

On some variants of Jensen's inequality

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Abstract.

Some variants of Jensen's discrete inequality are derived. These include interpolations of the basic relation for subadditive maps and of the generalised triangle inequality.

AMS Subject Classification: 26D15.

Key words and Phrases: Jensen's inequality, generalised triangle inequality, subadditive maps

1 Introduction

Let X be a real linear space and $C \subseteq X$ a convex set in X , that is, a set such that

$$x, y \in C \quad \text{and} \quad \lambda \in [0, 1] \quad \text{imply} \quad \lambda x + (1 - \lambda)y \in C.$$

If $f : C \rightarrow \mathbf{R}$ is convex, f satisfies

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$$

for all $x, y \in C$ and $\lambda \in [0, 1]$. If $p_i \geq 0$ ($i = 1, \dots, n$) with $P_n := \sum_{i=1}^n p_i > 0$ and $y_i \in C$ ($i = 1, \dots, n$), we have the Jensen inequality

$$f\left(\frac{1}{P_n} \sum_{i=1}^n p_i y_i\right) \leq \frac{1}{P_n} \sum_{i=1}^n p_i f(y_i)$$

(see [2] or [8, p. 6]). For some recent generalizations, refinements and applications the reader is referred to [1]–[7], [9] and [8, p. 20].

In this paper we show that several new results flow from simple but judicious applications of Abel's identity, which gives the following. Suppose X is a linear space, $x_i \in X$ ($i = 1, \dots, n$) and $s_n := \sum_{i=1}^n x_i$. If a_i is real ($i = 1, \dots, n$), then

$$\begin{aligned} \sum_{i=1}^n a_i x_i &= a_1 s_1 + \sum_{i=2}^n a_i (s_i - s_{i-1}) \\ &= \sum_{i=1}^{n-1} (a_i - a_{i+1}) s_i + a_n s_n. \end{aligned}$$

Consequences include an interpolation of the basic inequality for subadditive maps and of the generalised triangle inequality.

2 Results

We will start with the following theorem.

Theorem 2.1. *Let X be a linear space and $f : X \rightarrow \mathbf{R}$ a convex mapping, $x_1, \dots, x_n \in X$ and $0 \neq a_1 \geq a_2 \geq \dots \geq a_n \geq 0$. Then*

$$f\left(a_1^{-1} \sum_{i=1}^n a_i x_i\right) \leq a_1^{-1} \left\{ a_1 f(x_1) + \sum_{i=2}^n a_i \left[f\left(\sum_{j=1}^i x_j\right) - f\left(\sum_{j=1}^{i-1} x_j\right) \right] \right\}.$$

Proof. Choose $p_i := a_i - a_{i+1}$ ($1 \leq i < n$), $p_n := a_n$ and $y_i = s_i$ ($i = 1, \dots, n$) in Jensen's theorem. We derive

$$f \left[\frac{\sum_{i=1}^n (a_i - a_{i+1}) s_i}{\sum_{i=1}^n (a_i - a_{i+1})} \right] \leq \frac{\sum_{i=1}^n (a_i - a_{i+1}) f(s_i)}{\sum_{i=1}^n (a_i - a_{i+1})},$$

where for notational simplicity we have introduced $a_{n+1} := 0$. The desired result now follows by Abel's identity. \square

Corollary 2.2. *Let $g : X \rightarrow (0, \infty)$ be logarithmically concave, that is, let $\ln g$ be concave. Under the assumptions of the theorem*

$$g \left(a_1^{-1} \sum_{i=1}^n a_i x_i \right) \geq \left\{ [g(x_1)]^{a_1} \prod_{i=2}^n \left[\frac{g \left(\sum_{j=1}^i x_j \right)}{g \left(\sum_{j=1}^{i-1} x_j \right)} \right]^{a_i} \right\}^{1/a_1}.$$

The result follows from the theorem for the convex mapping $f = -\ln g$.

Suppose that the mapping $\varphi : X \rightarrow \mathbf{R}$ is subadditive, that is, for α, β nonnegative we have

$$\varphi(\alpha x + \beta y) \leq \alpha \varphi(x) + \beta \varphi(y).$$

By mathematical induction we have for all $\alpha_i \geq 0$ and $y_i \in X$ ($i = \dots, n$) that

$$\varphi \left(\sum_{i=1}^n \alpha_i y_i \right) \leq \sum_{i=1}^n \alpha_i \varphi(y_i).$$

This inequality may be interpolated as follows.

Corollary 2.3. *Let $\varphi : X \rightarrow \mathbf{R}$ be subadditive, $y_1, \dots, y_n \in X$ and $\alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_n \geq 0$. Then*

$$\begin{aligned} \varphi \left(\sum_{i=1}^n \alpha_i y_i \right) &\leq \alpha_1 \varphi(y_1) + \sum_{i=2}^n \alpha_i \left[\varphi \left(\sum_{j=1}^i y_j \right) - \varphi \left(\sum_{j=1}^{i-1} y_j \right) \right] \\ &\leq \sum_{i=1}^n \alpha_i \varphi(y_i). \end{aligned}$$

Proof. As φ is subadditive, it is convex. The first desired inequality follows from Theorem 2.1.

For the second, we observe that for $2 \leq i \leq n$,

$$\varphi \left(\sum_{j=1}^i y_j \right) - \varphi \left(\sum_{j=1}^{i-1} y_j \right) \leq \varphi(y_i).$$

Multiplying the i th inequality by α_i and summing over i provides the desired result.

Our second main result is the following.

Theorem 2.4. *Let $f : X \rightarrow \mathbf{R}$ be convex and $x_i \in X$ ($i = 1, \dots, n$). Suppose that m_i ($i = 1, \dots, n$) satisfy*

$$\sum_{j=1}^i m_j \geq 0 \quad (1 \leq i \leq n)$$

and

$$\sum_{i=1}^n (n+1-i)m_i > 0.$$

Then

$$f \left(\frac{\sum_{i=1}^n m_i x_i}{\sum_{i=1}^n (n+1-i)m_i} \right) \leq \frac{\sum_{i=1}^n \sum_{j=i}^n f(x_j - x_{j+1})}{\sum_{i=1}^n (n+1-i)m_i},$$

where again we put $x_{n+1} := 0$ for notational convenience.

Proof. Let $s_i = \sum_{j=1}^i m_j$ ($1 \leq i \leq n$). Then by Abel's identity

$$\begin{aligned} \sum_{i=1}^n m_i x_i &= s_1 x_1 + \sum_{i=2}^n (s_i - s_{i-1}) x_i \\ &= \sum_{i=1}^n s_i (x_i - x_{i+1}). \end{aligned}$$

Applying Jensen's inequality provides

$$f \left[\frac{\sum_{i=1}^n s_i (x_i - x_{i+1})}{\sum_{i=1}^n s_i} \right] \leq \frac{\sum_{i=1}^n s_i f(x_i - x_{i+1})}{\sum_{i=1}^n s_i}.$$

The numerator on the right-hand side may be written as

$$\sum_{i=1}^n m_i \sum_{j=i}^n f(x_j - x_{j+1})$$

and we have the desired result. \square

Corollary 2.5. *Let $g : X \rightarrow (0, \infty)$ be logarithmically concave. With the above assumptions*

$$g\left(\frac{\sum_{i=1}^n m_i x_i}{\sum_{i=1}^n (n+1-i)m_i}\right) \geq \left\{ \prod_{i=1}^n \left[\prod_{j=i}^n g(x_j - x_{j+1}) \right]^{m_i} \right\}^{1/\sum_{i=1}^n (n+1-i)m_i}.$$

The result follows from the theorem with the choice of convex mapping $f = -\ln g$.

3 Applications

We now derive some particular applications relating to homely choices of convex function.

1. Let $x_i > 0$ ($i = 1, \dots, n$) with $0 \neq a_1 \geq a_2 \geq \dots \geq a_n \geq 0$. Then

$$\sum_{i=1}^n a_i x_i \geq a_1 \left[x_1^{a_1} \prod_{i=2}^n \left(\frac{\sum_{j=1}^i x_j}{\sum_{j=1}^{i-1} x_j} \right)^{a_i} \right]^{1/a_1}.$$

The result follows from Corollary 2.2 with the mapping $g : (0, \infty) \rightarrow (0, \infty)$ given by $g(x) = x$.

Suppose $x_1 \geq x_2 \geq \dots \geq x_n \geq 0$ and $n_i \in \mathbf{R}$ with $m_1 \geq 0$. In the same way we have from Corollary 2.5 that

$$\frac{\sum_{i=1}^n m_i x_i}{\sum_{i=1}^n (n+1-i)m_i} \geq \left\{ \prod_{i=1}^n \left[\prod_{j=i}^n (x_j - x_{j+1}) \right]^{m_i} \right\}^{1/\sum_{i=1}^n (n+1-i)m_i}.$$

2. Let $x_i > 0$ ($i = 1, \dots, n$) and $0 \neq a_1 \geq a_2 \geq \dots \geq a_n \geq 0$. Then

$$a_1^2 \leq \left(\sum_{i=1}^n a_i x_i \right) \left(\frac{a_1}{x_1} - \sum_{i=2}^n \frac{a_i x_i}{\left(\sum_{j=1}^{i-1} x_j \right) \left(\sum_{k=1}^i x_k \right)} \right).$$

This follows from Theorem 2.1 applied to the convex mapping $f(x) = 1/x$ on the interval $(0, \infty)$.

3. Let $x_i \in \mathbf{R}$ and $a_1 \geq a_2 \geq \cdots \geq a_n \geq 0$. Then

$$\left(\sum_{i=1}^n a_i x_i \right)^2 \leq a_1 \left\{ a_1 x_1^2 + \sum_{i=2}^n a_i x_i \left[x_i + 2 \sum_{j=1}^{i-1} x_j \right] \right\}.$$

This follows from Theorem 2.1 applied for the convex mapping $f(x) = x^2$ ($x \in \mathbf{R}$).

4. Consider the mapping $f : \mathbf{R} \rightarrow \mathbf{R}$ given by $f(x) = \ln(1 + e^x)$. We have $f'(x) = e^x/(1 + e^x)$ and $f''(x) = e^x/(1 + e^x)^2$, which shows that f is convex on \mathbf{R} .

Let $0 \neq a_1 \geq a_2 \geq \cdots \geq a_n \geq 0$ and $x_1, \dots, x_n \in \mathbf{R}$. Then by Theorem 2.1

$$\begin{aligned} & \ln \left[1 + \exp \left(a_1^{-1} \sum_{i=1}^n a_i x_i \right) \right] \\ & \leq a_1 \ln[1 + e^{x_1}] + \sum_{i=2}^n a_i \left[\ln \left\{ 1 + \exp \left(\sum_{j=1}^i x_j \right) \right\} \right. \\ & \quad \left. - \ln \left\{ 1 + \exp \left(\sum_{j=1}^{i-1} x_j \right) \right\} \right] \\ & = \ln \left\{ (1 + e^{x_1})^{a_1} \prod_{i=2}^n \left[\frac{1 + \exp \left(\sum_{j=1}^i x_j \right)}{1 + \exp \left(\sum_{j=1}^{i-1} x_j \right)} \right]^{a_i} \right\}, \end{aligned}$$

whence

$$1 + \exp \left(a_1^{-1} \sum_{i=1}^n a_i x_i \right) \leq [1 + e^{x_1}]^{a_1} \prod_{i=2}^n \left[\frac{1 + \exp \left(\sum_{j=1}^i x_j \right)}{1 + \exp \left(\sum_{j=1}^{i-1} x_j \right)} \right]^{a_i}.$$

5. Let X be a real normed space and $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n \geq 0$. Then for $x_i \in X$ ($i = 1, \dots, n$) we have the refinement

$$\begin{aligned} \left\| \sum_{i=1}^n \alpha_i x_i \right\| & \leq \alpha_1 \|x_1\| + \sum_{i=2}^n \alpha_i \left(\left\| \sum_{j=1}^i x_j \right\| - \left\| \sum_{j=1}^{i-1} x_j \right\| \right) \\ & \leq \sum_{i=1}^n \alpha_i \|x_i\| \end{aligned}$$

of the generalised triangle inequality. The result follows from Corollary 2.3.

References

- [1] S. S. Dragomir, Some refinements of Ky Fan's inequality, *J. Math. Anal. Appl.* **163** (1992), 317–321.
- [2] S. S. Dragomir, Some refinements of Jensen's inequality, *J. Math. Anal. Appl.* **168** (1992), 518–522.
- [3] S. S. Dragomir, On some refinements of Jensen's inequality and applications, *Utilitas Math.* **43** (1993), 235–243.
- [4] S. S. Dragomir, Two mappings associated with Jensen's inequality, *Extracta Math.* **8** (1993), 102–105.
- [5] S. S. Dragomir, A further improvement of Jensen's inequality, *Tamkang J. Math.* **15** (1994), 29–36.
- [6] S. S. Dragomir and N. M. Ionescu, Some remarks on convex functions, *Anal. Num. Theor. Approx.* **21** (1992), 31–36.
- [7] S. S. Dragomir and D. M. Milošević, A sequence of mappings connected with Jensen's inequality and applications, *Mat. Vesn.* **44** (1992), 113–121.
- [8] D. S. Mitrinović, J. E. Pečarić and A. M. Fink, *Classical and New Inequalities in Analysis*. Kluwer Academic Publishers, Dordrecht/Boston/London (1993).
- [9] J. E. Pečarić and S. S. Dragomir, A refinement of Jensen inequality and applications, *Studia Univ. Babeş–Bolyai* **34** (1989), 15–19.