IMPROVED WEIGHTED OSTROWSKI-GRÜSS TYPE INEQUALITIES

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Abstract. Generalisations and improvements of product inequalities of Ostrowski-Grüss type for bounded differentiable mappings are given. Examples involving some specific weight functions are considered.

1. Introduction

In 1997, Dragomir and Wang [3] combined the Ostrowski inequality [6] with Grüss' inequality [4] to obtain a new result for bounded differentiable mappings which has subsequently been named the Ostrowski-Grüss inequality. Later, Matić et al. [5] pointed out how the Grüss inequality may be improved and hence established an improved Ostrowski-Grüss inequality. Recently, this inequality has itself been improved [2] and these results are shown in the following theorem.

Theorem 1. *Let $f : [a,b] \to \mathbb{R}$ be a differentiable mapping. If $\gamma \le f'(x) \le \Gamma$, $x \in [a,b]$ for some constants, $\gamma, \Gamma \in \mathbb{R}$, then*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt - \frac{f(b)-f(a)}{b-a}\left(x - \frac{a+b}{2}\right)\right|$$

$$\le \frac{1}{8}(b-a)(\Gamma-\gamma) \tag{1.1}$$

$$\le \frac{1}{4\sqrt{3}}(b-a)(\Gamma-\gamma) \tag{1.2}$$

$$\le \frac{1}{4}(b-a)(\Gamma-\gamma), \tag{1.3}$$

for all $x \in [a,b]$.

Inequality (1.3) is Dragomir and Wang's [3] original result, (1.2) is the improvement due to Matić et al. [5] and (1.1) is the further improvement by Cheng [2]. Cheng has also shown that the constant $\frac{1}{8}$ is sharp.

A product version of (1.2) and (1.3) was established in [7, p. 411] where a weighted Ostrowski inequality was employed. These results are given below.

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Theorem 2. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping and let $w:(a,b)\to(0,\infty)$ be integrable. If $\gamma\le f'(x)\le\Gamma$, $x\in[a,b]$ then*

$$\left|f(x)-\frac{1}{m}\int_a^b w(t)f(t)\,dt-\frac{f(b)-f(a)}{b-a}(x-\mu)\right|$$

$$\le\frac{1}{2}(\Gamma-\gamma)\frac{\sqrt{b-a}}{m}\left\{\int_a^b K^2(x,t)\,dt-\frac{m^2(x-\mu)^2}{b-a}\right\}^{\frac{1}{2}},\tag{1.4}$$

$$\le\frac{1}{4}(\Gamma-\gamma)(b-a),\tag{1.5}$$

for all $x\in[a,b]$. The parameters m,μ,K are defined as

$$m=\int_a^b w(t)\,dt,\quad \mu=\frac{\int_a^b tw(t)\,dt}{m},\quad K(x,t)=\begin{cases}\int_a^t w(u)\,du, & a\le t\le x\\ \int_b^t w(u)\,du, & x<t\le b.\end{cases}\tag{1.6}$$

In this paper we prove the weighted analogue of (1.1) and show that the bound is tighter than (1.4).

2. Preliminaries

The following measures of weight functions will be useful.

Definition 1. *Let $w:(a,b)\to(0,\infty)$ be integrable, i.e., $\int_a^b w(t)\,dt<\infty$. We denote the first moments to be m and M, where*

$$m(\alpha,\beta)=\int_\alpha^\beta w(t)\,dt,\qquad \textit{and}\qquad M(\alpha,\beta)=\int_\alpha^\beta tw(t)\,dt,\tag{2.1}$$

for $[\alpha,\beta]\subseteq[a,b]$. We also introduce the mean and define it as

$$\mu(\alpha,\beta)=\frac{M(\alpha,\beta)}{m(\alpha,\beta)}.\tag{2.2}$$

In addition, if the parameters (α,β) are removed, then the measure is assumed to be over the entire domain $[a,b]$. That is, $m(a,b)=m$, $M(a,b)=M$ and $\mu(a,b)=\mu$.

3. The Result

Before we proceed to the main result we need to establish the following lemma.

Lemma 1. *There exists $t^*\in[a,b]$ uniquely satisfying*

$$\frac{m}{b-a}|x-\mu|=\begin{cases}m(t^*,b), & a\le x\le\mu\\ m(a,t^*), & \mu<x\le b.\end{cases}\tag{3.1}$$

Proof. To begin, let us assume $a\le x\le\mu$ so that we need to show that there exists a unique t^* in $[a,b]$ such that

$$\frac{m}{b-a}(\mu-x)=m(t^*,b).\tag{3.2}$$

Consider

$$g(t)=\frac{m}{b-a}(\mu-x)-m(t,b),\qquad x\le t\le b.\tag{3.3}$$

From Definition 1 it is evident that g is strictly increasing on (x,b) and non-negative at $t=b$. Since g is obviously continuous then to show that $t^* \in [x,b]$ exists it will suffice to establish that $g(x) \le 0$. Now

$$\begin{aligned}\frac{m}{b-a}(\mu - x) &= \frac{1}{b-a}(M - xm) \\ &= \frac{1}{b-a}\left\{\int_a^x (t-x)\,w(t)\,dt + \int_x^b (t-x)\,w(t)\,dt\right\} \\ &\le \frac{1}{b-a}\int_x^b (t-x)\,w(t)\,dt \\ &\le \frac{b-x}{b-a} m(x,b) \\ &\le m(x,b).\end{aligned}$$

The argument for $\mu < x \le b$ follows similarly. □

The main result now follows.

Theorem 3. *Let $f : [a,b] \to \mathbb{R}$ be a differentiable mapping and let ω be as in Definition 1. If $\gamma \le f'(x) \le \Gamma$, $x \in [a,b]$ then*

$$\left| f(x) - \frac{1}{m}\int_a^b w(t)\,f(t)\,dt - \frac{f(b)-f(a)}{b-a}(x-\mu)\right| \le \frac{(\Gamma-\gamma)}{m}\int_x^{t^*} (t-x)\,w(t)\,dt, \tag{3.4}$$

for all $x \in [a,b]$ and t^* as defined in Lemma 1.

Proof. Define

$$p(x,t) = \frac{m}{b-a}(\mu - x) + \begin{cases} m(a,t), & a \le t \le x \\ m(b,t), & x < t \le b. \end{cases} \tag{3.5}$$

Then

$$\begin{aligned}\int_a^b p(x,t)\,f'(t)\,dt &= \frac{m}{b-a}(\mu - x)(f(b) - f(a)) + \int_a^x m(a,t)\,f'(t)\,dt \\ &\quad + \int_x^b m(b,t)\,f'(t)\,dt \\ &= \frac{f(b)-f(a)}{b-a} m(\mu - x) + mf(x) - \int_a^b w(t)\,f(t)\,dt.\end{aligned} \tag{3.6}$$

Now, if $a \le x \le \mu$ then

$$p(x,t) \ge 0 \quad \text{on} \quad t \in [a,x] \cup [t^*,b] \tag{3.7}$$

and

$$p(x,t) \le 0 \quad \text{on} \quad t \in (x,t^*). \tag{3.8}$$

Hence

$$\int_a^b p(x,t)\, f'(t)\, dt = \left(\int_a^x + \int_{t^*}^b\right) p(x,t)\, f'(t)\, dt + \int_x^{t^*} p(x,t)\, f'(t)\, dt \tag{3.9}$$
$$\leq \Gamma \left(\int_a^x + \int_{t^*}^b\right) p(x,t)\, dt + \gamma \int_x^{t^*} p(x,t)\, dt.$$

Now,

$$\begin{aligned}
&\left(\int_a^x + \int_{t^*}^b\right) p(x,t)\, dt \\
&= \int_a^x m(a,t) + \frac{m}{b-a}(\mu - x)\, dt + \int_{t^*}^b m(b,t) + \frac{m}{b-a}(\mu - x)\, dt \\
&= \frac{m}{b-a}(\mu - x)(x - a + b - t^*) + xm(a,x) \\
&\quad - M(a,x) + t^* m(t^*, b) - M(t^*, b) \\
&= \frac{m}{b-a}(\mu - x)(x + b - a) + xm(a,x) - M(a,x) - M(t^*, b) \\
&= m(t^*, b)(x + b - a) - M(t^*, b) + xm(a,x) - M(a,x) \\
&= \int_{t^*}^b (x + b - a - t)\, w(t)\, dt + \int_a^x (x - t)\, w(t)\, dt \\
&= m(x - \mu) - \int_x^{t^*} (x - t)\, w(t)\, dt + (b - a)\, m(t^*, b)
\end{aligned}$$

$$= \int_x^{t^*} (t - x)\, w(t)\, dt. \tag{3.10}$$

Also,

$$\begin{aligned}
\int_x^{t^*} p(x,t)\, dt &= \int_x^{t^*} \frac{m}{b-a}(\mu - x) + m(b,t)\, dt \\
&= \frac{m}{b-a}(\mu - x)(t^* - x) - (t^* - x)\, m(t^*, b) - \int_x^{t^*} (t - x)\, w(t)\, dt
\end{aligned}$$

$$= -\int_x^{t^*} (t - x)\, w(t)\, dt. \tag{3.11}$$

Combining (3.10) and (3.11) gives

$$\int_a^b p(x,t)\, f'(t)\, dt \leq (\Gamma - \gamma) \int_x^{t^*} (t - x)\, w(t)\, dt. \tag{3.12}$$

Similarly, we can show (3.12) to be true for $\mu < x \leq b$. In addition, a similar argument can show that

$$-\int_a^b p(x,t)\, f'(t)\, dt \leq (\Gamma - \gamma) \int_x^{t^*} (t - x)\, w(t)\, dt.$$

This completes the proof. □

It is of interest to compare the inequalities of (1.4) and (3.4). In the following we show that the bound in (3.4) is tighter than that in (1.4).

Corollary 1. *Let the conditions in Theorem 3 hold. The upper bound in inequality (3.4) is smaller than that in (1.4) for all $x \in [a,b]$.*

Proof. To begin, we recall the Chebyshev functional (see [1, p. 264] and [5]) that is commonly employed in Grüss type calculations.

Let f, g be two integrable mappings over $[a,b]$. The *Chebyshev functional*, $T(f,g)$, is defined by

$$T(f,g) := \frac{1}{b-a}\int_a^b f(t)g(t)\,dt - \frac{1}{(b-a)^2}\int_a^b f(t)\,dt \int_a^b g(t)\,dt. \tag{3.13}$$

The following properties of T will be useful

$$T(f,g) = T(g,f) \qquad \text{and} \qquad T(f+c,g) = T(f,g), \quad \text{where } c \text{ is constant.} \tag{3.14}$$

It is a simple matter to show that right hand side of (1.4) can be expressed as

$$(\Gamma - \gamma)\frac{b-a}{2m}T^{1/2}(K(x,t),K(x,t)), \tag{3.15}$$

where $K(x,t)$ is defined in (1.6). Thus, using (3.14) and the fact that

$$p(x,t) = \frac{m}{b-a}(\mu - x) + K(x,t) \qquad \text{and} \qquad \int_a^b p(x,t)\,dt = 0,$$

where $p(x,t)$ is defined in (3.5), we need to show that

$$\int_x^{t^*}(t-x)w(t)\,dt \le \frac{\sqrt{b-a}}{2}\left(\int_a^b p^2(x,t)\,dt\right)^{1/2}. \tag{3.16}$$

If we assume $a \le x \le \mu$, then from (3.7)-(3.8) and (3.10)-(3.11), we have

$$\begin{aligned}\int_x^{t^*}(t-x)w(t)\,dt &= \int_{t^*}^x p(x,t)\,dt \\ &= \frac{1}{2}\int_a^b |p(x,t)|\,dt \\ &\le \frac{\sqrt{b-a}}{2}\left(\int_a^b p^2(x,t)\,dt\right)^{1/2}.\end{aligned}$$

The last line being obtained via the Cauchy-Schwartz-Buniakowsky integral inequality.

A similar strategy can be employed for $\mu < x \le b$. This completes the proof. □

In the next section we evaluate (1.4) and (3.4) for the more popular weights and show that in these specific cases Theorem 3 provides a better bound.

4. Specific Weights

In the following, inequalities (1.4) and (3.4) are evaluated for some specific weights.

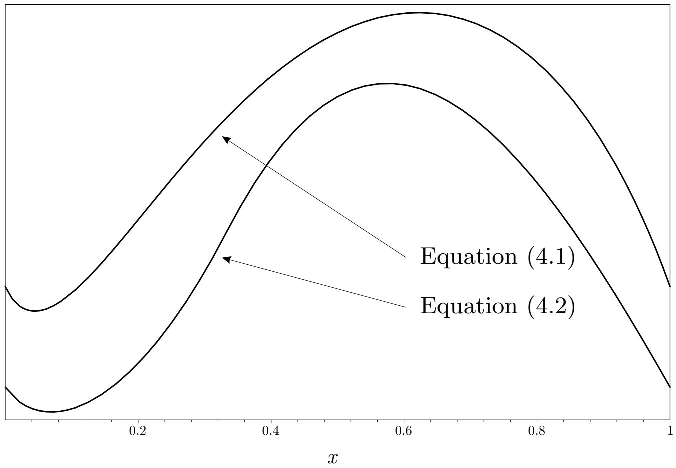

FIGURE 1. Plot of the bounds in (4.1) and (4.2).

4.1. **Uniform (Legendre).** Substituting $w(t) = 1$ into (1.4) gives the Matić et al [5] result

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt - \frac{f(b)-f(a)}{b-a}\left(x - \frac{a+b}{2}\right)\right| \le \frac{1}{4\sqrt{3}}(b-a)(\Gamma-\gamma)$$

and substituting into (3.4) returns the Cheng [2] result

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt - \frac{f(b)-f(a)}{b-a}\left(x - \frac{a+b}{2}\right)\right| \le \frac{1}{8}(b-a)(\Gamma-\gamma).$$

4.2. **Jacobi.** With $w(t) = \frac{1}{\sqrt{t}}$, $a = 0,\ b = 1$, (1.4) becomes

$$\left| f(x) - \frac{1}{2}\int_0^1 \frac{f(t)}{\sqrt{t}}dt - (f(1) - f(0))\left(x - \frac{1}{3}\right)\right| \le \frac{(\Gamma-\gamma)}{12}\sqrt{-36x^2 + 48x^{\frac{3}{2}} - 12x + 2} \tag{4.1}$$

and (3.4) is

$$\left| f(x) - \frac{1}{2}\int_0^1 \frac{f(t)}{\sqrt{t}}dt - (f(1) - f(0))\left(x - \frac{1}{3}\right)\right| \le (\Gamma-\gamma)\begin{cases} \frac{1}{3}x^3 - \frac{1}{3}x^2 + \frac{2}{3}x^{\frac{3}{2}} - \frac{2}{9}x + \frac{8}{81}, & 0 \le x \le \frac{1}{3}; \\ \frac{1}{3}x^3 - \frac{4}{3}x^2 + \frac{2}{3}x^{\frac{3}{2}} + \frac{4}{9}x - \frac{1}{8}, & \frac{1}{3} < x \le 1. \end{cases} \tag{4.2}$$

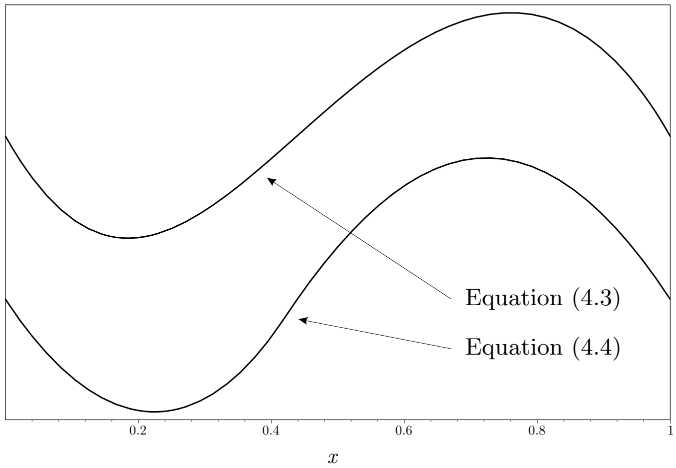

FIGURE 2. Plot of the bounds in (4.3) and (4.4).

Figure 1 shows a plot of the bounds in (4.1) and (4.2). It is clearly evident that (4.2) is tighter.

4.3. **Finite Laguerre.** We have $w(t) = e^{-t}$, $t \in [0,1]$. The results are

$$\left| f(x) - \frac{1}{1-e^{-1}} \int_0^1 e^{-t} f(t)\,dt - [f(1) - f(0)] \left(x - \frac{1-2e^{-1}}{1-e^{-1}} \right) \right| \tag{4.3}$$
$$\leq \frac{\Gamma - \gamma}{4(1-e^{-1})} \left(4\left(2e^{-1} - e^{-2} - 1\right) x^2 - 12\left(2e^{-1} - e^{-2} - 1\right) x + 8e^{-x}\left(1 - e^{-1}\right) - 10 + 16e^{-1} - 6e^{-2} \right)^{\frac{1}{2}}$$

$$\left| f(x) - \frac{1}{1-e^{-1}} \int_0^1 e^{-t} f(t)\,dt - [f(1) - f(0)] \left(x - \frac{1-2e^{-1}}{1-e^{-1}} \right) \right| \tag{4.4}$$
$$\leq \frac{\Gamma - \gamma}{1-e^{-1}} \begin{cases} (1-x)\ln\left[(1-x)\left(1-e^{-1}\right)\right] - (x-1)^2 + \frac{e^{-x}}{1-e^{-1}}, & x < \frac{1-2e^{-1}}{1-e^{-1}}; \\ (2-x)\ln\left[(2-x)\left(1-e^{-1}\right)\right] - (1-x)(2-x) + \frac{e^{-x}}{1-e^{-1}}, & x > \frac{1-2e^{-1}}{1-e^{-1}}; \end{cases}$$

where (4.3) is derived from (1.4) and (4.4) from (3.4). The bounds in (4.3) and (4.4) are shown graphically in Figure 2 and it is clearly evident that (4.4) is smaller than (4.3).

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