GENERALISATIONS OF A CERTAIN INEQUALITY USED IN THE THEORY OF FINITE DIFFERENCE EQUATIONS

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Abstract. In the present paper we establish some generalisations of a certain inequality used in the theory of finite difference and sum difference equations. The inequalities that we propose here can be used as handy tools in the theory of general classes of finite difference and sum-difference equations.

1. Introduction

In recent years considerable interest has been shown in developing various aspects of numerical analysis, both for its own sake and for its applications. Discrete iterative models have all along been the basis of many new developments that are presently taking place in various branches of numerical analysis.

Discrete inequalities has become a major tool in the qualitative analysis of such models. During the past twenty years or so many useful discrete inequalities have been found in the literature in order to study the various discrete models, see [1, 2], [4] – [10] and the references given therein. In 1973, the present author [5] established the following useful discrete inequality.

Lemma 1. *(Pachpatte [5]). Let $u(n)$, $h(n)$ and $k(n)$ be real-valued nonnegative functions defined on N_0, for which the inequality*

$$u(n) \leq u_0 + \sum_{s=0}^{n-1} h(s)\, u(s) + \sum_{s=0}^{n-1} h(s) \left(\sum_{t=0}^{s-1} k(t)\, u(t) \right),$$

holds for all $n \in N_0$, where u_0 is a nonnegative constant. Then

$$u(n) \leq u_0 \left[1 + \sum_{s=0}^{n-1} h(s) \prod_{t=0}^{s-1} [1 + h(t) + k(t)] \right],$$

for all $n \in N_0$.

The importance of this inequality lies in the fruitful utilization in various applications given in [5] for which the earlier inequalities in the literature do not apply directly. Many generalisations, extensions and variants of the above inequality and their various applications can be found in [6] – [8]. The main purpose of the present paper is to establish some new discrete generalisation and variants of the above inequality by using a fairly elementary analysis. For the integral inequalities similar to that we propose here, see [3, 4, 11, 12]. The inequalities established here can be used as tools in the study of qualitative nature of solutions of general classes

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of finite difference and sum-difference equations. We also present some immediate applications to convey the importance of our results to the literature.

2. Statement of Results

In what follows, we denote by $\mathbb{R}$ the set of real numbers and $N_0 = \{0, 1, 2, \dots\}$. For any function $u(m)$, $m \in N_0$, we define the operator Δ by $\Delta u(m) = u(m+1) - u(m)$ and for $i \geq 2$, $\Delta^i u(m) = \Delta\left(\Delta^{i-1} u(m)\right)$. The operators L_j are recursively defined by

$$L_0 u(m) = u(m), \quad L_j u(m) = \frac{1}{r_j(m)} \Delta L_{j-1} u(m), \quad j = 1, 2, \dots, n.$$

with $r_n(m) = 1$, where $u(m)$ and $r_j(m) > 0$ are some functions defined on N_0. We use the following notations for simplification of details of presentation. For $m \in N_0$, $s_0 = m$ and some functions $f_i(m) \geq 0$, $p_i(m) > 0$, $i = 1, 2, \dots, n$ we set

$$\begin{aligned}
&\sum_{k=1}^{n}\left(\sum_{s_1=0}^{m-1} p_1(s_1) \sum_{s_2=0}^{s_1-1} p_2(s_2) \cdots \sum_{s_{k-1}=0}^{s_{k-2}-1} p_{k-1}(s_{k-1}) \sum_{s_k=0}^{s_{k-1}-1} p_k(s_k) f_k(s_k)\right) \\
= \; & \sum_{s_1=0}^{m-1} p_1(s_1) f_1(s_1) + \sum_{s_1=0}^{m-1} p_1(s_1) \sum_{s_2=0}^{s_1-1} p_2(s_2) f_2(s_2) \\
& + \cdots \\
& + \sum_{s_1=0}^{m-1} p_1(s_1) \sum_{s_2=0}^{s_1-1} p_2(s_2) \cdots \sum_{s_{n-1}=0}^{s_{n-2}-1} p_{n-1}(s_{n-1}) \times \sum_{s_n=0}^{s_{n-1}-1} p_n(s_n) f_n(s_n).
\end{aligned}$$

For $m \in N_0$, $s_0 = m$ and some functions $q(m)$ and $r_j(m) > 0$, $j = 1, 2, \dots, n-1$, we set

$$\begin{aligned}
A[m, r, q(s_n)] &= A[m, r_1, \dots, r_{n-1}, q(s_n)] \\
&= \sum_{s_1=0}^{m-1} r_1(s_1) \sum_{s_2=0}^{s_1-1} r_2(s_2) \cdots \sum_{s_{n-1}=0}^{s_{n-2}-1} r_{n-1}(s_{n-1}) \times \sum_{s_n=0}^{s_{n-1}-1} q(s_n),
\end{aligned}$$

$$\begin{aligned}
\bar{A}[s_1, r, q(s_n)] &= \bar{A}[s_1, r_1, r_2, \dots, r_{n-1}, q(s_n)] \\
&= r_1(s_1) \sum_{s_1=0}^{s_2-1} r_2(s_2) \cdots \sum_{s_{n-1}=0}^{s_{n-2}-1} r_{n-1}(s_{n-1}) \times \sum_{s_n=0}^{s_{n-1}-1} q(s_n).
\end{aligned}$$

For $m_1 > m_2$, $m_1, m_2 \in N_0$ and any function $u(m)$ defined on N_0, we use the usual conventions

$$\sum_{s=m_1}^{m_2} u(s) = 0 \quad \text{and} \quad \prod_{s=m_1}^{m_2} u(s) = 1.$$

Our first result deals with some useful generalisations of the inequality given in Lemma 1.

Theorem 1. *Let $f_i(m) \geq 0$, $p_i(m) > 0$, for $i = 1, 2, \dots, n$, be real-valued functions defined on N_0 and c be a nonnegative real constant.*

(a_1) *Let $u(m) \geq 0$ be a real-valued function defined on N_0. If*

$$u(m) \leq c + \sum_{k=1}^{n} \left(\sum_{s_1=0}^{m-1} p_1(s_1) \sum_{s_2=0}^{s_1-1} p_2(s_2) \times \cdots \times \sum_{s_{k-1}=0}^{s_{k-2}-1} p_{k-1}(s_{k-1}) \sum_{s_k=0}^{s_{k-1}-1} p_k(s_k) f_k(s_k) u(s_k) \right), \tag{2.1}$$

for $m \in N_0$ with $s_0 = m$, then

$$u(m) \leq c \prod_{s_1=0}^{m-1} \left[1 + p_1(s_1) f_1(s_1) + p_1(s_1) \sum_{k=2}^{n} \left(\sum_{s_2=0}^{s_1-1} p_2(s_2) \times \cdots \times \sum_{s_{k-1}=0}^{s_{k-2}-1} p_{k-1}(s_{k-1}) \sum_{s_k=0}^{s_{k-1}-1} p_k(s_k) f_k(s_k) \right) \right], \tag{2.2}$$

for $m \in N_0$.

(a_2) *Let $u(m) \geq u_0 \geq 0$ be a real-valued function defined on N_0, u_0 is a real constant. Let $w(u)$ be a continuous nondecreasing real-valued function defined on $I = [u_0, \infty)$ and $w(u) > 0$ on (u_0, ∞), $w(u_0) = 0$. If*

$$u(m) \leq c + \sum_{k=1}^{n} \left(\sum_{s_1=0}^{m-1} p_1(s_1) \sum_{s_2=0}^{s_1-1} p_2(s_2) \times \cdots \times \sum_{s_{k-1}=0}^{s_{k-2}-1} p_{k-1}(s_{k-1}) \sum_{s_k=0}^{s_{k-1}-1} p_k(s_k) f_k(s_k) w(u(s_k)) \right), \tag{2.3}$$

for $m \in N_0$ with $s_0 = m$, then for $0 \leq m \leq m_1$,

$$u(m) \leq \Omega^{-1} \left[\Omega(c) + \sum_{k=1}^{n} \left(\sum_{s_1=0}^{m-1} p_1(s_1) \sum_{s_2=0}^{s_1-1} p_2(s_2) \times \cdots \times \sum_{s_{k-1}=0}^{s_{k-2}-1} p_{k-1}(s_{k-1}) \sum_{s_k=0}^{s_{k-1}-1} p_k(s_k) f_k(s_k) \right) \right], \tag{2.4}$$

where

$$\Omega(r) = \int_{r_0}^{r} \frac{ds}{w(s)}, \quad r \geq r_0 \quad \text{wuth} \quad r_0 > u_0, \tag{2.5}$$

Ω^{-1} is the inverse of Ω and $m_1 \in N_0$ be chosen so that

$$\Omega(c) + \sum_{k=1}^{n} \left(\sum_{s_1=0}^{m-1} p_1(s_1) \sum_{s_2=0}^{s_1-1} p_2(s_2) \times \cdots \times \sum_{s_{k-1}=0}^{s_{k-2}-1} p_{k-1}(s_{k-1}) \sum_{s_k=0}^{s_{k-1}-1} p_k(s_k) f_k(s_k) \right) \in Dom\left(\Omega^{-1}\right),$$

for all $m \in N_0$ lying in $0 \leq m \leq m_1$.

In the following theorem we establish the variants of inequalities given in Theorem 1 which may be convenient in some applications.

Theorem 2. *Let* $f_i(m) \geq 0$, $p_i(m) > 0$, *for* $i = 1, 2, \ldots, n$, *be real-valued functions defined on* N_0 *and* c *be a nonnegative real constant.*

(b_1) *Let* $u(m) \geq 0$ *be a real-valued function defined on* N_0. *If*

$$u^2(m) \leq c^2 + 2\sum_{k=1}^{n}\left(\sum_{s_1=0}^{m-1} p_1(s_1) \sum_{s_2=0}^{s_1-1} p_2(s_2) \times \cdots \times \sum_{s_{k-1}=0}^{s_{k-2}-1} p_{k-1}(s_{k-1}) \sum_{s_k=0}^{s_{k-1}-1} p_k(s_k) f_k(s_k) u(s_k)\right), \tag{2.6}$$

for $m \in N_0$ *with* $s_0 = m$, *then*

$$u(m) \leq c + \sum_{k=1}^{n}\left(\sum_{s_1=0}^{m-1} p_1(s_1) \sum_{s_2=0}^{s_1-1} p_2(s_2) \times \cdots \times \sum_{s_{k-1}=0}^{s_{k-2}-1} p_{k-1}(s_{k-1}) \sum_{s_k=0}^{s_{k-1}-1} p_k(s_k) f_k(s_k)\right), \tag{2.7}$$

for $m \in N_0$.

(b_2) *Let* $u(m) \geq u_0 \geq 0$ *be a real-valued function defined on* N_0, u_0 *is a real constant. Let* $w(u), \Omega, \Omega^{-1}$ *be as defined in* (a_2) *in Theorem 1. If*

$$u^2(m) \leq c^2 + 2\sum_{k=1}^{n}\left(\sum_{s_1=0}^{m-1} p_1(s_1) \sum_{s_2=0}^{s_1-1} p_2(s_2) \times \cdots \times \sum_{s_{k-1}=0}^{s_{k-2}-1} p_{k-1}(s_{k-1}) \sum_{s_k=0}^{s_{k-1}-1} p_k(s_k) f_k(s_k) u(s_k) w(u(s_k))\right), \tag{2.8}$$

for $m \in N_0$ *with* $s_0 = m$, *then for* $0 \leq m \leq m_2$,

$$u(m) \leq \Omega^{-1}\left[\Omega(c) + \sum_{k=1}^{n}\left(\sum_{s_1=0}^{m-1} p_1(s_1) \sum_{s_2=0}^{s_1-1} p_2(s_2) \times \cdots \times \sum_{s_{k-1}=0}^{s_{k-2}-1} p_{k-1}(s_{k-1}) \sum_{s_k=0}^{s_{k-1}-1} p_k(s_k) f_k(s_k)\right),\right] \tag{2.9}$$

where $m_2 \in N_0$ *is chosen so that*

$$\Omega(c) + \sum_{k=1}^{n}\left(\sum_{s_1=0}^{m-1} p_1(s_1) \sum_{s_2=0}^{s_1-1} p_2(s_2) \times \cdots \times \sum_{s_{k-1}=0}^{s_{k-2}-1} p_{k-1}(s_{k-1}) \sum_{s_k=0}^{s_{k-1}-1} p_k(s_k) f_k(s_k)\right) \in Dom\left(\Omega^{-1}\right),$$

for all $m \in N_0$ *lying in* $0 \leq m \leq m_2$.

Our next theorem deals with some new generalisations of Lemma 1 which can be used in certain applications.

Theorem 3. *Let $f(m) \geq 0$, $g(m) \geq 0$, $r_i(m) > 0$, for $i = 1, 2, \ldots, n-1$, be real-valued functions defined on N_0 and c be a nonnegative real constant.*

(c_1) *Let $u(m) \geq 0$ be a real-valued function defined on N_0. If*

$$u(m) \leq c + A[m, r, f(s_n)(u(s_n) + A[s_n, r, g(t_n) u(t_n)])], \tag{2.10}$$

for $m \in N_0$ with $s_0 = m$, then

$$u(m) \leq c\left[1 + A\left[m, r, f(s_n) \prod_{t_1=0}^{s_n-1} \left[1 + \bar{A}[t_1, r, [f(t_n) + g(t_n)]]\right]\right]\right], \tag{2.11}$$

for $m \in N_0$.

(c_2) *Let $u(m) \geq u_0 \geq 0$ be a real-valued function defined on N_0, u_0 is a real constant. Let $H(u)$ be a continuous nondecreasing real-valued function defined on $I = [u_0, \infty)$ and $H(u) > 0$ on (u_0, ∞), $H(u_0) = 0$. If*

$$u(m) \leq c + A[m, r, f(s_n)(u(s_n) + A[s_n, r, g(t_n) H(u(t_n))])], \tag{2.12}$$

for $m \in N_0$ with $s_0 = m$, then for $0 \leq m \leq m_3$,

$$u(m) \leq c + A\left[m, r, f(s_n) G^{-1}[G(c) + A[s_n, r, [f(t_n) + g(t_n)]]]\right], \tag{2.13}$$

where

$$G(\sigma) = \int_{\sigma_0}^{\sigma} \frac{ds}{s + H(s)}, \quad \sigma \geq \sigma_0 \quad \text{with} \quad \sigma_0 > u_0, \tag{2.14}$$

G^{-1} is the inverse of G and $m_3 \in N_0$ be chosen so that

$$G(c) + A[m, r, [f(t_n) + g(t_n)]] \in Dom\left(G^{-1}\right),$$

for $m \in N_0$ lying in $0 \leq m \leq m_3$.

3. Proof of Theorem 1

(a_1) We first assume that $c > 0$ and define a function $z(m)$ by the right side of (2.1). Then it is easy to observe that

$$\frac{\Delta z(m)}{p_1(m)} - f_1(m) u(m) = z_1(m), \tag{3.1}$$

where

$$z_1(m) = \sum_{k=2}^{n} \left(\sum_{s_2=0}^{m-1} p_2(s_2) \sum_{s_3=0}^{s_2-1} p_3(s_3) \times \cdots \times \sum_{s_{k-1}=0}^{s_{k-2}-1} p_{k-1}(s_{k-1}) \sum_{s_k=0}^{s_{k-1}-1} p_k(s_k) f_k(s_k) u(s_k) \right), \tag{3.2}$$

for $m \in N_0$ with $s_1 = m$. From (3.2) we observe that

$$\frac{\Delta z_1(m)}{p_2(m)} - f_2(m) u(m) = z_2(m), \tag{3.3}$$

where

$$z_2(m) = \sum_{k=3}^{n} \left(\sum_{s_3=0}^{m-1} p_3(s_3) \sum_{s_4=0}^{s_3-1} p_4(s_4) \times \cdots \times \sum_{s_{k-1}=0}^{s_{k-2}-1} p_{k-1}(s_{k-1}) \sum_{s_k=0}^{s_{k-1}-1} p_k(s_k) f_k(s_k) u(s_k) \right), \tag{3.4}$$

for $m \in N_0$ with $s_2 = m$. Continuing this way we have

$$\frac{\Delta z_{n-2}(m)}{p_{n-1}(m)} - f_{n-1}(m) u(m) = z_{n-1}(m), \tag{3.5}$$

where

$$z_{n-1}(m) = \sum_{s_n=0}^{m-1} p_n(s_n) f_n(s_n) u(s_n), \tag{3.6}$$

for $m \in N_0$. From (3.6) and using the fact that $u(m) \leq z(m)$ we get

$$\Delta z_{n-1}(m) \leq p_n(m) f_n(m) z(m). \tag{3.7}$$

If $h(s) \geq 0$ and $h(0) = 0$ for $s \in N_0$, and $z(m)$ is as defined above, then by using the summation by parts formula, it is easy to observe that

$$\begin{aligned} \sum_{s=0}^{m-1} \frac{\Delta h(s)}{z(s)} &= \left(\frac{h(m)}{z(m)} - \frac{h(0)}{z(0)} \right) - \sum_{s=0}^{m-1} \Delta \left(\frac{1}{z(s)} \right) h(s+1) \\ &= \frac{h(m)}{z(m)} + \sum_{s=0}^{m-1} \frac{\Delta z(s)}{z(s) z(s+1)} h(s+1) \\ &\geq \frac{h(m)}{z(m)}. \end{aligned}$$

Using (3.8) and (3.7) it is easy to observe that

$$\frac{z_{n-1}(m)}{z(m)} \leq \sum_{s_n=0}^{m-1} \frac{\Delta z_{n-1}(s_n)}{z(s_n)} \leq \sum_{s_n=0}^{m-1} p_n(s_n) f_n(s_n). \tag{3.8}$$

Using (3.8), (3.5) and (3.8) and the fact that $u(m) \leq z(m)$ we observe that

$$\begin{aligned} &\frac{z_{n-2}(m)}{z(m)} \qquad (3.9) \\ \leq\ & \sum_{s_{n-1}=0}^{m-1} \frac{\Delta z_{n-2}(s_{n-1})}{z(s_{n-1})} \\ =\ & \sum_{s_{n-1}=0}^{m-1} \frac{p_{n-1}(s_{n-1}) f_{n-1}(s_{n-1}) u(s_{n-1}) + p_{n-1}(s_{n-1}) z_{n-1}(s_{n-1})}{z(s_{n-1})} \\ \leq\ & \sum_{s_{n-1}=0}^{m-1} p_{n-1}(s_{n-1}) f_{n-1}(s_{n-1}) + \sum_{s_{n-1}=0}^{m-1} p_{n-1}(s_{n-1}) \sum_{s_n=0}^{s_{n-1}-1} p_n(s_n) f_n(s_n). \end{aligned}$$

Proceeding in this way we get

$$\frac{z_1(m)}{z(m)} \le \sum_{k=2}^{n} \left(\sum_{s_2=0}^{m-1} p_2(s_2) \sum_{s_3=0}^{s_2-1} p_3(s_3) \times \cdots \times \sum_{s_{k-1}=0}^{s_{k-2}-1} p_{k-1}(s_{k-1}) \sum_{s_k=0}^{s_{k-1}-1} p_k(s_k) f_k(s_k) \right). \tag{3.10}$$

From (3.1), (3.10) and using the fact that $u(m) \le z(m)$ we observe that

$$\frac{\Delta z(m)}{z(m)} \le p_1(m) f_1(m) + p_1(m) \sum_{k=2}^{n} \left(\sum_{s_2=0}^{m-1} p_2(s_2) \sum_{s_3=0}^{s_2-1} p_3(s_3) \times \cdots \times \sum_{s_{k-1}=0}^{s_{k-2}-1} p_{k-1}(s_{k-1}) \sum_{s_k=0}^{s_{k-1}-1} p_k(s_k) f_k(s_k) \right). \tag{3.11}$$

The inequality (3.11) implies the estimate (see [5, 6])

$$z(m) \le c \prod_{s_1=0}^{m-1} \left[1 + p_1(s_1) f_1(s_1) + p_1(s_1) \sum_{k=2}^{n} \left(\sum_{s_2=0}^{s_1-1} p_2(s_2) \times \cdots \times \sum_{s_{k-1}=0}^{s_{k-2}-1} p_{k-1}(s_{k-1}) \sum_{s_k=0}^{s_{k-1}-1} p_k(s_k) f_k(s_k) \right) \right]. \tag{3.12}$$

Now, using (3.12) in $u(m) \le z(m)$ we get the desired inequality in (2.2).

If $c \ge 0$, we carry out the above procedure with $c + \varepsilon$ instead of c, where $\varepsilon > 0$ is an arbitrary small constant, and subsequently pass to the limit as $\varepsilon \to 0$ to obtain (2.2). This completes the proof of (a_1).

(a_2) Assume that $c > 0$ and define a function $z(m)$ by the right side of (2.3). Then by following the same steps as in the proof of part (a_1) with suitable changes we get

$$\frac{\Delta z(m)}{w(z(m))} \le p_1(m) f_1(m) + p_1(m) \sum_{k=2}^{n} \left(\sum_{s_2=0}^{m-1} p_2(s_2) \sum_{s_3=0}^{s_2-1} p_3(s_3) \times \cdots \times \sum_{s_{k-1}=0}^{s_{k-2}-1} p_{k-1}(s_{k-1}) \sum_{s_k=0}^{s_{k-1}-1} p_k(s_k) f_k(s_k) \right). \tag{3.13}$$

From (2.5) and (3.13) we have

$$\begin{aligned}\Omega(z(m+1))-\Omega(z(m)) &= \int_{z(m)}^{z(m+1)}\frac{ds}{w(s)}\\ &\le \frac{\Delta z(m)}{w(z(m))}\\ &\le p_1(m)f_1(m)+p_1(m)\sum_{k=2}^{n}\left(\sum_{s_2=0}^{m-1}p_2(s_2)\sum_{s_3=0}^{s_2-1}p_3(s_3)\right.\\ &\qquad\left.\times\cdots\times\sum_{s_{k-1}=0}^{s_{k-2}-1}p_{k-1}(s_{k-1})\sum_{s_k=0}^{s_{k-1}-1}p_k(s_k)f_k(s_k)\right).\end{aligned}$$

Now set $m=s_1$ in (3.14) and sum over $s_1=0,1,2,\ldots,m-1$ to obtain the estimate

$$\Omega(z(m))\le\Omega(c)+\sum_{k=1}^{n}\left(\sum_{s_1=0}^{m-1}p_1(s_1)\sum_{s_2=0}^{s_1-1}p_2(s_2)\times\cdots\right.$$
$$\left.\times\sum_{s_{k-1}=0}^{s_{k-2}-1}p_{k-1}(s_{k-1})\sum_{s_k=0}^{s_{k-1}-1}p_k(s_k)f_k(s_k)\right).\tag{3.14}$$

The desired inequality in (2.4) follows from (3.14) and the fact that $u(m)\le z(m)$. The subdomain of N_0 for m is obvious.

The proof of the case when $c\ge 0$ can be completed as mentioned in the proof of part (a_1). This completes the proof of (a_2).

4. Proof of Theorem 2

(b_1) Assume that $c>0$ and define a function $z(m)$ by the right side of (2.6). Then it is easy to observe that

$$\frac{\Delta z(m)}{2p_1(m)}-f_1(m)u(m)=z_1(m),\tag{4.1}$$

where

$$z_1(m)=\sum_{k=2}^{n}\left(\sum_{s_2=0}^{m-1}p_2(s_2)\sum_{s_3=0}^{s_2-1}p_3(s_3)\right.$$
$$\left.\times\cdots\times\sum_{s_{k-1}=0}^{s_{k-2}-1}p_{k-1}(s_{k-1})\sum_{s_k=0}^{s_{k-1}-1}p_k(s_k)f_k(s_k)u(s_k)\right),\tag{4.2}$$

for $m\in N_0$ with $s_1=m$. Now by following the same steps as in the proof of part (a_1), below (3.2) up to (3.11) with suitable changes we have

$$\frac{\Delta z(m)}{2\sqrt{z(m)}}\le p_1(m)f_1(m)+p_1(m)\sum_{k=2}^{n}\left(\sum_{s_2=0}^{m-1}p_2(s_2)\sum_{s_3=0}^{s_2-1}p_3(s_3)\right.$$
$$\left.\times\cdots\times\sum_{s_{k-1}=0}^{s_{k-2}-1}p_{k-1}(s_{k-1})\sum_{s_k=0}^{s_{k-1}-1}p_k(s_k)f_k(s_k)\right).\tag{4.3}$$

By using the facts that $\sqrt{z(m)} > 0$, $\Delta z(m) \geq 0$, $\sqrt{z(m)} \leq \sqrt{z(m+1)}$ for $m \in N_0$ and (4.3) we observe that

$$\begin{aligned}
\Delta\sqrt{z(m)} &= \frac{z(m+1) - z(m)}{\sqrt{z(m+1)} + \sqrt{z(m)}} \\
&\leq \frac{\Delta z(m)}{2\sqrt{z(m)}} \\
&\leq p_1(m) f_1(m) + p_1(m) \sum_{k=2}^{n} \left(\sum_{s_2=0}^{m-1} p_2(s_2) \sum_{s_3=0}^{s_2-1} p_3(s_3) \right. \\
&\qquad \left. \times \cdots \times \sum_{s_{k-1}=0}^{s_{k-2}-1} p_{k-1}(s_{k-1}) \sum_{s_k=0}^{s_{k-1}-1} p_k(s_k) f_k(s_k) \right).
\end{aligned}$$

Now set $m = s_1$ in (4.4) and sum over $s_1 = 0, 1, 2, \ldots, m-1$ to obtain the estimate

$$\begin{aligned}
\sqrt{z(m)} \leq c + \sum_{k=1}^{n} &\left(\sum_{s_1=0}^{m-1} p_1(s_1) \sum_{s_2=0}^{s_1-1} p_2(s_2) \times \cdots \right. \\
&\left. \times \sum_{s_{k-1}=0}^{s_{k-2}-1} p_{k-1}(s_{k-1}) \sum_{s_k=0}^{s_{k-1}-1} p_k(s_k) f_k(s_k) \right).
\end{aligned} \tag{4.4}$$

The required inequality in (2.7) follows from (4.4) and the fact that $u(m) \leq \sqrt{z(m)}$. The proof of the case when $c \geq 0$ can be completed as mentioned in the proof of part (a_1). The proof of (b_1) is complete.

(b_2) Assume that $c > 0$ and define a function $z(m)$ by the right side of (2.8). Then it is easy to observe that

$$\frac{\Delta z(m)}{2p_1(m)} - f_1(m) u(m) w(u(m)) = z_1(m), \tag{4.5}$$

where

$$\begin{aligned}
z_1(m) = \sum_{k=2}^{n} &\left(\sum_{s_2=0}^{m-1} p_2(s_2) \sum_{s_3=0}^{s_2-1} p_3(s_3) \right. \\
&\left. \times \cdots \times \sum_{s_{k-1}=0}^{s_{k-2}-1} p_{k-1}(s_{k-1}) \sum_{s_k=0}^{s_{k-1}-1} p_k(s_k) f_k(s_k) u(s_k) w(u(s_k)) \right),
\end{aligned} \tag{4.6}$$

for $m \in N_0$ with $s_1 = m$. Now by following the same steps as in the proof of part (b_1) with suitable changes we have

$$\begin{aligned}
\sqrt{z(m)} \leq c + \sum_{k=1}^{n} &\left(\sum_{s_1=0}^{m-1} p_1(s_1) \sum_{s_2=0}^{s_1-1} p_2(s_2) \times \cdots \right. \\
&\left. \times \sum_{s_{k-1}=0}^{s_{k-2}-1} p_{k-1}(s_{k-1}) \sum_{s_k=0}^{s_{k-1}-1} p_k(s_k) f_k(s_k) w\left(\sqrt{z(s_k)}\right) \right).
\end{aligned} \tag{4.7}$$

Now an application of the inequality established in part (a_2) in Theorem 1 yields

$$\sqrt{z(m)} \le \Omega^{-1}\left[\Omega(c) + \sum_{k=1}^{n}\left(\sum_{s_1=0}^{m-1} p_1(s_1)\sum_{s_2=0}^{s_1-1} p_2(s_2) \times \cdots \times \sum_{s_{k-1}=0}^{s_{k-2}-1} p_{k-1}(s_{k-1})\sum_{s_k=0}^{s_{k-1}-1} p_k(s_k) f_k(s_k)\right)\right]. \tag{4.8}$$

The desired inequality in (2.9) follows from (4.8) and using the fact that $u(m) \le \sqrt{z(m)}$. The subdomain of N_0 for m is obvious. The proof of the case when $c \ge 0$ can be completed as mentioned in the proof of part (a_1) in Theorem 1. The proof is complete.

5. Proof of Theorem 3

(c_1) Assume that $c > 0$ and define a function $z(m)$ by the right side of (2.10). Then from the definition of $z(m)$ and using the fact that $u(m) \le z(m)$, it is easy to observe that

$$L_n z(m) \le f(m)\left(z(m) + A\left[m, r, g(t_n) z(t_n)\right]\right). \tag{5.1}$$

Define a function $v(m)$ by

$$v(m) = z(m) + A\left[m, r, g(t_n) z(t_n)\right]. \tag{5.2}$$

From (5.2) and using (5.1) and the facts that $z(m) \le v(m)$ and $v(m) \le v(m+1)$, we observe that

$$\Delta\left(\frac{L_{n-1}v(m)}{v(m)}\right) \le \left[f(m) + g(m)\right]. \tag{5.3}$$

Now, set $m = t_n$ in (5.3) and sum over $t_n = 0, 1, 2, \ldots, m-1$, to obtain the estimate

$$\frac{L_{n-1}v(m)}{v(m)} \le \sum_{t_n=0}^{m-1}\left[f(t_n) + g(t_n)\right]. \tag{5.4}$$

Here we have used the fact that $L_{n-1}v(0) = 0$. Again, as above, from (5.4) we observe that

$$\Delta\left(\frac{L_{n-2}v(m)}{v(m)}\right) \le r_{n-1}(m)\sum_{t_n=0}^{m-1}\left[f(t_n) + g(t_n)\right]. \tag{5.5}$$

From (5.5) we obtain the estimate

$$\frac{L_{n-2}v(m)}{v(m)} \le \sum_{t_{n-1}=0}^{m-1} r_{n-1}(t_{n-1})\sum_{t_n=0}^{t_{n-1}-1}\left[f(t_n) + g(t_n)\right].$$

Here we have used the fact that $L_{n-2}v\,(0)=0$. Continuing in this way we obtain

$$(5.6)\qquad \frac{\Delta v\,(m)}{v\,(m)} \le r_1\,(m)\sum_{t_2=0}^{m-1} r_2\,(t_2)\sum_{t_3=0}^{t_2-1} r_3\,(t_3) \times\cdots\times \sum_{t_{n-1}=0}^{t_{n-2}-1} r_{n-1}\,(t_{n-1})\sum_{t_n=0}^{t_{n-1}-1}\left[f\,(t_n)+g\,(t_n)\right].$$

The above inequality implies the estimate (see [5, 6])

$$v\,(m) \le c\prod_{t_1=0}^{m-1}\left[1+\bar{A}\left[t_1,r,\left[f\,(t_n)+g\,(t_n)\right]\right]\right].$$

Using this bound on $v\,(m)$ in (5.1) we have

$$(5.7)\qquad L_n z\,(m) \le cf\,(m)\prod_{t_1=0}^{m-1}\left[1+\bar{A}\left[t_1,r,\left[f\,(t_n)+g\,(t_n)\right]\right]\right].$$

The inequality (5.7) implies the estimate (see [9])

$$(5.8)\qquad z\,(m) \le c\left[1+A\left[m,r,f\,(s_n)\prod_{t_1=0}^{s_n-1}\left[1+\bar{A}\left[t_1,r,\left[f\,(t_n)+g\,(t_n)\right]\right]\right]\right]\right].$$

Now, by using (5.8) in $u\,(m)\le z\,(m)$ we get the required inequality in (2.11). The proof of the case when $c\ge 0$ can be completed as mentioned in the proof of Part (a_1) in Theorem 1.

(c_2) Assume that $c>0$ and define a function $z\,(m)$ by the right side of (2.12). Then from the definition of $z\,(m)$ and using the fact that $u\,(m)\le z\,(m)$ we observe that

$$(5.9)\qquad L_n z\,(m) \le f\,(m)\left(z\,(m)+A\left[m,r,g\,(t_n)\,H\,(z\,(t_n))\right]\right).$$

Define a function $v\,(m)$ by

$$(5.10)\qquad v\,(m) = z\,(m)+A\left[m,r,g\,(t_n)\,H\,(z\,(t_n))\right].$$

From (5.10) and using (5.9) and the fact that $z\,(m)\le v\,(m)$, we observe that

$$\begin{aligned} L_n v\,(m) &= L_n z\,(m)+g\,(m)\,H\,(z\,(m)) \\ &\le f\,(m)\,v\,(m)+g\,(m)\,H\,(v\,(m)) \\ &\le \left[f\,(m)+g\,(m)\right](v\,(m)+H\,(v\,(m))). \end{aligned}$$

From (5.11) and using the fact that $v\,(m)\le v\,(m+1)$ it is easy to observe that

$$(5.11)\qquad \Delta\left(\frac{L_{n-1}v\,(m)}{v\,(m)+H\,(v\,(m))}\right) \le \left[f\,(m)+g\,(m)\right].$$

Now, by following the same steps as in the proof of Part (c_1), below (5.3) up to (5.6) with suitable modifications we have

$$\frac{\Delta v(m)}{v(m)+H(v(m))} \leq r_1(m)\sum_{t_2=0}^{m-1} r_2(t_2)\sum_{t_3=0}^{t_2-1} r_3(t_3) \times \cdots \times \sum_{t_{n-1}=0}^{t_{n-2}-1} r_{n-1}(t_{n-1})\sum_{t_n=0}^{t_{n-1}-1}\left[f(t_n)+g(t_n)\right]. \tag{5.12}$$

From (2.14) and (5.12) we observe that

$$\begin{aligned} G(v(m+1))-G(v(m)) &= \int_{v(m)}^{v(m+1)} \frac{ds}{s+H(s)} \\ &\leq \frac{\Delta v(m)}{v(m)+H(v(m))} \\ &\leq r_1(m)\sum_{t_2=0}^{m-1} r_2(t_2)\sum_{t_3=0}^{t_2-1} r_3(t_3) \\ &\quad \times \cdots \times \sum_{t_{n-1}=0}^{t_{n-2}-1} r_{n-1}(t_{n-1})\sum_{t_n=0}^{t_{n-1}-1}\left[f(t_n)+g(t_n)\right]. \end{aligned}$$

Now set $m=t_1$ in (5.13) and sum over $t_1=0,1,2,\ldots,m-1$, to obtain the estimate

$$G(v(m)) \leq G(v(c)) + A\left[m,r,\left[f(t_n)+g(t_n)\right]\right]. \tag{5.13}$$

Now using the bound on $v(m)$ from (5.13) in (5.9) we get

$$L_n z(m) \leq f(m)\,G^{-1}\left[G(c)+A\left[m,r,\left[f(t_n)+g(t_n)\right]\right]\right]. \tag{5.14}$$

The inequality (5.14) implies the estimate (see [9])

$$z(m) \leq c + f(m)\,A\left[m,r,f(s_n)\,G^{-1}\left[G(c)+A\left[s_n,r,\left[f(t_n)+g(t_n)\right]\right]\right]\right]. \tag{5.15}$$

Using (5.15) in $u(m) \leq z(m)$, we get the required inequality in (2.13). The subdomain of N_0 for m is obvious. The proof of the case when $c \geq 0$ can be completed as mentioned in the proof of Part (a_1) of Theorem 1. The proof is complete.

6. Some Applications

In this section we give some applications of the inequality (a_1) in Theorem 1 to study the boundedness and uniqueness of the solutions of a certain sum-difference equation for which the inequalities available in the existing literature do not apply directly. These applications are given as examples.

Example 1. *As a first application, we obtain the bound on the solution of the following sum-difference equation*

$$(6.1)\quad y(m) = F\Bigg(m, q_1(m) + \sum_{s_1=0}^{m-1} p_1(s_1) \sum_{s_2=0}^{s_1-1} p_2(s_2) \times \cdots \times \sum_{s_{n-1}=0}^{s_{n-2}-1} p_{n-1}(s_{n-1}) \sum_{s_n=0}^{s_{n-1}-1} p_n(s_n) h(s_n, y(s_n)),$$
$$q_2(m) + \sum_{s_2=0}^{m-1} p_2(s_2) \sum_{s_3=0}^{s_2-1} p_3(s_3) \times \cdots \times \sum_{s_{n-1}=0}^{s_{n-2}-1} p_{n-1}(s_{n-1}) \sum_{s_n=0}^{s_{n-1}-1} p_n(s_n) h(s_n, y(s_n)),$$
$$\vdots$$
$$q_{n-1}(m) + \sum_{s_{n-1}=0}^{m-1} p_{n-1}(s_{n-1}) \sum_{s_n=0}^{s_{n-1}-1} p_n(s_n) h(s_n, y(s_n)),$$
$$q_n(m) + \sum_{s_n=0}^{m-1} p_n(s_n) h(s_n, y(s_n))\Bigg),$$

where $q_i(m)$, $p_i(m) > 0$ for $i = 1, 2, \ldots, n$ are real-valued functions defined on N_0, and h and F are real-valued functions defined on $N_0 \times \mathbb{R}$ and $N_0 \times \mathbb{R}^n$ respectively. The nonlinear difference equations of the form

$$(6.2)\qquad L_n x(m) = F(m, L_0 x(m), L_1 x(m), \ldots, L_{n-2} x(m), L_{n-1} x(m)),$$

with the given initial conditions

$$(6.3)\qquad L_{i-1} x(0) = C_{i-1}, \quad i = 1, 2, \ldots, n,$$

can be reduced to the particular case of equation (6.1), where C_{i-1} are real constants and the operators L_j, $0 \le j \le n$ are defined as in Section 2.

We assume that

$$(6.4)\qquad \sum_{k=1}^{n} |q_k(m)| \le c,$$

$$(6.5)\qquad |h(m, y(m))| \le f(m) |y(m)|,$$

$$(6.6)\qquad |F(m, z_1, z_2, \ldots, z_{n-1}, z_n)| \le \sum_{k=1}^{n} |z_k|,$$

where $c \ge 0$ is a real constant and $f(m) \ge 0$ is a real-valued function defined on N_0. From (6.1) and (6.4) – (6.6) we observe that

$$(6.7)\quad |y(m)| \le c + \sum_{k=1}^{n} \Bigg(\sum_{s_1=0}^{m-1} p_1(s_1) \sum_{s_2=0}^{s_1-1} p_2(s_2) \times \cdots \times \sum_{s_{k-1}=0}^{s_{k-2}-1} p_{k-1}(s_{k-1}) \sum_{s_k=0}^{s_{k-1}-1} p_k(s_k) f(s_k) |y(s_k)| \Bigg).$$

Now, an application of the inequality given in (a_1) *in Theorem 1 with* $f_i(m) = f(m)$, $i = 1, 2, \dots, n$, *yields*

$$|y(m)| \le c \prod_{s_1=0}^{m-1} \left[1 + p_1(s_1) f_1(s_1) + p_1(s_1) \sum_{k=2}^{n} \left(\sum_{s_2=0}^{s_1-1} p_2(s_2) \times \cdots \times \sum_{s_{k-1}=0}^{s_{k-2}-1} p_{k-1}(s_{k-1}) \sum_{s_k=0}^{s_{k-1}-1} p_k(s_k) f(s_k) \right) \right]. \tag{6.8}$$

The inequality (6.8) gives the bound on the solution $y(m)$ *of the equation (6.1) in terms of known functions. Thus, if the right side of (6.8) is bounded, then we assert that the solution of equation (6.1) is bounded for* $m \in N_0$.

Example 2. *As a second application, we shall discuss the uniqueness of the solutions of equation (6.1). We assume that the functions* h *and* F *in (6.1) satisfy the following conditions*

$$|h(m, y_1(m)) - h(m, y_2(m))| \le f(m) |y_1(m) - y_2(m)|, \tag{6.9}$$

$$|F(m, z_1, z_2, \dots, z_{n-1}, z_n) - F(m, \bar{z}_1, \bar{z}_2, \dots, \bar{z}_{n-1}, \bar{z}_n)| \le \sum_{k=1}^{n} |z_k - \bar{z}_k|, \tag{6.10}$$

where $f(m)$ *is as in Example 1. If* $y_1(m)$ *and* $y_2(m)$ *are any two solutions of equation (6.1), then from (6.1) and (6.9), (6.10) we have*

$$|y_1(m) - y_2(m)| \le \varepsilon + \sum_{k=1}^{n} \left(\sum_{s_1=0}^{m-1} p_1(s_1) \sum_{s_2=0}^{s_1-1} p_2(s_2) \times \cdots \times \sum_{s_{k-1}=0}^{s_{k-2}-1} p_{k-1}(s_{k-1}) \sum_{s_k=0}^{s_{k-1}-1} p_k(s_k) f(s_k) |y_1(s_k) - y_2(s_k)| \right), \tag{6.11}$$

where $\varepsilon > 0$ *is an arbitrary small constant. Now an application of the inequality* (a_1) *in Theorem 1 with* $f_i(m) = f(m)$, $i = 1, 2, \dots, n$, *yields*

$$|y_1(m) - y_2(m)| \le \varepsilon \prod_{s_1=0}^{m-1} \left[1 + p_1(s_1) f_1(s_1) + p_1(s_1) \sum_{k=2}^{n} \left(\sum_{s_2=0}^{s_1-1} p_2(s_2) \times \cdots \times \sum_{s_{k-1}=0}^{s_{k-2}-1} p_{k-1}(s_{k-1}) \sum_{s_k=0}^{s_{k-1}-1} p_k(s_k) f(s_k) \right) \right]. \tag{6.12}$$

Since $\varepsilon > 0$ *is arbitrary we have* $y_1(m) = y_2(m)$, *i.e., there is at most one solution of the equation (6.1).*

In concluding this paper we note that there are many possible applications of the inequalities established in this paper to certain classes of finite difference and sum-difference equations. Various other applications of these will appear elsewhere.

References

[1] A. CORDUNEAU, A discrete integral inequality of the convolution type, *Bull. Inst. Politechnic Iasi. Ser. I: Math. Mech. Fiz.,* **33** (1987), 35-40.

[2] S.S. DRAGOMIR, On discrete generalisation of Pachpatte's inequality and applications, *Bull. Math. Soc. Sci. Math. Roumanie,* **36** (1992), 45-58.

[3] M. MEDVED, Bihari type inequalities with multiple integral and delay, *Periodica Math. Hungar.,* **27** (1993), 207-212.

[4] D.S. MITRINOVIĆ and J.E. PEČARIĆ, *Differential and Integral Inequalities*, Naucna Knjiga Belgrade, 1988.

[5] B.G. PACHPATTE, Finite difference inequalities and their applications, *Proc. Nat. Acad. Sci. India,* **43**(A) (9173), 348-356.

[6] B.G. PACHPATTE, On some new discrete inequalities and their applications to a class of sum-difference equations, *Ann. Sti. Univ. Al. I. Cusa Iasi,* **24** (1978), 315-326.

[7] B.G. PACHPATTE, Finite difference inequalities and discrete time control systems, *Indian J. Pure Appl. Math.,* **9** (1978), 1282-1290.

[8] B.G. PACHPATTE, On some fundamental discrete inequalities in two independent variables, *Tamkang. J. Math.,* **12** (1981), 21-33.

[9] B.G. PACHPATTE, On some new discrete inequalities related to a certain integral inequality, *Libertas Mathematica,* **13** (1993), 85-97.

[10] B.G. PACHPATTE, On some new inequalities related to certain inequalities in the theory of differential equations, *J. Math. Anal. Appl.,* **189** (1995), 128-144.

[11] M. RAB, Linear integral inequalities, *Arch. Math. Scripta Fac. Sci. Nat. Ujep Brunersis,* **15** (1979), 37-46.

[12] E.C. YOUNG, On integral inequalities of Gronwall-Bellman type, *Proc. Amer. Math. Soc.,* **94** (1985), 636-640.

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