ON A NEW HILBERT TYPE INEQUALITY AND ITS APPLICATIONS

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ABSTRACT. This paper gives a new Hilbert's type inequality with the same best constant factor π. As its applications, we give its equivalent form and some particular results.

1. INTRODUCTION

If $a_n, b_n \geq 0 \quad (n \in N_0 = N \cup \{0\})$, such that $0 < \sum_{n=0}^{\infty} a_n^2 < \infty$, and $0 < \sum_{n=0}^{\infty} b_n^2 < \infty$, then the famous Hilbert's inequality (see Hardy et al. [1]) is given by

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{a_m b_n}{m+n+1} < \pi\left(\sum_{n=0}^{\infty}a_n^2\sum_{n=0}^{\infty}b_n^2\right)^{\frac{1}{2}}, \tag{1.1}$$

where the constant factor π is the best possible. Its equivalent form is

$$\sum_{n=0}^{\infty}\left(\sum_{m=0}^{\infty}\frac{a_m}{m+n+1}\right)^2 < \pi^2\sum_{n=0}^{\infty}a_n^2, \tag{1.2}$$

where the constant factor π^2 is still the best possible (see [2]).

Inequality (1.1) is important in analysis and its applications (see [3]). Recently, Yang and Debnath [2, 4] and Yang [5, 6] gave (1.1) some new extensions and improvements. Kuang and Debnath [7] considered its strengthened versions and some generalisations, and Pachpatte [8] built some new inequalities similar to (1.1).

The major objective of this paper is to obtain a new inequality similar to (1.1) but other than [8], which relates to the best constant factor π and the double series form as

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{\ln m + \ln n + \frac{3}{4}} = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{\ln e^{\frac{3}{4}}mn}.$$

For this, we build some lemmas and estimate the following weight coefficient

$$\omega(n) = \sum_{m=1}^{\infty}\frac{1}{m\ln e^{\frac{3}{4}}mn}\left(\frac{\ln e^{\frac{3}{8}}n}{\ln e^{\frac{3}{8}}m}\right)^{\frac{1}{2}}, \quad (n \in N). \tag{1.3}$$

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2. Some Lemmas

Let f have its first four derivatives on $[1,\infty)$, $(-1)^n f^{(n)}(x) > 0$ $(n = 0,1,2,3,4)$, $\int_1^\infty f(x)\,dx < \infty$, and $f(x),\ f'(x) \to 0$ $(x \to \infty)$, then (see [7, (2.1)])

$$\sum_{k=1}^{\infty} f(k) < \int_1^\infty f(x)\,dx + \frac{1}{2}f(1) - \frac{1}{12}f'(1). \tag{2.1}$$

Lemma 1. *For $n \in N$, define the function $R(n)$ as*

$$\begin{aligned} R(n) &= \left(\frac{3}{8\ln e^{\frac{3}{8}}n}\right)^{\frac{1}{2}} \int_0^{3/\left(8\ln e^{3/8}n\right)} \frac{1}{(1+u)\,u^{\frac{1}{2}}}du \\ &\quad -\frac{25}{36\ln e^{\frac{3}{4}}n} - \frac{1}{12\left(\ln e^{\frac{3}{4}}n\right)^2}. \end{aligned} \tag{2.2}$$

Then we have $R(n) > 0$ $(n \in N)$.

Proof. Integrating by parts, we find

$$\begin{aligned} & \int_0^{3/\left(8\ln e^{3/8}n\right)} \frac{1}{(1+u)\,u^{\frac{1}{2}}}du \\ = \;& 2\int_0^{3/\left(8\ln e^{3/8}n\right)} \frac{1}{(1+u)}du^{\frac{1}{2}} \\ = \;& 2\left(\frac{1}{1+u}\right)u^{\frac{1}{2}}\Big|_0^{3/\left(8\ln e^{3/8}n\right)} - 2\int_0^{3/\left(8\ln e^{3/8}n\right)} u^{\frac{1}{2}}d\frac{1}{(1+u)} \\ = \;& \frac{2\ln e^{\frac{3}{8}}n}{\ln e^{\frac{3}{4}}n}\left(\frac{3}{8\ln e^{\frac{3}{8}}n}\right)^{\frac{1}{2}} + 2\int_0^{3/\left(8\ln e^{3/8}n\right)} u^{\frac{1}{2}}\frac{1}{(1+u)^2}du \\ = \;& \frac{2\ln e^{\frac{3}{8}}n}{\ln e^{\frac{3}{4}}n}\left(\frac{3}{8\ln e^{\frac{3}{8}}n}\right)^{\frac{1}{2}} + \frac{4}{3}\int_0^{3/\left(8\ln e^{3/8}n\right)} \frac{1}{(1+u)^2}du^{\frac{3}{2}} \\ = \;& \frac{2\ln e^{\frac{3}{8}}n}{\ln e^{\frac{3}{4}}n}\left(\frac{3}{8\ln e^{\frac{3}{8}}n}\right)^{\frac{1}{2}} \\ & +\frac{4}{3}\left[\frac{u^{\frac{3}{2}}}{(1+u)^2}\Big|_0^{3/\left(8\ln e^{3/8}n\right)} + 2\int_0^{3/\left(8\ln e^{3/8}n\right)} \frac{1}{(1+u)^3}u^{\frac{3}{2}}du\right] \\ > \;& \frac{2\ln e^{\frac{3}{8}}n}{\ln e^{\frac{3}{4}}n}\left(\frac{3}{8\ln e^{\frac{3}{8}}n}\right)^{\frac{1}{2}} + \frac{4}{3}\cdot\frac{u^{\frac{3}{2}}}{(1+u)^2}\Big|_0^{3/\left(8\ln e^{3/8}n\right)} \\ = \;& \frac{3}{4\ln e^{\frac{3}{4}}n}\left(\frac{3}{8\ln e^{\frac{3}{8}}n}\right)^{-\frac{1}{2}} + \frac{3}{16\left(\ln e^{\frac{3}{4}}n\right)^2}\left(\frac{3}{8\ln e^{\frac{3}{8}}n}\right)^{-\frac{1}{2}}. \end{aligned}$$

Hence, by (2.2), we have

$$\begin{aligned} R(n) &> \frac{3}{4\ln e^{\frac{3}{4}}n}+\frac{3}{16\left(\ln e^{\frac{3}{4}}n\right)^2}-\frac{25}{36}\cdot\frac{1}{\ln e^{\frac{3}{4}}n}-\frac{1}{12\left(\ln e^{\frac{3}{4}}n\right)^2} \\ &= \frac{1}{18\ln e^{\frac{3}{4}}n}+\frac{5}{48\left(\ln e^{\frac{3}{4}}n\right)^2}>0 \quad (n\in N). \end{aligned}$$

The lemma is proved. ∎

Lemma 2. *If $\omega(n)$ is defined by (1.3), then we have $\omega(n)<\pi$ for $n\in N$.*

Proof. For fixed $n\in N$, setting $f_n(x)$ as

$$f_n(x)=\frac{1}{x\ln e^{\frac{3}{4}}nx}\left(\frac{\ln e^{\frac{3}{8}}n}{\ln e^{\frac{3}{8}}x}\right)^{\frac{1}{2}}, \quad x\in[1,\infty),$$

then we find $f_n(x)$ is with the assumptions of (2.1),

$$f_n(1)=\frac{1}{\ln e^{\frac{3}{4}}n}\left(\frac{8\ln e^{\frac{3}{8}}n}{3}\right)^{\frac{1}{2}},$$

and

$$\begin{aligned} f_n'(1) &= \left[-\frac{1}{x^2\ln e^{\frac{3}{4}}nx}\left(\frac{\ln e^{\frac{3}{8}}n}{\ln e^{\frac{3}{8}}x}\right)^{\frac{1}{2}}-\frac{1}{x^2\ln^2 e^{\frac{3}{4}}nx}\left(\frac{\ln e^{\frac{3}{8}}n}{\ln e^{\frac{3}{8}}x}\right)^{\frac{1}{2}}\right. \\ &\qquad \left.-\frac{1}{2x^2\ln e^{\frac{3}{4}}nx}\cdot\frac{\left(\ln e^{\frac{3}{8}}n\right)^{\frac{1}{2}}}{\left(\ln e^{\frac{3}{8}}x\right)^{\frac{3}{2}}}\right]_{x=1} \\ &= -\left(\frac{7}{3\ln e^{\frac{3}{4}}n}+\frac{1}{\ln^2 e^{\frac{3}{4}}n}\right)\left(\frac{8\ln e^{\frac{3}{8}}n}{3}\right)^{\frac{1}{2}}. \end{aligned}$$

Putting $u=\frac{\ln e^{\frac{3}{8}}x}{\ln e^{\frac{3}{8}}n}$, we obtain

$$\begin{aligned} &\int_1^\infty f_n(x)\,dx \\ =&\int_1^\infty \frac{1}{x\left(\ln e^{\frac{3}{8}}n\right)\left[1+\left(\ln e^{\frac{3}{8}}x\right)/\left(\ln e^{\frac{3}{8}}n\right)\right]}\left(\frac{\ln e^{\frac{3}{8}}n}{\ln e^{\frac{3}{8}}x}\right)^{\frac{1}{2}}dx \\ =&\int_{3/\left(8\ln e^{3/8}n\right)}^\infty \frac{1}{1+u}\left(\frac{1}{u}\right)^{\frac{1}{2}}du=\pi-\int_0^{3/\left(8\ln e^{3/8}n\right)}\frac{1}{1+u}\left(\frac{1}{u}\right)^{\frac{1}{2}}du. \end{aligned}$$

Hence by (2.1), (2.2) and the above results, we have

$$\begin{aligned}\omega(n) &= \sum_{m=1}^{\infty} f_n(m) < \int_1^{\infty} f_n(x)\,dx + \frac{1}{2} f_n(1) - \frac{1}{12} f_n'(1) \\ &= \pi - \int_0^{3/\left(8\ln e^{3/8} n\right)} \frac{1}{1+u}\left(\frac{1}{u}\right)^{\frac{1}{2}} du + \frac{1}{2\ln e^{\frac{3}{4}} n}\left(\frac{8\ln e^{\frac{3}{8}} n}{3}\right)^{\frac{1}{2}} \\ &\quad + \frac{1}{12}\left(\frac{7}{3\ln e^{\frac{3}{4}} n} + \frac{1}{\ln^2 e^{\frac{3}{4}} n}\right)\left(\frac{8\ln e^{\frac{3}{8}} n}{3}\right)^{\frac{1}{2}} \\ &= \pi - \left(\frac{8\ln e^{\frac{3}{8}} n}{3}\right)^{\frac{1}{2}} R(n).\end{aligned}$$

In view of $R(n) > 0$ in Lemma 1, we have $\omega(n) < \pi$. The lemma is proved. ∎

Lemma 3. *For* $0 < \varepsilon < 1$, *we have*

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{1}{mn\ln e^{\frac{3}{4}} mn} \cdot \frac{1}{\left(\ln e^{\frac{3}{8}} m\right)^{(1+\varepsilon)/2}} \cdot \frac{1}{\left(\ln e^{\frac{3}{8}} n\right)^{(1+\varepsilon)/2}} \tag{2.3}$$
$$\geq \frac{1}{\varepsilon}(\pi + o(1)) \quad \left(\varepsilon \to 0^+\right).$$

Proof. For $x, y \geq 1$, setting $u = \frac{\ln e^{\frac{3}{8}} x}{\ln e^{\frac{3}{8}} y}$, we find

$$\begin{aligned}&\int_1^{\infty} \frac{1}{x\ln e^{\frac{3}{4}} xy}\left(\frac{\ln e^{\frac{3}{8}} y}{\ln e^{\frac{3}{8}} x}\right)^{\frac{1+\varepsilon}{2}} dx \\ &= \int_{3/\left(8\ln e^{3/8} y\right)}^{\infty} \frac{1}{1+u}\left(\frac{1}{u}\right)^{\frac{1+\varepsilon}{2}} du \\ &= \int_0^{\infty} \frac{1}{1+u}\left(\frac{1}{u}\right)^{\frac{1+\varepsilon}{2}} du - \int_0^{3/\left(8\ln e^{3/8} y\right)} \frac{1}{1+u}\left(\frac{1}{u}\right)^{\frac{1+\varepsilon}{2}} du \\ &\geq \int_0^{\infty} \frac{1}{1+u}\left(\frac{1}{u}\right)^{\frac{1+\varepsilon}{2}} du - \int_0^{3/\left(8\ln e^{3/8} y\right)} \left(\frac{1}{u}\right)^{\frac{1+\varepsilon}{2}} du \\ &= \int_0^{\infty} \frac{1}{1+u}\left(\frac{1}{u}\right)^{\frac{1+\varepsilon}{2}} du - \frac{2}{1-\varepsilon}\left(\frac{3}{8\ln e^{3/8} y}\right)^{\frac{1-\varepsilon}{2}}.\end{aligned}$$

Hence we have

$$\begin{aligned}&\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{1}{mn\ln e^{\frac{3}{4}} mn} \cdot \frac{1}{\left(\ln e^{\frac{3}{8}} m\right)^{(1+\varepsilon)/2}} \cdot \frac{1}{\left(\ln e^{\frac{3}{8}} n\right)^{(1+\varepsilon)/2}} \\ &> \int_1^{\infty}\int_1^{\infty} \frac{1}{xy\ln e^{\frac{3}{4}} xy} \cdot \frac{1}{\left(\ln e^{\frac{3}{8}} x\right)^{(1+\varepsilon)/2}\left(\ln e^{\frac{3}{8}} y\right)^{(1+\varepsilon)/2}} dxdy\end{aligned}$$

$$
\begin{aligned}
&= \int_1^\infty \frac{1}{y}\left(\frac{1}{\ln e^{\frac{3}{8}}y}\right)^{1+\varepsilon}\left[\int_1^\infty \frac{1}{x\ln e^{\frac{3}{4}}xy}\left(\frac{\ln e^{\frac{3}{8}}y}{\ln e^{\frac{3}{8}}x}\right)^{\frac{1+\varepsilon}{2}}dx\right]dy \\
&\geq \int_1^\infty \frac{1}{y}\left(\frac{1}{\ln e^{\frac{3}{8}}y}\right)^{1+\varepsilon}dy\int_0^\infty \frac{1}{1+u}\left(\frac{1}{u}\right)^{\frac{1+\varepsilon}{2}}du \\
&\quad -\frac{2}{1-\varepsilon}\left(\frac{3}{8}\right)^{\frac{1-\varepsilon}{2}}\int_1^\infty \frac{1}{y}\left(\frac{1}{\ln e^{\frac{3}{8}}y}\right)^{\frac{1+\varepsilon}{2}+1} \\
&= \frac{1}{\varepsilon}\left(\frac{8}{3}\right)^{\varepsilon}(\pi+o(1))-\left(\frac{2}{1-\varepsilon}\right)\left(\frac{2}{1+\varepsilon}\right)\left(\frac{8}{3}\right)^{\varepsilon} \\
&\geq \frac{1}{\varepsilon}\left(\frac{8}{3}\right)^{\varepsilon}(\pi+o(1))-O(1)=\frac{1}{\varepsilon}(\pi+o(1))\quad\left(\varepsilon\to 0^+\right).
\end{aligned}
$$

The lemma is proved. ∎

3. Main Result and Some Applications

Theorem 1. *If $r, s \in R$, and $a_n, b_n \geq 0$, such that*

$$0<\sum_{n=1}^{\infty} n^{1-2r}a_n^2<\infty$$

and

$$0<\sum_{n=1}^{\infty} n^{1-2s}b_n^2<\infty,$$

then we have

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m^r n^s\ln e^{\frac{3}{4}}mn}<\pi\left(\sum_{n=1}^{\infty}n^{1-2r}a_n^2\sum_{n=1}^{\infty}n^{1-2s}b_n^2\right)^{\frac{1}{2}}, \tag{3.1}$$

where the constant factor π is the best possible. In particular,

(a) *for $r=s=\frac{1}{2}$, we have*

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{\sqrt{mn}\ln e^{\frac{3}{4}}mn}<\pi\left(\sum_{n=1}^{\infty}a_n^2\sum_{n=1}^{\infty}b_n^2\right)^{\frac{1}{2}}; \tag{3.2}$$

(b) *for $r=s=1$, we have*

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{mn\ln e^{\frac{3}{4}}mn}<\pi\left(\sum_{n=1}^{\infty}\frac{a_n^2}{2}\sum_{n=1}^{\infty}\frac{b_n^2}{2}\right)^{\frac{1}{2}}; \tag{3.3}$$

(c) *for $r=s=0$, we have*

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{\ln e^{\frac{3}{4}}mn}<\pi\left(\sum_{n=1}^{\infty}na_n^2\sum_{n=1}^{\infty}nb_n^2\right)^{\frac{1}{2}}. \tag{3.4}$$

Proof. By Cauchy's inequality and (1.3), we have

$$
\begin{aligned}
&\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m^r n^s \ln e^{\frac{3}{4}}mn}\\
=\ &\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left[\frac{a_m}{\left(\ln e^{\frac{3}{4}}mn\right)^{\frac{1}{2}}}\left(\frac{\ln e^{\frac{3}{8}}m}{\ln e^{\frac{3}{8}}n}\right)^{\frac{1}{4}}\left(\frac{m^{\frac{1}{2}-r}}{n^{\frac{1}{2}}}\right)\right]\\
&\qquad\times\left[\frac{b_n}{\left(\ln e^{\frac{3}{4}}mn\right)^{\frac{1}{2}}}\left(\frac{\ln e^{\frac{3}{8}}n}{\ln e^{\frac{3}{8}}m}\right)^{\frac{1}{4}}\left(\frac{n^{\frac{1}{2}-s}}{m^{\frac{1}{2}}}\right)\right]\\
\le\ &\left[\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m^2}{\ln e^{\frac{3}{4}}mn}\left(\frac{\ln e^{\frac{3}{8}}m}{\ln e^{\frac{3}{8}}n}\right)^{\frac{1}{2}}\left(\frac{m^{1-2r}}{n}\right)\right.\\
&\qquad\left.\times\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{b_n^2}{\ln e^{\frac{3}{4}}mn}\left(\frac{\ln e^{\frac{3}{8}}n}{\ln e^{\frac{3}{8}}m}\right)^{\frac{1}{2}}\left(\frac{n^{1-2s}}{m}\right)\right]^{\frac{1}{2}}\\
=\ &\left(\sum_{m=1}^{\infty}\omega(m)\,m^{1-2r}a_m^2\sum_{n=1}^{\infty}\omega(n)\,n^{1-2s}b_n^2\right)^{\frac{1}{2}}.
\end{aligned}
$$

In view of $\omega(n) < \pi$ in Lemma 2, we have (3.1).

For $0 < \varepsilon < 1$, setting $\tilde{a}_n$ and $\tilde{b}_n$ as:

$$
\tilde{a}_n = \frac{1}{n^{1-r}\left(\ln e^{\frac{3}{8}}n\right)^{\frac{1+\varepsilon}{2}}},\quad \tilde{b}_n = \frac{1}{n^{1-s}\left(\ln e^{\frac{3}{8}}n\right)^{\frac{1+\varepsilon}{2}}},\quad n \in N,
$$

then we have

$$
\begin{aligned}
\sum_{n=1}^{\infty}n^{1-2r}\tilde{a}_n^2 &= \sum_{n=1}^{\infty}n^{1-2s}\tilde{b}_n^2 = \sum_{n=1}^{\infty}\frac{1}{n\left(\ln e^{\frac{3}{8}}n\right)^{1+\varepsilon}}\\
&= \sum_{n=1}^{2}\frac{1}{n\left(\ln e^{\frac{3}{8}}n\right)^{1+\varepsilon}} + \sum_{n=3}^{\infty}\frac{1}{n\left(\ln e^{\frac{3}{8}}n\right)^{1+\varepsilon}}\\
&< \sum_{n=1}^{2}\frac{1}{n\left(\ln e^{\frac{3}{8}}n\right)^{1+\varepsilon}} + \int_{e^{5/8}}^{\infty}\frac{1}{x\left(\ln e^{\frac{3}{8}}x\right)^{1+\varepsilon}}dx\\
&= \frac{1}{\left(\frac{3}{8}\right)^{1+\varepsilon}} + \frac{1}{2\left(\ln 2e^{\frac{3}{8}}\right)^{1+\varepsilon}} + \frac{1}{\varepsilon}\\
&= \frac{1}{\varepsilon}\left(1+o(1)\right)\quad\left(\varepsilon\to 0^+\right).
\end{aligned}
\tag{3.5}
$$

If the constant factor π in (3.1) is not the best possible, then there exists a positive number $K < \pi$, such that (3.1) is valid if we change π to K. In particular, we have

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\tilde{a}_m\tilde{a}_n}{m^r n^s \ln \alpha mn} < K\left(\sum_{n=1}^{\infty} n^{1-2s}\tilde{a}_n^2 \sum_{n=1}^{\infty} n^{1-2r}\tilde{b}_n^2\right)^{\frac{1}{2}}.$$

Hence by (2.3) and (3.5), we have

$$(\pi + o(1)) \le \varepsilon \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\tilde{a}_m\tilde{a}_n}{m^r n^s \ln \alpha mn} < K(1+o(1)) \quad \left(\varepsilon \to 0^+\right).$$

It follows that $\pi \le K$. This contradicts the fact that $K < \pi$. Hence the constant factor π in (3.1) is the best possible. The theorem is proved. ∎

Remark 1. *Inequality (3.2) is similar to the following (see* [1, Theorem 321]*):*

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\lambda_m^{1/2}\mu_n^{1/2}}{\Lambda_m + M_n} < \pi\left(\sum_{n=1}^{\infty} a_n^2 \sum_{n=1}^{\infty} b_n^2\right)^{\frac{1}{2}} \tag{3.6}$$
$$\left(\Lambda_m = \sum_{i=1}^{m}\lambda_i, \quad M_n = \sum_{i=1}^{n}\mu_i, \quad \lambda_i, \mu_i > 0\right).$$

Inequality (3.3) is more profound than the Mulholland's inequality as (see [9]*):*

$$\sum_{n=2}^{\infty}\sum_{m=2}^{\infty}\frac{a_m b_n}{mn \ln mn} < \pi\left(\sum_{n=2}^{\infty}\frac{a_n^2}{n}\sum_{n=2}^{\infty}\frac{b_n^2}{n}\right)^{\frac{1}{2}}, \tag{3.7}$$

and inequality (3.4) is similar to (1.1). Inequality (3.1) is a new Hilbert's type inequality with the same best constant factor π and two parameters r and s.

Theorem 2. *If $r \in R$, $a_n \ge 0$, and $0 < \sum_{n=1}^{\infty} n^{1-2r} a_n^2 < \infty$, then we have*

$$\sum_{n=1}^{\infty}\frac{1}{n}\left(\sum_{m=1}^{\infty}\frac{a_m}{m^r \ln e^{\frac{3}{4}} mn}\right)^2 < \pi^2 \sum_{n=1}^{\infty} n^{1-2r} a_n^2, \tag{3.8}$$

where the constant factor π^2 is the best possible. Inequality (3.8) is equivalent to (3.1). In particular,

(a) *for $r = \frac{1}{2}$, we have*

$$\sum_{n=1}^{\infty}\frac{1}{n}\left(\sum_{m=1}^{\infty}\frac{a_m}{\sqrt{m}\ln e^{\frac{3}{4}} mn}\right)^2 < \pi^2 \sum_{n=1}^{\infty} a_n^2; \tag{3.9}$$

(b) *for $r = 1$, we have*

$$\sum_{n=1}^{\infty}\frac{1}{n}\left(\sum_{m=1}^{\infty}\frac{a_m}{m\ln e^{\frac{3}{4}} mn}\right)^2 < \pi^2 \sum_{n=1}^{\infty}\frac{a_n^2}{n}; \tag{3.10}$$

(c) *for $r = 0$, we have*

$$\sum_{n=1}^{\infty}\frac{1}{n}\left(\sum_{m=1}^{\infty}\frac{a_m}{\ln e^{\frac{3}{4}} mn}\right)^2 < \pi^2 \sum_{n=1}^{\infty} n a_n^2. \tag{3.11}$$

Proof. Since

$$0<\sum_{n=1}^{\infty} n^{1-2r}a_n^2<\infty,$$

there exists $k_0\geq 1$, such that for any $k>k_0$, we have

$$0<\sum_{n=1}^{k} n^{1-2r}a_n^2<\infty,$$

and

$$b_n(k)=\frac{1}{n^{\frac{1}{2}}}\sum_{n=1}^{k}\frac{a_m}{m^r\ln e^{\frac{3}{4}}mn}>0\quad(n\in N).$$

By (3.1), for $s=\frac{1}{2}$, and setting $b_n(k)=a_n\equiv 0\ (n>k)$, we have

$$(3.12)\qquad 0<\left[\sum_{n=1}^{k} b_n^2(k)\right]^2=\left[\sum_{n=1}^{k}\frac{1}{n}\left(\sum_{m=1}^{k}\frac{a_m}{m^r\ln e^{\frac{3}{4}}mn}\right)^2\right]^2$$

$$=\left[\sum_{n=1}^{k}\sum_{m=1}^{k}\frac{a_m b_n(k)}{m^r n^{\frac{1}{2}}\ln e^{\frac{3}{4}}mn}\right]^2<\pi^2\sum_{n=1}^{k} n^{1-2r}a_n^2\sum_{n=1}^{k} b_n^2(k).$$

Thus we find

$$(3.13)\qquad 0<\sum_{n=1}^{k}\frac{1}{n}\left(\sum_{m=1}^{k}\frac{a_m}{m^r\ln e^{\frac{3}{4}}mn}\right)^2=\sum_{n=1}^{k} b_n^2(k)<\pi^2\sum_{n=1}^{k} n^{1-2r}a_n^2.$$

It follows that

$$0<\sum_{n=1}^{\infty} b_n^2(\infty)\leq\pi^2\sum_{n=1}^{\infty} n^{1-2r}a_n^2<\infty.$$

Hence by (3.1), for $k\to\infty$, neither (3.12) nor (3.13) takes the form of equality. We have (3.8).

On the other hand, if (3.8) holds, by Cauchy's inequality, we have

$$(3.14)\qquad \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m^r n^s\ln e^{\frac{3}{4}}mn}$$

$$=\sum_{n=1}^{\infty}\left(\frac{1}{n^{\frac{1}{2}}}\sum_{m=1}^{\infty}\frac{a_m}{m^r\ln e^{\frac{3}{4}}mn}\right)\left(n^{\frac{1}{2}-s}b_n\right)$$

$$\leq\left[\sum_{n=1}^{\infty}\frac{1}{n}\left(\sum_{m=1}^{\infty}\frac{a_m}{m^r\ln e^{\frac{3}{4}}mn}\right)^2\sum_{n=1}^{\infty} n^{1-2s}b_n^2\right]^{\frac{1}{2}}.$$

By (3.8), we have (3.1).

Hence inequalities (3.1) and (3.8) are equivalent. If the constant factor π^2 in (3.8) is not the best possible, we may show that the constant factor π in (3.1) is not the best possible by using (3.14). This is a contradiction. The theorem is proved. ∎

Remark 2. *Inequality (3.11) is similar to (1.2), which is equivalent to (3.4). Since inequality (3.1) and its equivalent form (3.8) are all with the best constant factors, we give some new results.*

References

[1] Hardy, G.H., Littlewood, J.E, Polya, G., *Inequalities,* Cambridge Univ. Press, 1934.

[2] Yang, Bicheng, Debnath, L., On a new generalisation of Hardy-Hilbert's inequality and its applications, *J. Math. Anal. Appl.,* (1999) 233: 484-497.

[3] Mitrinović, D.S., Pečarić, J.E, Fink, A.M., *Inequalities Involving Functions and their Integrals and Derivatives,* Kluwer Academic Publishers, 1991.

[4] Yang, Bicheng, Debnath, L., Some inequality involving π and an application to Hilbert's inequality, *Applied Math. Letters,* 1999, 12:101-105.

[5] Yang, Bicheng, On a strengthened version of the more accurate Hardy-Hilbert's inequality, *Acta Math. Sinica,* 1999, **42**(6):1103-1110.

[6] Yang, Bicheng, On a strengthened Hardy-Hilbert's inequality, *J. Ineq. Pure and Appl. Math.,* 2000, **1**(2), Art. 22.

[7] Kuang, Jichang, Debnath, L., On new generalisations of Hilbert's inequality and their applications, *J. Math. Anal. Appl., 2000, 245:248-265.*

[8] Pachpatte, B.G., On some inequalities similar to Hilbert's inequality, *J. Math. Anal. Appl., 1998, 226:166-179.*

[9] Mulholland, H.P., Some theorems on Dirichlet series with coefficients and related integrals, *Proc. London Math. Soc.,* 1929, **29**(2):281-292.

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