SOME OSROWSKI TYPE INEQUALITIES FOR DOUBLE INTEGRALS OF FUNCTIONS WHOSE PARTIAL DERIVATIVES SATISFY CERTAIN CONVEXITY PROPERTIES

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Abstract. In this paper we point out some new two-dimensional integral inequalities, for functions whose partial derivatives satisfy certain convexity properties.

1. Introduction

Barnett and Dragomir [1], Milovanović [5], Pachpatte [6] and Hanna et al. [4] developed two dimensional integral inequalities whose error bounds were expressed in term of Lebesgue norms of the integrand partial derivatives. Here we consider a function whose first partial derivatives exist and satisfy certain convexity properties over a given rectangular region. The work in this paper is presented in the following order. In Section 2, a two variable integral identity for first differentiable mapping is developed. In Section 3, we derive some double integral inequalities. Error bounds are expressed in term of the L_p norms of the first and first mixed partial derivatives of the integrand.

2. Integral Identity

The following identity is interesting in itself as well:

Lemma 1. *Let $f : [a,b] \times [c,d] \to \mathbb{R}$ be a function so that the partial derivatives $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial^2 f}{\partial x \partial y}$ to be denoted by D_x, D_y and D_{xy} herein respectively, are continuous on $[a,b] \times [c,d]$, then for any $(x_0, y_0) \in [a,b] \times [c,d]$ we have the identity:*

$$
\begin{aligned}
(2.1) \quad f(x_0, y_0) &= \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(t,s)\, ds dt \\
&+ \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d (y_0 - s) \left(\int_0^1 D_y [t, (1-\mu) y_0 + \mu s]\, d\mu \right) ds dt \\
&+ \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d (x_0 - t) \left(\int_0^1 D_x [(1-\lambda) x_0 + \lambda t, s]\, d\lambda \right) ds dt
\end{aligned}
$$

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$$+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d (x_0-t)(y_0-s)$$
$$\times\left(\int_0^1\int_0^1 D_{xy}\left[(1-\lambda)x_0+\lambda t,(1-\mu)y_0+\mu s\right]d\mu d\lambda\right)dsdt.$$

Proof. We will use the following identity, which has been established in [3]

$$Q(x)=\frac{1}{b-a}\int_a^b Q(t) + \frac{1}{b-a}\int_a^b (x-t)\left(\int_0^1 Q'\left[(1-\lambda)x+\lambda t\right]d\lambda\right)dt, \tag{2.2}$$

where $x\in[a,b]$ and $Q:[a,b]\to\mathbb{R}$ is an absolutely continuous function on $[a,b]$. For the sake of completeness we give here a short proof for (2.2).

For any $x,t\in[a,b]$, $x\neq t$,

$$\frac{Q(x)-Q(t)}{x-t}=\frac{\int_t^x Q'(u)du}{x-t}=\int_0^1 Q'[(1-\lambda)t+\lambda x]d\lambda$$

where we used the change of variables,

$$u=(1-\lambda)t+\lambda x,\qquad u\in[0,1].$$

Then

$$Q(x)=Q(t)+(x-t)\int_0^1 Q'[(1-\lambda)t+\lambda x]d\lambda$$

and this holds for any $x,t\in[a,b]$. Integrating over $t\in[a,b]$ and dividing by $(b-a)$ we get (2.2).

Now, fix $y_0\in[c,d]$, then by (2.2) we have the equality

$$f(x_0,y_0)=\frac{1}{b-a}\int_a^b f(t,y_0)\,dy + \frac{1}{b-a}\int_a^b (x_0-t)\left(\int_0^1 D_x\left[(1-\lambda)x_0+\lambda t,y_0\right]d\lambda\right)dt \tag{2.3}$$

for any $t\in[a,b]$. Applying (2.2) over the second variable, we may write:

$$f(t,y_0)=\frac{1}{d-c}\int_a^b f(t,y_0)\,dy + \frac{1}{d-c}\int_c^d (y_0-s)\left(\int_0^1 D_y\left[t,(1-\mu)y_0+\mu s\right]d\mu\right)dt. \tag{2.4}$$

Integrating (2.4) over $t\in[a,b]$ and using Fubini's theorem we deduce

$$\frac{1}{b-a}\int_a^b f(t,y_0)\,dy = \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)\,dsdt + \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d (y_0-s)\left(\int_0^1 D_y\left[t,(1-\mu)y_0+\mu s\right]d\mu\right)dsdt. \tag{2.5}$$

If we fix $x_0, t \in [a,b]$, then applying (2.2) again, we obtain

$$\begin{aligned} D_x\left[(1-\lambda)x_0+\lambda t, y_0\right] &= \frac{1}{d-c}\int_c^d D_x\left[(1-\lambda)x_0+\lambda t, s\right]ds \\ &+ \frac{1}{d-c}\int_c^d (y_0-s)\left(\int_0^1 D_{xy}\left[(1-\lambda)x_0+\lambda t,(1-\mu)y_0+\mu s\right]d\mu\right)ds. \end{aligned} \tag{2.6}$$

We then integrate (2.6) over $\lambda \in [0,1]$, and, on inverting the order of integrals, we get that

$$\begin{aligned} \int_0^1 D_x\left[(1-\lambda)x_0+\lambda t, y_0\right]d\lambda &= \frac{1}{d-c}\int_c^d D_x\left[(1-\lambda)x_0+\lambda t, s\right]ds \\ &+ \frac{1}{d-c}\int_c^d (y_0-s)\left(\int_0^1\int_0^1 D_{xy}\left[(1-\lambda)x_0+\lambda t,(1-\mu)y_0+\mu s\right]d\mu d\lambda\right)ds. \end{aligned} \tag{2.7}$$

Consequently, we deduce

$$\begin{aligned} &\frac{1}{(b-a)}\int_c^d (x_0-t)\left(\int_0^1 D_x\left[(1-\lambda)x_0+\lambda t, y_0\right]d\lambda\right)dt \\ &= \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d (x_0-t)\left(\int_0^1 D_x\left[(1-\lambda)x_0+\lambda t, s\right]d\lambda\right)dsdt \\ &\quad + \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d (x_0-t)(y_0-s) \\ &\quad \times\left(\int_0^1\int_0^1 D_{xy}\left[(1-\lambda)x_0+\lambda t,(1-\mu)y_0+\mu s\right]d\mu d\lambda\right)dsdt. \end{aligned} \tag{2.8}$$

Finally, using (2.3), (2.5) and (2.8) we obtain the desired equality (2.1). ∎

3. Some Integral Inequalities

3.1. Mapping Whose First Derivative Belongs to $L_\infty[[a,b]\times[c,d]]$.

In this section we tap the equalities of the previous section and develop inequalities for the depiction of the two dimensional integral of a function with respect to the derivatives at a multiple number of points over the given region.

We start with the following result.

Theorem 1. *Let $f:[a,b]\times[c,d]\to\mathbb{R}$ be a function such that the partial derivatives D_x, D_y and D_{xy} exist and are continuous on $[a,b]\times[c,d]$. If $|D_x|$ is convex over first direction, $|D_y|$ is convex over the second direction and $|D_{xy}|$ is convex in both*

directions, then we have the inequality

$$\begin{aligned}
(3.1)\quad & \left|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)\,dsdt - f(x_0,y_0)\right| \\
&\leq \frac{[\|D_x(x_0,.)\|_\infty + \|D_x\|_\infty]}{2(b-a)}\left[\frac{1}{4}(b-a)^2 + \left(x_0 - \frac{a+b}{2}\right)^2\right] \\
&+ \frac{[\|D_y(.,y_0)\|_\infty + \|D_y\|_\infty]}{2(d-c)}\left[\frac{1}{4}(d-c)^2 + \left(y_0 - \frac{c+d}{2}\right)^2\right] \\
&+ \frac{[\|D_{xy}(x_0,.)\|_\infty + |D_{xy}(x_0,y_0)| + \|D_{xy}(.,y_0)\|_\infty + \|D_{xy}\|_\infty]}{4(b-a)(d-c)} \\
&\times\left[\frac{1}{4}(b-a)^2 + \left(x_0 - \frac{a+b}{2}\right)^2\right]\left[\frac{1}{4}(d-c)^2 + \left(y_0 - \frac{c+d}{2}\right)^2\right].
\end{aligned}$$

for all $(x_0,y_0)\in[a,b]\times[c,d]$, *where*

$$\|D_x(x_0,.)\|_\infty := \sup_{s\in[c,d]}|D_x(x_0,s)| < \infty, \quad \|D_y(.,y_0)\|_\infty := \sup_{t\in[a,b]}|D_y(t,y_0)| < \infty,$$

$$\|D_x\|_\infty := \sup_{(t,s)\in[a,b]\times[c,d]}|D_x(t,s)| < \infty,$$

$$\|D_{xy}(x_0,.)\|_\infty := \sup_{s\in[c,d]}|D_{xy}(x_0,s)| < \infty, \quad \|D_{xy}(.,y_0)\|_\infty := \sup_{t\in[a,b]}|D_{xy}(t,y_0)| < \infty$$

and

$$\|D_{xy}\|_\infty := \sup_{(t,s)\in[a,b]\times[c,d]}|D_{xy}(t,s)| < \infty.$$

Proof. Using Lemma 1 we get from (2.1)

$$\begin{aligned}
(3.2)\quad & \left|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)\,dsdt - f(x_0,y_0)\right| \\
&= \left|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d (x_0-t)\left(\int_0^1 D_x[(1-\lambda)x_0+\lambda t,s]\,d\lambda\right)dsdt\right. \\
&+ \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d (y_0-s)\left(\int_0^1 D_y[t,(1-\mu)y_0+\mu s]\,d\mu\right)dsdt \\
&+ \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d (x_0-t)(y_0-s) \\
&\times\left.\left(\int_0^1\int_0^1 D_{xy}[(1-\lambda)x_0+\lambda t,(1-\mu)y_0+\mu s]\,d\mu d\lambda\right)dsdt\right|
\end{aligned}$$

By the triangle inequality it follows that

$$\left|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)\,dsdt - f(x_0,y_0)\right| \tag{3.3}$$
$$\leq \left|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d (x_0-t)\left(\int_0^1 D_x\left[(1-\lambda)x_0+\lambda t,s\right]d\lambda\right)dsdt\right|$$
$$+\left|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d (y_0-s)\left(\int_0^1 D_y\left[t,(1-\mu)y_0+\mu s\right]d\mu\right)dsdt\right|$$
$$+\left|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d (x_0-t)(y_0-s)\right.$$
$$\left.\times\left(\int_0^1\int_0^1 D_{xy}\left[(1-\lambda)x_0+\lambda t,(1-\mu)y_0+\mu s\right]d\mu d\lambda\right)dsdt\right|.$$

Now, since $|D_x(.,s)|$ is convex over the $t-$direction for any $s\in[c,d]$, using the properties of the modulus, we then have that

$$\left|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d (x_0-t)\left(\int_0^1 D_x\left[(1-\lambda)x_0+\lambda t,s\right]d\lambda\right)dsdt\right| \tag{3.4}$$
$$\leq \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d |x_0-t|\left(\int_0^1 \left|D_x\left[(1-\lambda)x_0+\lambda t,s\right]\right|d\lambda\right)dsdt$$

$$\leq \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d |x_0-t| \tag{3.5}$$
$$\times\left(\int_0^1\left[(1-\lambda)\left|D_x(x_0,s)\right|+\lambda\left|D_x(t,s)\right|\right]d\lambda\right)dsdt$$

$$\leq \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d \left|\frac{x_0-t}{2}\right|\left(\|D_x(x_0,.)\|_\infty+\|D_x\|_\infty\right)dsdt \tag{3.6}$$
$$=\frac{\left[\|D_x(x_0,.)\|_\infty+\|D_x\|_\infty\right]}{2(b-a)(d-c)}\int_a^b\int_c^d |x_0-t|\,dsdt$$
$$=\frac{\left[\|D_x(x_0,.)\|_\infty+\|D_x\|_\infty\right]}{2(b-a)}\left[\frac{1}{4}(b-a)^2+\left(x_0-\frac{a+b}{2}\right)^2\right]$$

since

$$\begin{aligned}\int_a^b\int_c^d |x_0-t|\,dsdt &= \int_c^d ds\int_a^b |x_0-t|\,dt\\ &= (d-c)\left[\int_a^{x_0}(x_0-t)\,dt+\int_{x_0}^b (t-x_0)\,dt\right].\end{aligned}$$

In a similar manner, since $|D_x(t,.)|$ is convex over the $s-$direction for any $t \in (a,b)$, we obtain

$$\left|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d (y_0-s)\left(\int_0^1 D_y[t,(1-\mu)y_0+\mu s]\,d\mu\right)dsdt\right| \tag{3.7}$$
$$\leq \frac{[\|D_y(.,y_0)\|_\infty+\|D_y\|_\infty]}{2(d-c)}\left[\frac{1}{4}(d-c)^2+\left(y_0-\frac{c+d}{2}\right)^2\right].$$

Now, taking into account that $|D_{xy}(.,.)|$ is also convex over both t and s we can prove that

$$\left|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d (x_0-t)(y_0-s)\right. \tag{3.8}$$
$$\left.\times\left(\int_0^1\int_0^1 D_{xy}[(1-\lambda)x_0+\lambda t,(1-\mu)y_0+\mu s]\,d\mu d\lambda\right)dsdt\right|$$
$$\leq \frac{[\|D_{xy}(x_0,.)\|_\infty+|D_{xy}(x_0,y_0)|+\|D_{xy}(.,y_0)\|_\infty+\|D_{xy}\|_\infty]}{4(b-a)(d-c)}$$
$$\times\left[\frac{1}{4}(b-a)^2+\left(x_0-\frac{a+b}{2}\right)^2\right]\left[\frac{1}{4}(d-c)^2+\left(y_0-\frac{c+d}{2}\right)^2\right].$$

Utilizing (3.6), (3.7), (3.8) and substituting into (3.2), we obtain the desired result. Thus, the theorem is proved. ∎

Taking into account that x_0 and y_0 are free parameters, we can produce "mid-point" and "boundary-point" type results by choosing appropriate values for x_0 and y_0.

The following corollary will give the best result in the class.

Corollary 1. *Under the assumptions of Theorem 1, we have the inequality:*

$$\left|\int_a^b\int_c^d f(t,s)dsdt-(b-a)(d-c)f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right| \tag{3.9}$$
$$\leq \frac{\|D_x(\frac{a+b}{2},.)\|_\infty+\|D_x\|_\infty}{8}(b-a)^2(d-c)$$
$$+\frac{\|D_y(.,\frac{c+d}{2})\|_\infty+\|D_y\|_\infty}{8}(b-a)(d-c)^2$$
$$+\frac{(b-a)^2(d-c)^2}{64}\left[\|D_{xy}(\frac{a+b}{2},.)\|_\infty\right.$$
$$\left.+|D_{xy}(\frac{a+b}{2},\frac{c+d}{2})|+\|D_{xy}(.,\frac{c+d}{2})\|_\infty+\|D_{xy}\|_\infty\right].$$

3.2. Mappings whose First Derivative Belongs to $L_p[[a,b]\times[c,d]]$.

In this section we point out an inequality for double integrals in terms of the $\|\cdot\|_p-$norm of the first derivatives.

Theorem 2. *Let $f:[a,b]\times[c,d]\to\mathbb{R}$ be a function such that the partial derivatives D_x, D_y, D_{xy} exist and are continuous on $[a,b]\times[c,d]$. If $|D_x|$ is convex over first*

direction, $|D_y|$ is convex over the second direction and $|D_{xy}|$ is convex in both directions, then

$$\begin{aligned}
&\left|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)\,dsdt - f(x_0,y_0)\right| \\
\leq\ & \frac{\left[(b-a)^{\frac{1}{p}}\left\|D_x(x_0,.)\right\|_p + \|D_x\|_p\right]}{2(b-a)(d-c)^{\frac{q-1}{q}}}\left[\frac{(x_0-a)^{q+1}+(b-x_0)^{q+1}}{(q+1)}\right]^{\frac{1}{q}} \\
&+\frac{\left[(d-c)^{\frac{1}{p}}\left\|D_y(.,y_0)\right\|_p + \|D_y\|_p\right]}{2(b-a)^{\frac{q-1}{q}}(d-c)}\left[\frac{(y_0-c)^{q+1}+(d-y_0)^{q+1}}{(q+1)}\right]^{\frac{1}{q}} \\
&+\frac{[G_1(x_0)+G_2(x_0,y_0)+G_3(y_0)+\|D_{xy}\|_p]}{4(b-a)(d-c)} \\
&\times\left[\frac{(x_0-a)^{q+1}+(b-x_0)^{q+1}}{(q+1)}\right]^{\frac{1}{q}}\left[\frac{(y_0-c)^{q+1}+(d-y_0)^{q+1}}{(q+1)}\right]^{\frac{1}{q}}.
\end{aligned}$$

for all $(x_0,y_0)\in[a,b]\times[c,d]$ where

$$G_1(x_0) := (b-a)^{\frac{1}{p}}\left\|D_{xy}(x_0,.)\right\|_p,$$

$$G_2(x_0,y_0) := (b-a)^{\frac{1}{p}}(d-c)^{\frac{1}{p}}|D_{xy}(x_0,y_0)|,$$

$$G_3(y_0) := (d-c)^{\frac{1}{p}}\left\|D_{xy}(.,y_0)\right\|_p,$$

$$\|D_x(x_0,.)\|_p := \left(\int_c^d |D_x(x_0,s)|^p\ ds\right)^{\frac{1}{p}} < \infty,$$

$$\|D_y(.,y_0)\|_p := \left(\int_a^b |D_y(t,y_0)|^p\ dt\right)^{\frac{1}{p}} < \infty,$$

$$\|D_x\|_p := \left(\int_a^b\int_c^d |D_x(t,s)|^p\ dsdt\right)^{\frac{1}{p}} < \infty,$$

$$\|D_{xy}(x_0,.)\|_p := \left(\int_c^d |D_{xy}(x_0,s)|^p\ ds\right)^{\frac{1}{p}} < \infty,$$

$$\|D_{xy}(.,y_0)\|_p := \left(\int_a^b |D_{xy}(t,y_0)|^p\ dt\right)^{\frac{1}{p}} < \infty,$$

and

$$\|D_{xy}\|_p := \left(\int_a^b\int_c^d |D_{xy}(t,s)|^p\ dsdt\right)^{\frac{1}{p}} < \infty.$$

where $\frac{1}{p}+\frac{1}{q}=1$ for all $(1<p<\infty)$.

Proof. We can write (3.3) as

$$\left|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)\,dsdt - f(x_0,y_0)\right| \leq M_1(x_0)+M_2(y_0)+M_3(x_0,y_0),$$

where

$$M_1(x_0) := \left| \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d (x_0 - t) \left(\int_0^1 D_x [(1-\lambda) x_0 + \lambda t, s] \, d\lambda \right) dsdt \right|,$$

$$M_2(y_0) := \left| \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d (y_0 - s) \left(\int_0^1 D_y [t, (1-\mu) y_0 + \mu s] \, d\mu \right) dsdt \right|$$

and

$$\begin{aligned} M_3(x_0, y_0) &:= \left| \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d (x_0 - t)(y_0 - s) \right. \\ &\quad \left. \times \left(\int_0^1 \int_0^1 D_{xy} [(1-\lambda) x_0 + \lambda t, (1-\mu) y_0 + \mu s] \, d\mu d\lambda \right) dsdt \right|. \end{aligned}$$

Now

$$\begin{aligned} M_1(x_0) &= \left| \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d (x_0 - t) \left(\int_0^1 D_x [(1-\lambda) x_0 + \lambda t, s] \, d\lambda \right) dsdt \right| \\ &\leq \frac{1}{2(b-a)(d-c)} \int_a^b \int_c^d |x_0 - t| \left(|D_x(x_0, s)| + |D_x(t, s)| \right) dsdt \end{aligned}$$

then

$$M_1(x_0) \leq \frac{1}{2(b-a)(d-c)} \left[I_{x_0}^{(1)} + I_{x_0}^{(2)} \right], \tag{3.10}$$

where

$$I_{x_0}^{(1)} := \int_a^b \int_c^d |x_0 - t| \, |D_x(x_0, s)| \, dsdt, \tag{3.11}$$

$$I_{x_0}^{(2)} := \int_a^b \int_c^d |x_0 - t| \, |D_x(t, s)| \, dsdt. \tag{3.12}$$

Applying the Hölder inequality for double integrals for $I_{x_0}^{(1)}$, we find that

$$\begin{aligned} I_{x_0}^{(1)} &\leq \left(\int_a^b \int_c^d |x_0 - t|^q \, dsdt \right)^{\frac{1}{q}} \left(\int_a^b \int_c^d |D_x(x_0, s)|^p \, dsdt \right)^{\frac{1}{p}} \\ &= \left((d-c) \int_a^b |x_0 - t|^q \, dt \right)^{\frac{1}{q}} \left((b-a) \int_c^d |D_x(x_0, s)|^p \, ds \right)^{\frac{1}{p}} \\ &= (d-c)^{\frac{1}{q}} \left[\int_a^{x_0} (x_0 - t)^q \, dt + \int_{x_0}^b (t - x_0)^q \, dt \right]^{\frac{1}{q}} \times (b-a)^{\frac{1}{p}} \left\| D_x(x_0, .) \right\|_p \\ &= (d-c)^{\frac{1}{q}} \left[\frac{-(x_0 - t)^{q+1}}{q+1} \bigg|_a^{x_0} + \frac{(t - x_0)^{q+1}}{q+1} \bigg|_{x_0}^b \right]^{\frac{1}{q}} \times (b-a)^{\frac{1}{p}} \left\| D_x(x_0, .) \right\|_p, \end{aligned}$$

then we get

$$I_{x_0}^{(1)} \leq (b-a)^{\frac{1}{p}} (d-c)^{\frac{1}{q}} \left[\frac{(x_0 - a)^{q+1} + (b - x_0)^{q+1}}{(q+1)} \right]^{\frac{1}{q}} \left\| D_x(x_0, .) \right\|_p. \tag{3.13}$$

Similarly

$$I_{x_0}^{(2)} \leq (d-c)^{\frac{1}{q}} \left[\frac{(x_0-a)^{q+1}+(b-x_0)^{q+1}}{(q+1)} \right]^{\frac{1}{q}} \|D_x\|_p . \tag{3.14}$$

Using (3.13) and (3.14) and substituting in (3.10), M_1 becomes

$$M_1(x_0) \leq \frac{\left[(b-a)^{\frac{1}{p}} \|D_x(x_0,.)\|_p + \|D_x\|_p\right]}{2(b-a)(d-c)^{\frac{q-1}{q}}} \left[\frac{(x_0-a)^{q+1}+(b-x_0)^{q+1}}{(q+1)} \right]^{\frac{1}{q}} . \tag{3.15}$$

In similar way

$$M_2(y_0) \leq \frac{\left[(d-c)^{\frac{1}{p}} \|D_y(.,y_0)\|_p + \|D_y\|_p\right]}{2(b-a)^{\frac{q-1}{q}}(d-c)} \left[\frac{(y_0-c)^{q+1}+(d-y_0)^{q+1}}{(q+1)} \right]^{\frac{1}{q}}$$

and

$$\begin{aligned} M_3(x_0,y_0) \quad &\leq \quad \frac{[G_1(x_0)+G_2(x_0,y_0)+G_3(y_0)+\|D_{xy}\|_p]}{4(b-a)(d-c)} \\ &\quad \times \left[\frac{(x_0-a)^{q+1}+(b-x_0)^{q+1}}{(q+1)} \right]^{\frac{1}{q}} \left[\frac{(y_0-c)^{q+1}+(d-y_0)^{q+1}}{(q+1)} \right]^{\frac{1}{q}} . \end{aligned}$$

Thus the theorem is proved. ∎

The best inequality in the class can be produced at $x_0 = \frac{a+b}{2}$ and $y_0 = \frac{c+d}{2}$ as in the following corollary

Corollary 2. *Taking into account the conditions and assumptions in Theorem 2, the following inequality holds*

$$\begin{aligned} &\left| \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(t,s)\,dsdt - f\left(\frac{a+b}{2},\frac{c+d}{2}\right) \right| \\ \leq \quad & \frac{\left[(b-a)^{\frac{1}{p}} \left\|D_x\left(\frac{a+b}{2},.\right)\right\|_p + \|D_x\|_p\right]}{4(d-c)^{\frac{q-1}{q}}} \left[\frac{(b-a)}{q+1}\right]^{\frac{1}{q}} \\ & + \frac{\left[(d-c)^{\frac{1}{p}} \left\|D_y\left(.,\frac{c+d}{2}\right)\right\|_p + \|D_y\|_p\right]}{4(b-a)^{\frac{q-1}{q}}} \left[\frac{(d-c)}{q+1}\right]^{\frac{1}{q}} \\ & + \frac{\left[G_1(\frac{a+b}{2}) + G_2(\frac{a+b}{2},\frac{d+c}{2}) + G_3(\frac{d+c}{2}) + \|D_{xy}\|_p\right]}{16} \\ & \times \left[\frac{(b-a)}{q+1}\right]^{\frac{1}{q}} \left[\frac{(d-c)}{q+1}\right]^{\frac{1}{q}} \end{aligned} \tag{3.16}$$

where

$$G_1\left(\frac{a+b}{2}\right) := (b-a)^{\frac{1}{p}} \left\| D_{xy}\left(\frac{a+b}{2},.\right) \right\|_p ,$$

$$G_2\left(\frac{a+b}{2},\frac{d+c}{2}\right) := (b-a)^{\frac{1}{p}} (d-c)^{\frac{1}{p}} \left| D_{xy}\left(\frac{a+b}{2},\frac{d+c}{2}\right) \right| ,$$

and

$$G_3(\frac{d+c}{2}) := (d-c)^{\frac{1}{p}} \left\| D_{xy}\left(., \frac{d+c}{2}\right)\right\|_p .$$

3.3. Mappings whose First Derivative Belongs to $L_1[[a,b]\times[c,d]]$.

In this section an inequality of Ostrowski type involving two-dimensional integrals for functions whose partial derivatives belong to L_1 and satisfy certain convexity properties can be produced as shown in the following theorem.

Theorem 3. *With the assumption in Theorem 1, we have*

$$\begin{aligned}
(3.17)\quad & \left|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)\,dsdt - f(x_0,y_0)\right| \\
&\le \frac{[(b-a)\|D_x(x_0,.)\|_1 + \|D_x\|_1]}{2(b-a)(d-c)}\left[\frac{1}{2}(b-a)+\left|x_0-\frac{a+b}{2}\right|\right] \\
&+ \frac{[(d-c)\|D_y(.,y_0)\|_1 + \|D_y\|_1]}{2(b-a)(d-c)}\left[\frac{1}{2}(d-c)+\left|y_0-\frac{c+d}{2}\right|\right] \\
&+ \frac{[S_1(x_0)+S_2(x_0,y_0)+S_3(y_0)+\|D_{xy}\|_1]}{4(b-a)(d-c)} \\
&\times\left[\frac{1}{2}(b-a)+\left|x_0-\frac{a+b}{2}\right|\right]\left[\frac{1}{2}(d-c)+\left|y_0-\frac{c+d}{2}\right|\right].
\end{aligned}$$

for all $(x_0,y_0)\in[a,b]\times[c,d]$, *where*

$$S_1(x_0) := (b-a)\|D_{xy}(x_0,.)\|_1,$$
$$S_2(x_0,y_0) := (b-a)(d-c)|D_{xy}(x_0,y_0)|,$$
$$S_3(y_0) := (d-c)\|D_{xy}(.,y_0)\|_1,$$
$$\|D_x(x_0,.)\|_1 := \int_c^d |D_x(x_0,s)|ds < \infty, \quad \|D_y(.,y_0)\|_1 := \int_a^b |D_y(t,y_0)|dt < \infty,$$
$$\|D_x\|_1 := \int_a^b\int_c^d |D_x(t,s)|\,dsdt < \infty,$$
$$\|D_{xy}(x_0,.)\|_1 := \int_c^d |D_{xy}(x_0,s)|ds < \infty, \quad \|D_{xy}(.,y_0)\|_1 := \int_a^b |D_{xy}(t,y_0)|dt < \infty,$$

and

$$\|D_{xy}\|_1 := \int_a^b\int_c^d |D_x(t,s)|\,dsdt < \infty.$$

Proof. Utilizing the equations (3.2) to (3.5) we can write

$$\begin{aligned}
(3.18)\quad & \left|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d (x_0-t)\left(\int_0^1 D_x[(1-\lambda)x_0+\lambda t,s]\,d\lambda\right)dsdt\right| \\
&\le \frac{[(b-a)\|D_x(x_0,.)\|_1 + \|D_x\|_1]}{2(b-a)(d-c)}\sup_{t\in[a,b]}|x_0-t| \\
&\le \frac{[(b-a)\|D_x(x_0,.)\|_1 + \|D_x\|_1]}{2(b-a)(d-c)}\left[\frac{1}{2}(b-a)+\left|x_0-\frac{a+b}{2}\right|\right]
\end{aligned}$$

since

$$\sup_{t\in[a,b]}|x_0-t| = \mathrm{m}ax\{x_0-a, b-x_0\} = \left[\frac{1}{2}(b-a)+\left|x_0-\frac{a+b}{2}\right|\right]$$

where we have used the fact that

$$\mathrm{m}ax\{X,Y\} = \frac{X+Y}{2}+\left|\frac{Y-X}{2}\right|.$$

Similarly, we can deduce that

$$\left|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d (y_0-s)\left(\int_0^1 D_y\left[t,(1-\mu)y_0+\mu s\right]d\mu\right)dsdt\right| \le \frac{\left[(d-c)\left\|D_y(.,y_0)\right\|_1+\left\|D_y\right\|_1\right]}{2(b-a)(d-c)}\left[\frac{1}{2}(d-c)+\left|y_0-\frac{c+d}{2}\right|\right] \tag{3.19}$$

and

$$\begin{aligned}\left|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d (x_0-t)(y_0-s)\right.&\\ \times\left(\int_0^1\int_0^1 D_{xy}\left[(1-\lambda)x_0+\lambda t,(1-\mu)y_0+\mu s\right]d\mu d\lambda\right)dsdt\Bigg|&\\ \le \frac{\left[S_1(x_0)+S_2(x_0,y_0)+S_3(y_0)+\left\|D_{xy}\right\|_1\right]}{4(b-a)(d-c)}&\\ \times\left[\frac{1}{2}(b-a)+\left|x_0-\frac{a+b}{2}\right|\right]\left[\frac{1}{2}(d-c)+\left|y_0-\frac{c+d}{2}\right|\right].&\end{aligned} \tag{3.20}$$

Utilizing (3.18), (3.19) and (3.20), substituting into (3.3), thus the Theorem 3 is completely proved. ∎

Again the best result can be given at $x_0=\frac{a+b}{2}$ and $y_0=\frac{c+d}{2}$, as shown in the following corollary

Corollary 3. *Under the above assumptions, we have*

$$\begin{aligned}\left|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)\,dsdt - f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|&\\ \le \frac{\left[(b-a)\left\|D_x\left(\frac{a+b}{2},.\right)\right\|_1+\left\|D_x\right\|_1\right]}{4(d-c)}&\\ +\frac{\left[(d-c)\left\|D_y\left(.,\frac{c+d}{2}\right)\right\|_1+\left\|D_y\right\|_1\right]}{4(b-a)}&\\ +\frac{\left[S_1(\frac{a+b}{2})+S_2(\frac{a+b}{2},\frac{c+d}{2})+S_3(\frac{c+d}{2})+\left\|D_{xy}\right\|_1\right]}{16}.&\end{aligned} \tag{3.21}$$

where

$$S_1(\frac{a+b}{2}) := (b-a)\left\|D_{xy}\left(\frac{a+b}{2},.\right)\right\|_1,$$

$$S_2(\frac{a+b}{2},\frac{c+d}{2}) := (b-a)(d-c)\left|D_{xy}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|,$$

and

$$S_3\left(\frac{c+d}{2}\right) := (d-c)\left\|D_{xy}\left(\cdot, \frac{c+d}{2}\right)\right\|_1 .$$

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