A generalisation of Lin divergence and the derivation of a new information divergence, *Elec. and Comm. in Japan,* **78**(7), (1995), 37-40.

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