Computation of Constants in the Strengthened Cauchy Inequality for Box Spline-Wavelets

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Abstract

In this paper we give explicit estimates of the constants γ in the strengthened Cauchy–Schwarz inequality for box spline–wavelets. The results are obtained by a combination of functional analysis methods and numerical computations.

1. Introduction

Let H be a Hilbert space. The usual Cauchy–Schwarz inequality is refined by the strengthened one in the sense that it states the existence of a constant $\gamma \in [0, 1)$ such that

$$|(v, w)| \leq \gamma \|v\| \|w\| \tag{1.1}$$

for $v \in V$, $w \in W$, where V, W are linear subspaces of H with $V \cap W = \{0\}$ and γ depends only on V and W, and not on the choice of the functions v and w. The smallest such quantity γ may be called the cosine of the angle between the spaces V and W. When H is finite dimensional, then obviously $\gamma < 1$, whereas for an infinite dimensional H the validity of (1.1) with $\gamma \in [0, 1)$ depends on V and W; in both cases a non–trivial problem consists in giving precise estimates for γ.

In view of some applications to Numerical Analysis (see the references in [3]) it is interesting to take as H a function space and consider wavelet subspaces V, W. In this connection since now, as far as we know, the only existing result is a strengthened Cauchy–Schwarz inequality for general biorthogonal wavelets defined on the real line and on an interval (that is for the one–dimensional case) [3]; see also [4] for a particular family of spline–wavelets in the bidimensional case.

In the present paper we set $H = L^2(\mathbf{R})$ and we give explicit estimates for the constants γ in the case of box spline–wavelets, which represent the simplest non–orthonormal examples; see [1,2]. For benefit of non–expert readers we recall the definition and set what V and W will be in our case. Consider first

$$\varphi(x) = \begin{cases} 1, & \text{if } 0 \leq x < 1 \\ 0, & \text{otherwise} \end{cases}$$

$$\psi^3(x) = -\frac{1}{8}\varphi(2x+2)-\frac{1}{8}\varphi(2x+1)+\varphi(2x)-\varphi(2x-1)+\frac{1}{8}\varphi(2x-2)+\frac{1}{8}\varphi(2x-3),$$

$$\begin{aligned}\psi^5(x) &= \frac{3}{128}\varphi(2x+4)+\frac{3}{128}\varphi(2x+3)-\frac{11}{64}\varphi(2x+2)\\ &- \frac{11}{64}\varphi(2x+1)+\varphi(2x)-\varphi(2x-1)+\frac{11}{64}\varphi(2x-2)\\ &+ \frac{11}{64}\varphi(2x-3)-\frac{3}{128}\varphi(2x-4)-\frac{3}{128}\varphi(2x-5),\end{aligned}$$

and let for $j \in \mathbf{R}$

$$V_j = \text{span}_H\{\varphi_{jk},\, k \in \mathbf{Z}\}$$
$$W_j^3 = \text{span}_H\{\psi_{jk}^3,\, k \in \mathbf{Z}\},$$
$$W_j^5 = \text{span}_H\{\psi_{jk}^5,\, k \in \mathbf{Z}\}$$

where

$$\varphi_{jk} = 2^{\frac{j}{2}}\varphi(2^j x - k)$$
$$\psi_{jk}^{3,5} = 2^{\frac{j}{2}}\psi^{3,5}(2^j x - k)$$

(for the details of the construction see [1,2]).

We shall give estimates for γ in the two cases: $V = V_j$, $W = W_j^3$ and $V = V_j$, $W = W_j^5$. Namely, we shall prove

Theorem 1.1. *Define*

$$\gamma_{3,5} = \sup \frac{|(v,w)|}{\|v\|\|w\|}$$

v running in V_j, w in $W_j^{3,5}$. Then

$$\gamma_3 \leq 0.2424242425,$$

corresponding to $\gamma_3 \leq \frac{8}{33}$, and

$$\gamma_5 \leq 0.3624657741.$$

The proof of the theorem will be a combination of functional analysis and numerical computations, performed with Maple.

2. Abstract estimates of Cauchy constants

We give here the following preliminary result, valid in any Hilbert space H.

Theorem 2.1. *Let V and W be closed linear subspaces of H. Let $\{\varphi_k\}_{k\in\mathbf{Z}}$ be an orthogonal basis for V. Assume further that $\{\psi_k\}_{k\in\mathbf{Z}}$ is a basis for W,* *quasi–orthogonal in the following sense:* $(\psi_j, \psi_k) = 0$ *if* $|k-j| > N$ *for a suitable integer* N*, and define*

$$\frac{(\psi_j, \psi_k)}{\|\psi_j\| \|\psi_k\|} = \delta_{jk}, \ \textit{for } 0 < |k-j| \leq N. \tag{2.1}$$

Assume

$$\delta_{j,j-1} = \delta_{j,j+1} = \delta_1, \dots, \delta_{j,j-N} = \delta_{j,j+N} = \delta_N \tag{2.2}$$

are independent on j*. Suppose now* $(\varphi_j, \psi_k) = 0$ *for* $j < k - N'$ *or* $j > k + N''$*, for some* N'*,* N'' *with* $N' + N'' = N$*; define then*

$$\gamma_{jk} = \frac{(\varphi_j, \psi_k)}{\|\varphi_j\| \|\psi_k\|} \ \textit{for } k - N' \leq j \leq k + N'' \tag{2.3}$$

and set

$$\gamma_{k-N',k} = \gamma_{-N'}, \dots, \gamma_{k+N'',k} = \gamma_{N''} \tag{2.4}$$

independent on k*. Then*

$$|(v, w)| \leq \gamma \|v\| \|w\|$$

with

$$\gamma \leq \left(\sum_{-N' \leq h \leq N''} \gamma_h \right) \frac{1}{\sqrt{1 - 2\left(\sum_{0<h\leq N} \delta_h\right)}}. \tag{2.5}$$

Proof. Take $v = \sum_{j \in \mathbf{Z}} v_j$, with $v_j = c_j \varphi_j$, $w = \sum_{k \in \mathbf{Z}} w_k$, with $w_k = c'_k \psi_k$. We have from (2.3)

$$\begin{aligned} |(v, w)| &= \left| \left(\sum_j v_j, \sum_k w_k \right) \right| \\ &\leq \sum_k \left(\sum_j |(v_j, w_k)| \right) \\ &\leq \sum_k \left(\sum_{k-N' \leq j \leq k+N''} \gamma_{jk} \|v_j\| \|w_k\| \right). \end{aligned}$$

Then from (2.4) it follows

$$\begin{aligned} |(v, w)| &\leq \sum_{-N' \leq h \leq N''} \left(\gamma_h \sum_k \|w_k\| \|v_{k+h}\| \right) \\ &\leq \sum_{-N' \leq h \leq N''} \left(\gamma_h \sqrt{\sum_k \|w_k\|^2} \sqrt{\sum_k \|v_{k+h}\|^2} \right). \end{aligned}$$

Using the orthogonality of φ_j, $j \in \mathbf{Z}$, we obtain

$$|(v,w)| \le \sum_{-N' \le h \le N''} \gamma_h \|v\| \sqrt{\sum_k \|w_k\|^2}. \tag{2.6}$$

We shall prove now that

$$\sum_k \|w_k\|^2 \le \frac{1}{1 - 2\left(\sum_{0<h\le N} \delta_h\right)} \|w\|^2, \tag{2.7}$$

which combined with (2.6) proves (2.5). To this end we begin to write

$$\begin{aligned} \|w\|^2 &= (w,w) = \left(\sum_j w_j, \sum_k w_k\right) \\ &= \sum_k (w_k, w_k) + \sum_{j,k,j\ne k} (w_j, w_k) \end{aligned}$$

so we have

$$\|w\|^2 = \sum_k \|w_k\|^2 + \sum_{j,k,0<|k-j|\le N} (w_j, w_k)\,. \tag{2.8}$$

On the other hand from (2.1) and (2.2) we deduce

$$\begin{aligned} \left| \sum_{k,j,0<|k-j|\le N} (w_j, w_k) \right| &\le \sum_{k,j,0<|k-j|\le N} \delta_{jk} \|w_j\| \|w_k\| \\ &\le \frac{1}{2} \sum_{k,j,0<|k-j|\le N} \delta_{|k-j|} \|w_j\|^2 + \frac{1}{2} \sum_{k,j,0<|k-j|\le N} \delta_{|k-j|} \|w_k\|^2 \\ &= \sum_{0<|h|\le N} \delta_{|h|} \left(\sum_k \|w_{k-h}\|^2 \right) \end{aligned}$$

then we have

$$\left| \sum_{k,j,0<|k-j|\le N} (w_j, w_k) \right| \le 2 \left(\sum_{0<h<N} \delta_h \right) \left(\sum_k \|w_k\|^2 \right). \tag{2.9}$$

Combining (2.8) and (2.9) we obtain

$$\sum_k \|w_k\|^2 - 2 \left(\sum_{0<h\le N} \delta_h \right) \left(\sum_k \|w_k\|^2 \right) = \left(1 - 2 \left(\sum_{0<h\le N} \delta_h \right) \right) \left(\sum_k \|w_k\|^2 \right) \le \|w\|^2.$$

Hence (2.7) follows; this concludes the proof of theorem 2.1.

△

3. Proof of the theorem 1.1.

We first observe that it is sufficient to argue on the case $V = V_0$, $W = W_0^{3,5}$, as it follows easily by the change of variable $y = 2^j x$. We then apply theorem 2.1 taking

$$\varphi_k = \varphi(x-k), \quad \psi_k = \psi^{3,5}(x-k).$$

The assumptions of theorem 2.1 are satisfied with $N = 2$ or 4. In (2.1) and (2.2) we have now

case $W = W^3$:

$$\begin{aligned} \delta_1^3 &= 0 \\ \delta_2^3 &= 0.01515151515, \end{aligned}$$

case $W = W^5$:

$$\begin{aligned} \delta_1^5 &= 0.007599309154 \\ \delta_2^5 &= 0.02786413356 \\ \delta_3^5 &= 0.007599309154 \\ \delta_4^5 &= 0.0005181347150. \end{aligned}$$

In (2.3) and (2.4) we have

case $W = W^3$: $N_3' = N_3'' = 1$ and

$$\begin{aligned} \gamma_{-1}^3 &= 0.1230914910 \\ \gamma_0^3 &= 0 \\ \gamma_1^3 &= 0.1230914910, \end{aligned}$$

case $W = W^5$: $N_5' = N_5'' = 2$ and

$$\begin{aligned} \gamma_{-2}^5 &= 0.02276257268 \\ \gamma_{-1}^5 &= 0.1669255330 \\ \gamma_0^5 &= 0 \\ \gamma_1^5 &= 0.1669255330 \\ \gamma_2^5 &= 0.02276257268. \end{aligned}$$

Inserting in (2.5) we obtain the conclusion.

References

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