MONOTONICITY OF SEQUENCES INVOLVING CONVEX AND CONCAVE FUNCTIONS

CHAO-PING CHEN, FENG QI, PIETRO CERONE, AND SEVER S. DRAGOMIR

Abstract. Let f be an increasing and convex (concave) function on $[0,1)$ and ϕ a positive increasing concave function on $[0,\infty)$ such that $\phi(0)=0$ and the sequence $\bigl\{\phi(i+1)\bigl(\frac{\phi(i+1)}{\phi(i)}-1\bigr)\bigr\}_{i\in\mathbb{N}}$ decreases $\bigl($the sequence $\bigl\{\phi(i)\bigl(\frac{\phi(i)}{\phi(i+1)}-1\bigr)\bigr\}_{i\in\mathbb{N}}$ increases$\bigr)$. Then the sequence $\bigl\{\frac{1}{\phi(n)}\sum_{i=0}^{n-1}f\bigl(\frac{\phi(i)}{\phi(n)}\bigr)\bigr\}_{n\in\mathbb{N}}$ is increasing.

1. Introduction

Let f be a strictly increasing convex (or concave) function in $(0,1]$, J.-Ch. Kuang in [8] verified that

$$\frac{1}{n}\sum_{k=1}^{n}f\Bigl(\frac{k}{n}\Bigr)>\frac{1}{n+1}\sum_{k=1}^{n+1}f\Bigl(\frac{k}{n+1}\Bigr)>\int_0^1 f(x)\,\mathrm{d}x. \tag{1}$$

In [15], the second author generalized the results in [8] and obtained the following main result and some corollaries:

Let f be a strictly increasing convex (or concave) function in $(0,1]$, then the sequence $\frac{1}{n}\sum_{i=k+1}^{n+k}f\bigl(\frac{i}{n+k}\bigr)$ is decreasing in n and k and has a lower bound $\int_0^1 f(t)\,\mathrm{d}t$, that is,

$$\frac{1}{n}\sum_{i=k+1}^{n+k}f\Bigl(\frac{i}{n+k}\Bigr)>\frac{1}{n+1}\sum_{i=k+1}^{n+k+1}f\Bigl(\frac{i}{n+k+1}\Bigr)>\int_0^1 f(t)\,\mathrm{d}t, \tag{2}$$

where k is a nonnegative integer, n a natural number.

With the help of these conclusions, we can deduce Alzer's inequality (see [8]), Minc-Sathre's inequality (see [16]), and other inequalities involving the sum of

2000 *Mathematics Subject Classification.* Primary 26D15; Secondary 26A51.

Key words and phrases. Inequality, concave function, monotonicity.

The first two authors were supported in part by NSF (#10001016) of China, SF for the Prominent Youth of Henan Province, SF of Henan Innovation Talents at Universities, NSF of Henan Province (#004051800), SF for Pure Research of Natural Science of the Education Department of Henan Province (#1999110004), Doctor Fund of Jiaozuo Institute of Technology, China.

This paper was typeset using $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-LaTeX.

powers of positive numbers or the ratios of the arithmetic means of n numbers (see [18, 22]). These inequalities have been investigated by many mathematicians. For more information, please refer to the references in this paper. Some results in another direction can be found in [3] and the book online [4, pp. 20–26].

Considering the convexity of a given function or sequence and using the Hermite-Hadamard inequality in [7, 11], the following results were obtained in [19].

Theorem A. *Let f be an increasing and convex (concave) function defined on $[0,1]$, $\{a_i\}_{i\in\mathbb{N}}$ an increasing positive sequence such that $\big\{i\big(\frac{a_i}{a_{i+1}}-1\big)\big\}_{i\in\mathbb{N}}$ decreases $\big($the sequence $\big\{i\big(\frac{a_{i+1}}{a_i}-1\big)\big\}_{i\in\mathbb{N}}$ increases$\big)$, then the sequence $\big\{\frac{1}{n}\sum_{i=1}^{n} f\big(\frac{a_i}{a_n}\big)\big\}_{n\in\mathbb{N}}$ is decreasing. That is*

$$\frac{1}{n}\sum_{i=1}^{n} f\Big(\frac{a_i}{a_n}\Big) \geq \frac{1}{n+1}\sum_{i=1}^{n+1} f\Big(\frac{a_i}{a_{n+1}}\Big) \geq \int_0^1 f(t)\,\mathrm{d}t. \tag{3}$$

Theorem B. *Let f be an increasing and convex (concave) positive function defined on $[0,1]$, and φ be an increasing convex positive function defined on $[0,\infty)$ such that $\varphi(0)=0$ and $\big\{\varphi(i)\big[\frac{\varphi(i)}{\varphi(i+1)}-1\big]\big\}_{i\in\mathbb{N}}$ decreases, then $\big\{\frac{1}{\varphi(n)}\sum_{i=1}^{n} f\big(\frac{\varphi(i)}{\varphi(n)}\big)\big\}_{n\in\mathbb{N}}$ is decreasing. That is*

$$\frac{1}{\varphi(n)}\sum_{i=1}^{n} f\left(\frac{\varphi(i)}{\varphi(n)}\right) \geq \frac{1}{\varphi(n+1)}\sum_{i=1}^{n+1} f\left(\frac{\varphi(i)}{\varphi(n+1)}\right). \tag{4}$$

Taking particular sequences $\{a_i\}_{i\in\mathbb{N}}$ and special functions f and φ in Theorem A and Theorem B, many new inequalities between ratios of mean values are obtained. Further, Alzer's inequality, Minc-Sathre's inequality, and the like, may be recovered under the current setting.

In this article, using a similar approach to that in [19], the following theorems are obtained.

Theorem 1. *Let f be an increasing and convex (concave) function defined on $[0,1]$. Then the sequences $\big\{\frac{1}{n}\sum_{i=1}^{n} f\big(\frac{i}{n}\big)\big\}_{n\in\mathbb{N}}$ decreases and $\big\{\frac{1}{n}\sum_{i=0}^{n-1} f\big(\frac{i}{n}\big)\big\}_{n\in\mathbb{N}}$ increases, and*

$$\begin{aligned}\frac{1}{n}\sum_{i=1}^{n} f\Big(\frac{i}{n}\Big) \geq \frac{1}{n+1}\sum_{i=1}^{n+1} f\Big(\frac{i}{n+1}\Big) \geq \int_0^1 f(t)\,\mathrm{d}t \\ \geq \frac{1}{n+1}\sum_{i=0}^{n} f\Big(\frac{i}{n+1}\Big) \geq \frac{1}{n}\sum_{i=0}^{n-1} f\Big(\frac{i}{n}\Big).\end{aligned} \tag{5}$$

Theorem 2. *Let f be an increasing and convex (concave) function defined on $[0,1)$, the sequence $\{a_i\}_{i\in\mathbb{N}}$ be a positive increasing sequence such that the sequence $\bigl\{i\bigl(\frac{a_{i+1}}{a_i}-1\bigr)\bigr\}_{i\in\mathbb{N}}$ decreases $\bigl($the sequence $\bigl\{i\bigl(\frac{a_i}{a_{i+1}}-1\bigr)\bigr\}_{i\in\mathbb{N}}$ increases$\bigr)$. Then the sequence $\bigl\{\frac{1}{n}\sum_{i=1}^{n-1}f\bigl(\frac{a_i}{a_n}\bigr)\bigr\}_{n\in\mathbb{N}}$ is increasing, and*

$$\int_0^1 f(t)\,\mathrm{d}t \ge \frac{1}{n+1}\sum_{i=0}^{n} f\Bigl(\frac{a_i}{a_{n+1}}\Bigr) \ge \frac{1}{n}\sum_{i=0}^{n-1} f\Bigl(\frac{a_i}{a_n}\Bigr), \tag{6}$$

where $a_0=0$.

Theorem 3. *Let f be an increasing and convex (concave) function defined on $[0,1]$ and ϕ be a positive increasing concave function defined on $[0,\infty)$ such that $\phi(0)=0$ and the sequence $\bigl\{\phi(i+1)\bigl(\frac{\phi(i+1)}{\phi(i)}-1\bigr)\bigr\}_{i\in\mathbb{N}}$ decreases $\bigl($the sequence $\bigl\{\phi(i)\bigl(\frac{\phi(i)}{\phi(i+1)}-1\bigr)\bigr\}_{i\in\mathbb{N}}$ increases$\bigr)$. Then the sequence $\bigl\{\frac{1}{\phi(n)}\sum_{i=0}^{n-1}f\bigl(\frac{\phi(i)}{\phi(n)}\bigr)\bigr\}_{n\in\mathbb{N}}$ is increasing, that is,*

$$\frac{1}{\phi(n+1)}\sum_{i=0}^{n} f\left(\frac{\phi(i)}{\phi(n+1)}\right) \ge \frac{1}{\phi(n)}\sum_{i=0}^{n-1} f\left(\frac{\phi(i)}{\phi(n)}\right). \tag{7}$$

2. Proofs of Theorems

Proof of Theorem 1. The first inequality in (5) is equivalent to inequality (1). Now we will prove the last inequality in (5).

The last inequality in (5) is equivalent to

$$\begin{aligned}
(n+1)\sum_{i=0}^{n-1} f\Bigl(\frac{i}{n}\Bigr) &\le n\sum_{i=0}^{n} f\Bigl(\frac{i}{n+1}\Bigr),\\
f(0)+(n+1)\sum_{i=1}^{n-1} f\Bigl(\frac{i}{n}\Bigr) &\le n\sum_{i=1}^{n} f\Bigl(\frac{i}{n+1}\Bigr),\\
\sum_{i=1}^{n}\Bigl[if\Bigl(\frac{i-1}{n}\Bigr)+(n-i)f\Bigl(\frac{i}{n}\Bigr)\Bigr] &\le n\sum_{i=1}^{n} f\Bigl(\frac{i}{n+1}\Bigr),\\
\sum_{i=1}^{n}\Bigl[\frac{i}{n}f\Bigl(\frac{i-1}{n}\Bigr)+\Bigl(1-\frac{i}{n}\Bigr)f\Bigl(\frac{i}{n}\Bigr)\Bigr] &\le \sum_{i=1}^{n} f\Bigl(\frac{i}{n+1}\Bigr).
\end{aligned} \tag{8}$$

It is easy to see that

$$\frac{i(i-1)+(n-i)i}{n^2} < \frac{i}{n+1}, \tag{9}$$

$$\frac{(i+1)^2+(n-i)i}{(n+1)^2} \ge \frac{i}{n}. \tag{10}$$

Since the function f is increasing, from (9) and (10), it follows that

$$f\left(\frac{i(i-1)+(n-i)i}{n^2}\right) \le f\Big(\frac{i}{n+1}\Big), \tag{11}$$

$$f\left(\frac{(i+1)^2+(n-i)i}{(n+1)^2}\right) \ge f\Big(\frac{i}{n}\Big). \tag{12}$$

If f is concave, then we have

$$\frac{i}{n} f\Big(\frac{i-1}{n}\Big) + \Big(1-\frac{i}{n}\Big) f\Big(\frac{i}{n}\Big) \le f\left(\frac{i(i-1)+(n-i)i}{n^2}\right). \tag{13}$$

Combining of (11) with (13) yields

$$\frac{i}{n} f\Big(\frac{i-1}{n}\Big) + \Big(1-\frac{i}{n}\Big) f\Big(\frac{i}{n}\Big) \le f\Big(\frac{i}{n+1}\Big). \tag{14}$$

This implies that the last line in (8) is valid.

If f is convex, then

$$\begin{aligned}
&\sum_{i=1}^{n}\Big[\frac{i}{n+1} f\Big(\frac{i}{n+1}\Big) + \frac{n-i+1}{n+1} f\Big(\frac{i-1}{n+1}\Big)\Big] \\
&\ge \sum_{i=1}^{n} f\Big(\frac{i}{n+1}\cdot\frac{i}{n+1} + \frac{n-i+1}{n+1}\cdot\frac{i-1}{n+1}\Big) \\
&= \sum_{i=0}^{n-1} f\left(\frac{(i+1)^2+(n-i)i}{(n+1)^2}\right).
\end{aligned} \tag{15}$$

Combining (12) with (15) yields

$$\begin{aligned}
\frac{n}{n+1}\sum_{i=0}^{n} f\Big(\frac{i}{n+1}\Big) &= \frac{n}{n+1} f(0) + \frac{n}{n+1}\sum_{i=1}^{n} f\Big(\frac{i}{n+1}\Big) \\
&= \sum_{i=1}^{n}\Big[\frac{i}{n+1} f\Big(\frac{i}{n+1}\Big) + \frac{n-i+1}{n+1} f\Big(\frac{i-1}{n+1}\Big)\Big] \\
&\ge \sum_{i=0}^{n-1} f\Big(\frac{i}{n}\Big).
\end{aligned} \tag{16}$$

The proof is complete. □

Proof of Theorem 2. The right inequality in (6) can be rewritten as

$$\begin{aligned}
(n+1)\sum_{i=0}^{n-1} f\Big(\frac{a_i}{a_n}\Big) &\le n\sum_{i=0}^{n} f\Big(\frac{a_i}{a_{n+1}}\Big),\\
f(0)+(n+1)\sum_{i=1}^{n-1} f\Big(\frac{a_i}{a_n}\Big) &\le n\sum_{i=1}^{n} f\Big(\frac{a_i}{a_{n+1}}\Big),\\
\sum_{i=1}^{n}\Big[if\Big(\frac{a_{i-1}}{a_n}\Big)+(n-i)f\Big(\frac{a_i}{a_n}\Big)\Big] &\le n\sum_{i=1}^{n} f\Big(\frac{a_i}{a_{n+1}}\Big),\\
\sum_{i=1}^{n}\Big[\frac{i}{n}f\Big(\frac{a_{i-1}}{a_n}\Big)+\Big(1-\frac{i}{n}\Big)f\Big(\frac{a_i}{a_n}\Big)\Big] &\le \sum_{i=1}^{n} f\Big(\frac{a_i}{a_{n+1}}\Big).
\end{aligned} \tag{17}$$

If the sequence $\big\{i\big(\frac{a_{i+1}}{a_i}-1\big)\big\}_{i\in\mathbb{N}}$ is decreasing, then

$$\frac{(i+1)a_{i+1}+(n-i)a_i}{(n+1)a_{n+1}} \ge \frac{a_i}{a_n}. \tag{18}$$

In fact, inequality (18) is equivalent to

$$(i+1)\Big(\frac{a_{i+1}}{a_i}-1\Big) \ge (n+1)\Big(\frac{a_{n+1}}{a_n}-1\Big).$$

Let $x_i = i\big(\frac{a_{i+1}}{a_i}-1\big)$, then $\{x_i\}_{i\in\mathbb{N}}$ decreases, therefore

$$\begin{aligned}
&(i+1)\Big(\frac{a_{i+1}}{a_i}-1\Big)-(n+1)\Big(\frac{a_{n+1}}{a_n}-1\Big)\\
&=\frac{(i+1)x_i}{i}-\frac{(n+1)x_n}{n}\\
&=(x_i-x_n)+\Big(\frac{x_i}{i}-\frac{x_n}{n}\Big)\\
&\ge 0.
\end{aligned}$$

On the other hand, if the sequence $\big\{i\big(\frac{a_i}{a_{i+1}}-1\big)\big\}_{i\in\mathbb{N}}$ increases, then

$$i\Big(\frac{a_{i-1}}{a_i}-1\Big) \le n\Big(\frac{a_n}{a_{n+1}}-1\Big). \tag{19}$$

In fact, we have

$$i\Big(\frac{a_{i-1}}{a_i}-1\Big) \le (i-1)\Big(\frac{a_{i-1}}{a_i}-1\Big) \le n\Big(\frac{a_n}{a_{n+1}}-1\Big).$$

The inequality (19) can be rewritten as

$$\frac{ia_{i-1}+(n-i)a_i}{na_n} \le \frac{a_i}{a_{n+1}}. \tag{20}$$

Since the function f is increasing, it follows from inequalities (18) and (20) that

$$f\left(\frac{(i+1)a_{i+1}+(n-i)a_i}{(n+1)a_{n+1}}\right) \ge f\Big(\frac{a_i}{a_n}\Big) \tag{21}$$

and

$$f\Big(\frac{ia_{i-1}+(n-i)a_i}{na_n}\Big)\le f\Big(\frac{a_i}{a_{n+1}}\Big), \tag{22}$$

respectively.

If f is a positive increasing convex function and the sequence $\big\{i\big(\frac{a_{i+1}}{a_i}-1\big)\big\}_{i\in\mathbb{N}}$ decreases, then from (17) and (21),

$$\begin{aligned}\frac{n}{n+1}\sum_{i=0}^{n}f\Big(\frac{a_i}{a_{n+1}}\Big)&=\frac{n}{n+1}f(0)+\frac{n}{n+1}\sum_{i=1}^{n}f\Big(\frac{a_i}{a_{n+1}}\Big)\\&=\sum_{i=1}^{n}\Big[\frac{i}{n+1}f\Big(\frac{a_i}{a_{n+1}}\Big)+\frac{n-i+1}{n+1}f\Big(\frac{a_{i-1}}{a_{n+1}}\Big)\Big]\\&\ge\sum_{i=1}^{n}f\left(\frac{ia_i+(n-i+1)a_{i-1}}{(n+1)a_{n+1}}\right)\\&=\sum_{i=0}^{n-1}f\left(\frac{(i+1)a_{i+1}+(n-i)a_i}{(n+1)a_{n+1}}\right)\\&\ge\sum_{i=0}^{n-1}f\Big(\frac{a_i}{a_n}\Big),\end{aligned} \tag{23}$$

where we define $a_0=0$.

If f is a positive increasing concave function and the sequence $\big\{i\big(\frac{a_i}{a_{i+1}}-1\big)\big\}_{i\in\mathbb{N}}$ increases, then from (17) and (22),

$$\begin{aligned}&\sum_{i=1}^{n}\Big[\frac{i}{n}f\Big(\frac{a_{i-1}}{a_n}\Big)+\Big(1-\frac{i}{n}\Big)f\Big(\frac{a_i}{a_n}\Big)\Big]\\&\qquad\le\sum_{i=1}^{n}f\Big(\frac{ia_{i-1}+(n-i)a_i}{na_n}\Big)\le\sum_{i=1}^{n}f\Big(\frac{a_i}{a_{n+1}}\Big).\end{aligned} \tag{24}$$

The proof is complete. □

Proof of Theorem 3. Firstly, suppose that the function f is an increasing convex function and the sequence $\big\{\phi(i+1)\big(\frac{\phi(i+1)}{\phi(i)}-1\big)\big\}_{i\in\mathbb{N}}$ is decreasing. Then

$$\phi(i+1)\left(\frac{\phi(i+1)}{\phi(i)}-1\right)\ge\phi(n+1)\left(\frac{\phi(n+1)}{\phi(n)}-1\right), \tag{25}$$

which is equivalent to

$$\frac{\phi^2(i+1)+[\phi(n+1)-\phi(i+1)]\phi(i)}{\phi^2(n+1)}\ge\frac{\phi(i)}{\phi(n)}. \tag{26}$$

Therefore

$$f\left(\frac{\phi^2(i+1)+[\phi(n+1)-\phi(i+1)]\phi(i)}{\phi^2(n+1)}\right)\geq f\left(\frac{\phi(i)}{\phi(n)}\right), \tag{27}$$

since the function f is increasing.

Further, by standard convexity arguments, it follows that

$$\begin{aligned}
&\sum_{i=1}^{n}\left[\frac{\phi(i)}{\phi(n+1)}f\left(\frac{\phi(i)}{\phi(n+1)}\right)+\frac{\phi(n+1)-\phi(i)}{\phi(n+1)}f\left(\frac{\phi(i-1)}{\phi(n+1)}\right)\right]\\
&\geq\sum_{i=1}^{n}f\left(\frac{\phi^2(i)+[\phi(n+1)-\phi(i)]\phi(i-1)}{\phi^2(n+1)}\right)\\
&=\sum_{i=0}^{n-1}f\left(\frac{\phi^2(i+1)+[\phi(n+1)-\phi(i+1)]\phi(i)}{\phi^2(n+1)}\right)\\
&\geq\sum_{i=0}^{n-1}f\left(\frac{\phi(i)}{\phi(n)}\right),
\end{aligned} \tag{28}$$

and

$$\begin{aligned}
&\sum_{i=1}^{n}\left[\frac{\phi(i)}{\phi(n+1)}f\left(\frac{\phi(i)}{\phi(n+1)}\right)+\frac{\phi(n+1)-\phi(i)}{\phi(n+1)}f\left(\frac{\phi(i-1)}{\phi(n+1)}\right)\right]\\
&=\sum_{i=0}^{n-1}\frac{\phi(n+1)-\phi(i+1)+\phi(i)}{\phi(n+1)}f\left(\frac{\phi(i)}{\phi(n+1)}\right)+\frac{\phi(n)}{\phi(n+1)}f\left(\frac{\phi(n)}{\phi(n+1)}\right)\\
&\leq\frac{\phi(n)}{\phi(n+1)}\sum_{i=0}^{n-1}f\left(\frac{\phi(i)}{\phi(n+1)}\right)+\frac{\phi(n)}{\phi(n+1)}f\left(\frac{\phi(n)}{\phi(n+1)}\right)\\
&=\frac{\phi(n)}{\phi(n+1)}\sum_{i=0}^{n}f\left(\frac{\phi(i)}{\phi(n+1)}\right).
\end{aligned} \tag{29}$$

Combining of (28) with (29) yields

$$\frac{\phi(n)}{\phi(n+1)}\sum_{i=0}^{n}f\left(\frac{\phi(i)}{\phi(n+1)}\right)\geq\sum_{i=0}^{n-1}f\left(\frac{\phi(i)}{\phi(n)}\right)$$

and so inequality (7) holds.

Secondly, let f be an increasing concave function and the sequence $\big\{\phi(i)\big(\frac{\phi(i)}{\phi(i+1)}-1\big)\big\}_{i\in\mathbb{N}}$ be increasing. Then

$$\phi(n)\left(\frac{\phi(n)}{\phi(n+1)}-1\right)\geq\phi(i-1)\left(\frac{\phi(i-1)}{\phi(i)}-1\right), \tag{30}$$

which is equivalent to

$$\frac{\phi(i)}{\phi(n+1)}\geq\frac{\phi^2(i-1)+[\phi(n)-\phi(i-1)]\phi(i)}{\phi^2(n)}, \tag{31}$$

and hence

$$f\left(\frac{\phi(i)}{\phi(n+1)}\right) \geq f\left(\frac{\phi^2(i-1)+[\phi(n)-\phi(i-1)]\phi(i)}{\phi^2(n)}\right). \tag{32}$$

Thus from (32)

$$\begin{aligned}
\sum_{i=1}^{n} f\left(\frac{\phi(i)}{\phi(n+1)}\right) &\geq \sum_{i=1}^{n} f\left(\frac{\phi^2(i-1)+[\phi(n)-\phi(i-1)]\phi(i)}{\phi^2(n)}\right) \\
&\geq \sum_{i=1}^{n}\left[\frac{\phi(i-1)}{\phi(n)} f\left(\frac{\phi(i-1)}{\phi(n)}\right)+\frac{\phi(n)-\phi(i-1)}{\phi(n)} f\left(\frac{\phi(i)}{\phi(n)}\right)\right], \text{ (since } f \text{ is concave)}, \\
&\geq \sum_{i=1}^{n}\left[\frac{\phi(i-1)}{\phi(n)} f\left(\frac{\phi(i-1)}{\phi(n)}\right)+\frac{\phi(n+1)-\phi(i)}{\phi(n)} f\left(\frac{\phi(i)}{\phi(n)}\right)\right], \text{ (since } \phi \text{ is concave)}.
\end{aligned} \tag{33}$$

Inequality (33) can be rewritten as

$$\begin{aligned}
&\phi(n) \sum_{i=1}^{n} f\left(\frac{\phi(i)}{\phi(n+1)}\right) \\
&\geq \sum_{i=1}^{n}\left[\phi(i-1) f\left(\frac{\phi(i-1)}{\phi(n)}\right)+[\phi(n+1)-\phi(i)] f\left(\frac{\phi(i)}{\phi(n)}\right)\right] \\
&= \phi(n+1) \sum_{i=1}^{n} f\left(\frac{\phi(i)}{\phi(n)}\right)-\phi(n) f(1),
\end{aligned} \tag{34}$$

which is equivalent to

$$\begin{aligned}
\phi(n+1) \sum_{i=1}^{n} f\left(\frac{\phi(i)}{\phi(n)}\right) &\leq \phi(n) \sum_{i=1}^{n+1} f\left(\frac{\phi(i)}{\phi(n+1)}\right), \\
\frac{1}{\phi(n)} \sum_{i=1}^{n} f\left(\frac{\phi(i)}{\phi(n)}\right) &\leq \frac{1}{\phi(n+1)} \sum_{i=1}^{n+1} f\left(\frac{\phi(i)}{\phi(n+1)}\right).
\end{aligned} \tag{35}$$

Therefore

$$\begin{aligned}
&\frac{1}{\phi(n+1)} \sum_{i=1}^{n} f\left(\frac{\phi(i)}{\phi(n+1)}\right)-\frac{1}{\phi(n)} \sum_{i=1}^{n-1} f\left(\frac{\phi(i)}{\phi(n)}\right) \\
&\geq\left[\frac{1}{\phi(n)}-\frac{1}{\phi(n+1)}\right] f(1) \\
&\geq\left[\frac{1}{\phi(n)}-\frac{1}{\phi(n+1)}\right] f(0),
\end{aligned} \tag{36}$$

which implies the inequality (7).

The proof is complete. □

3. Corollaries

From these theorems, we can obtain many new inequalities related to Alzer's inequality and others or, similar inequalities to those in [19].

If $f(x) = x^r$ for $x \in [0,1]$ and $r > 0$, then it follows from Theorem 1 that

Corollary 1. *Let $n \in \mathbb{N}$, then, for all real number $r > 0$, we have*

$$\left(\frac{\frac{1}{n}\sum_{i=1}^{n-1} i^r}{\frac{1}{n+1}\sum_{i=1}^{n} i^r}\right)^{1/r} \le \frac{n}{n+1} \le \left(\frac{\frac{1}{n}\sum_{i=1}^{n} i^r}{\frac{1}{n+1}\sum_{i=1}^{n+1} i^r}\right)^{1/r}. \tag{37}$$

The right hand inequality in (37) is called Alzer's inequality.

Taking $f(x) = \ln(1+x)$ and $f(x) = \ln\frac{x}{1+x}$ for $x \in [0,1]$ in Theorem 2 produces

Corollary 2. *If $\{a_i\}_{i\ge 0}$ is a positive increasing sequence such that $a_0 = 0$ and the sequence $\left\{i\left(\frac{a_i}{a_{i+1}} - 1\right)\right\}_{i\in\mathbb{N}}$ increases, then*

$$\frac{a_n}{a_{n+1}} \ge \frac{\sqrt[n]{\prod_{i=0}^{n-1}(a_i + a_n)}}{\sqrt[n+1]{\prod_{i=0}^{n}(a_i + a_{n+1})}} \ge \frac{\sqrt[n]{\prod_{i=0}^{n-1} a_i}}{\sqrt[n+1]{\prod_{i=0}^{n} a_i}}. \tag{38}$$

Similarly, if $f(x) = \ln(1+x)$ for $x \in [0,1]$, we have from Theorem 3

Corollary 3. *Let ϕ be a positive increasing cancave function defined on $[0,\infty)$ such that $\phi(0) = 0$ and the sequence $\left\{\phi(i)\left(\frac{\phi(i)}{\phi(i+1)} - 1\right)\right\}_{i\in\mathbb{N}}$ increases, then*

$$\frac{[\phi(n)]^{n/\phi(n)}}{[\phi(n+1)]^{(n+1)/\phi(n+1)}} \ge \frac{\sqrt[\phi(n)]{\prod_{i=0}^{n-1}[\phi(n) - \phi(i)]}}{\sqrt[\phi(n+1)]{\prod_{i=0}^{n}[\phi(n+1) - \phi(i)]}}. \tag{39}$$

Remark 1. Theorem A and Theorem 2 together give upper and lower bounds for integral $\int_0^1 f(t)\,dt$. Further, Theorem B and Theorem 3 may be combined to give, with the stated conditions holding,

$$\begin{aligned}\frac{\phi(n+1)}{\phi(n)}\sum_{i=0}^{n-1} f\left(\frac{\phi(i)}{\phi(n)}\right) - f(0) &\le \sum_{i=1}^{n} f\left(\frac{\phi(i)}{\phi(n+1)}\right) \\ &\le \frac{\phi(n+1)}{\phi(n)}\sum_{i=1}^{n} f\left(\frac{\phi(i)}{\phi(n)}\right) - f(1). \end{aligned} \tag{40}$$

Acknowledgements. This paper was finalized while the second author visited RGMIA between November 1, 2001 and January 31, 2002, as a Visiting Professor with grants from the Victoria University of Technology and Jiaozuo Institute of Technology.

References

[1] H. Alzer, *On an inequality of H. Minc and L. Sathre*, J. Math. Anal. Appl. **179** (1993), 396–402.

[2] T. H. Chan, P. Gao and F. Qi, *On a generalization of Martins' inequality*, RGMIA Res. Rep. Coll. **4** (2001), no. 1, Article 12. Available online at `http://rgmia.vu.edu.au/v4n1.html`.

[3] S. S. Dragomir, *Some remarks on Hadamard's inequalities for convex functions*, Extracta Math. **9** (1994), no. 2, 88–94.

[4] S. S. Dragomir and C. E. M. Pearce, *Selected Topics on Hermite-Hadamard Type Inequalities and Applications*, RGMIA Monographs, 2000. Available online at `http://rgmia.vu.edu.au/monographs/hermite_hadamard.html`.

[5] N. Elezović and J. Pečarić, *On Alzer's inequality*, J. Math. Anal. Appl. **223** (1998), 366–369.

[6] B.-N. Guo and F. Qi, *An algebraic inequality, II*, RGMIA Res. Rep. Coll. **4** (2001), no. 1, Article 8. Available online at `http://rgmia.vu.edu.au/v4n1.html`.

[7] J.-Ch. Kuang, *Chángyòng Bùděngshì* (*Applied Inequalities*), 2nd edition, Hunan Education Press, Changsha, China, 1993. (Chinese)

[8] J.-Ch. Kuang, *Some extensions and refinements of Minc-Sathre inequality*, Math. Gaz. **83** (1999), 123–127.

[9] J. S. Martins, *Arithmetic and geometric means, an applications to Lorentz sequence spaces*, Math Nachr. **139** (1988), 281–288.

[10] H. Minc and L. Sathre, *Some inequalities involving* $(n!)^{1/r}$, Proc. Edinburgh Math. Soc. **14** (1964/65), 41–46.

[11] D. S. Mitrinović, J. E. Pečarić and A. M. Fink, *Classical and New Inequalities in Analysis*, Kluwer Academic Publishers, Dordrecht/Boston/London, 1993.

[12] N. Ozeki, *On some inequalities*, J. College Arts Sci. Chiba Univ. **4** (1965), no. 3, 211–214. (Japanese)

[13] F. Qi, *An algebraic inequality*, J. Inequal. Pure Appl. Math. **2** (2001), no. 1, Article 13. Available online at `http://jipam.vu.edu.au/v2n1/006_00.html`. RGMIA Res. Rep. Coll. **2** (1999), no. 1, Article 8, 81–83. Available online at `http://rgmia.vu.edu.au/v2n1.html`.

[14] F. Qi, *Generalization of H. Alzer's inequality*, J. Math. Anal. Appl. **240** (1999), 294–297.

[15] F. Qi, *Generalizations of Alzer's and Kuang's inequality*, Tamkang J. Math. **31** (2000), no. 3, 223–227. RGMIA Res. Rep. Coll. **2** (1999), no. 6, Article 12. Available online at `http://rgmia.vu.edu.au/v2n6.html`.

[16] F. Qi, *Inequalities and monotonicity of sequences involving* $\sqrt[n]{(n+k)!/k!}$, RGMIA Res. Rep. Coll. **2** (1999), no. 5, Article 8, 685–692. Available online at `http://rgmia.vu.edu.au/v2n5.html`.

[17] F. Qi and L. Debnath, *On a new generalization of Alzer's inequality*, Internat. J. Math. Math. Sci. **23** (2000), no. 12, 815–818.

[18] F. Qi and B.-N. Guo, *An inequality between ratio of the extended logarithmic means and ratio of the exponential means*, Taiwanese Journal of Mathematics (2002), in the press.

[19] F. Qi and B.-N. Guo, *Monotonicity of sequences involving convex function and sequence*, RGMIA Res. Rep. Coll. **3** (2000), no. 2, Article 14. Available online at http://rgmia.vu.edu.au/v3n2.html.

[20] F. Qi and Q.-M. Luo, *Generalization of H. Minc and J. Sathre's inequality*, Tamkang J. Math. **31** (2000), no. 2, 145–148. RGMIA Res. Rep. Coll. **2** (1999), no. 6, Article 14. Available online at http://rgmia.vu.edu.au/v2n6.html.

[21] J. Sándor, *On an inequality of Alzer*, J. Math. Anal. Appl. **192** (1995), 1034–1035.

[22] J. Sándor, *Comments on an inequality for the sum of powers of positive numbers*, RGMIA Res. Rep. Coll. **2** (1999), no. 2, 259–261. Available online at http://rgmia.vu.edu.au/v2n2.html.

[23] N. Towghi, *Notes on integral inequalities*, RGMIA Res. Rep. Coll. **4** (2001), no. 2, Article 10. Available online at http://rgmia.vu.edu.au/v4n2.html.

[24] N. Towghi and F. Qi, *An inequality for the ratios of the arithmetic means of functions with a positive parameter*, RGMIA Res. Rep. Coll. **4** (2001), no. 2, Article 15. Available online at http://rgmia.vu.edu.au/v4n2.html.

[25] J. S. Ume, *An elementary proof of H. Alzer's inequality*, Math. Japon. **44** (1996), no. 3, 521–522.

(Chen) Department of Mathematics, Jiaozuo Institute of Technology, Jiaozuo City, Henan 454000, China

(Qi) Department of Mathematics, Jiaozuo Institute of Technology, Jiaozuo City, Henan 454000, China

E-mail address: qifeng@jzit.edu.cn

URL: http://rgmia.vu.edu.au/qi.html

(Cerone) School of Communications and Informatics, Victoria University of Technology, P. O. Box 14428, Melbourne City MC, Victoria 8001, Australia

E-mail address: pc@matilda.vu.edu.au

URL: http://rgmia.vu.edu.au/cerone

(Dragomir) School of Communications and Informatics, Victoria University of Technology, P. O. Box 14428, Melbourne City MC, Victoria 8001, Australia

E-mail address: sever.dragomir@vu.edu.au

URL: http://rgmia.vu.edu.au/SSDragomirWeb.html