NEW GENERALIZATIONS OF THE TELYAKOVSKII'S INEQUALITIES

Tomovski Živorad

Faculty of Mathematical and Natural Sciences, Skopje, Macedonia

E-mail address: tomovski@iunona.pmf.ukim.edu.mk

Abstract. Generalizations of the Telyakovskii type inequalities [1] are made, by considering the condition $S_{p\alpha r},\ 1 < p \leq 2,\ \alpha \geq 0$, $r \in \{0, 1, 2, \ldots [\alpha]\}$ instead of S.

1. Introduction and main results

Let

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \tag{1.1}$$

$$g(x) = \sum_{n=1}^{\infty} a_n \sin nx \tag{1.2}$$

be the cosine and sine trigonometric series.

The following L^1-integrability class S for series (1.1) and (1.2) was defined by Telyakovskii in [1].

A sequence $\{a_n\}_{n=0}^{\infty}$ of real numbers belong to the class S, or $\{a_n\} \in S$, if $a_n \to 0$ as $n \to \infty$ and there exists a sequence of numbers $\{A_n\}_{n=0}^{\infty}$ such that:

a) $A_n \downarrow 0$

b) $\sum\limits_{n=0}^{\infty} A_n < \infty$.

c) $|\Delta a_n| \leq A_n$, for all n.

In the same paper [1], Telyakovskii has proved the following Theorems.

Theorem A. *Let the coefficients of the series $f(x)$ satisfy the condition S. Then the series is a Fourier series and the following inequality holds:*

$$\int_0^{\pi} |f(x)|\, dx \leq C \sum_{n=0}^{\infty} A_n\,,$$

where C is an absolute constant.

Theorem B. *Let the coefficients of the series $g(x)$ satisfy the condition S. Then the following relation holds for $p = 1, 2, 3, \dots$.*

$$\int\limits_{\pi/(p+1)}^{\pi} |g(x)|\, dx = \sum_{n=1}^{p} \frac{|a_n|}{n} + O\left(\sum_{n=1}^{\infty} A_n\right).$$

In particular $g(x)$ is a Fourier series iff $\sum\limits_{n=1}^{\infty} \frac{|a_n|}{n} < \infty$.

On the other hand, a null sequence $\{a_n\}_{n=0}^{\infty}$ of real numbers belongs to the class $S_{p\alpha r}$ if for some $\alpha \geq 0$, $r \in \{0, 1, 2, \dots [\alpha]\}$ and some monotonically decreasing sequence $\{A_n\}_{n=0}^{\infty}$ such that $\sum\limits_{n=0}^{\infty} n^{\alpha} A_n < \infty$, the condition $\frac{1}{n^{p(\alpha-r)+1}} \sum\limits_{k=1}^{n} \frac{|\Delta a_k|^p}{A_k^p} = O(1)$, $p > 1$ holds.

It is obvious that $S_{p\alpha r}$ is a stronger class than that of S.

The Dirichlet's kernel, conjugate Dirichlet's kernel and modified Dirichlet's kernel are defined respectively by

$$D_n(t) = \frac{1}{2} + \sum_{k=1}^{n} \cos\, kt = \frac{\sin\left(n + \frac{1}{2}\right)t}{2\, \sin\, \frac{t}{2}}$$

$$\tilde{D}_n(t) = \sum_{k=1}^{n} \sin\, kt = \frac{\cos\, \frac{t}{2} - \cos\left(n + \frac{1}{2}\right)t}{2\, \sin\, \frac{t}{2}}$$

$$\overline{D}_n(t) = -\frac{1}{2}\, \mathrm{ctg}\, \frac{t}{2} + \tilde{D}_n(t) = -\frac{\cos\left(n + \frac{1}{2}\right)t}{2\, \sin\, \frac{t}{2}},$$

In this paper we shall prove the following Theorems.

Theorem 1. *Let the coefficients of the series* (1.1) *belong to the class $S_{p\alpha r}$, $1 < p \leq 2$, $\alpha \geq 0$, $r \in \{0, 1, 2, \dots [\alpha]\}$. Then the r-th derivate of the series* (1.1) *is a Fourier series of some $f^{(r)} \in L^1(0, \pi)$ and the following inequality holds:*

$$\int\limits_{0}^{\pi} |f^{(r)}(x)| dx \leq M_{p,\alpha} \sum_{n=0}^{\infty} n^{\alpha} A_n\,,$$

where $M_{p,\alpha}$ is a positive constant depends on p and α.

Theorem 2. *Let the coefficients of the series* (1.2) *belong to the class $S_{p\alpha r}$, $1 < p \leq 2$, $\alpha \geq 0$, $r \in \{0, 1, 2, \dots [\alpha]\}$. Then the r-th derivate of the series* (1.2) *converges to a function and for $m = 1, 2, 3, \dots$ the following inequality holds:*

$$\int\limits_{\pi/(m+1)}^{\pi} |g^{(r)}(x)| dx \leq M \sum_{j=1}^{m} |a_j| j^{r-1} + O_{p,\alpha}\left(\sum_{k=1}^{\infty} k^{\alpha} A_k\right), \tag{1.3}$$

where $0 < M = M(\alpha) < \infty$ *and* $O_{p,\alpha}$ *depends on* p*, and* α*. Moreover, if* $\sum_{n=1}^{\infty} n^{r-1}|a_n| < \infty$*, then the* r*-th derivate of the series* (1.2) *is a Fourier series of some* $g^{(r)} \in L^1(0,\pi)$ *and*

$$\int_0^\pi |g^{(r)}(x)|dx \leq O_\alpha \left(\sum_{j=1}^{\infty} |a_j| j^{r-1} \right) + O_{p,\alpha} \left(\sum_{j=1}^{\infty} j^\alpha A_j \right), \tag{1.4}$$

2. Lemmas

For the proof of our new theorems we need the following lemmas:

Lemma 1. *Let* $\{\alpha_j\}_{j=1}^k$ *be a sequence of real numbers. Then the following relation holds for* $1 < p \leq 2$*,* $v = 0, 1, 2, \ldots, r$ *and* $r = 0, 1, 2, \ldots$

$$V_k = \int_{\pi/k}^{\pi} \left| \sum_{j=1}^{k} \alpha_j \frac{\left(j + \frac{1}{2}\right)^v \sin\left[\left(j + \frac{1}{2}\right)x + \frac{v\pi}{2}\right]}{\left(\sin\left(\frac{x}{2}\right)\right)^{r+1-v}} \right| dx =$$

$$= O_p \left[k^{r+1} \left(\frac{1}{k} \sum_{j=1}^{k} |\alpha_j|^p \right)^{1/p} \right],$$

where O_p *depends only on* p*.*

Proof. Applying first Holder inequality, yields:

$$V_k = \int_{\pi/k}^{\pi} \frac{1}{\left(\sin\left(\frac{x}{2}\right)\right)^{r+1-v}} \left| \sum_{j=1}^{k} \alpha_j \left(j + \frac{1}{2}\right)^v \sin\left[\left(j + \frac{1}{2}\right)x + \frac{v\pi}{2}\right] \right| dx \leq$$

$$\leq \left[\int_{\pi/k}^{\pi} \frac{dx}{\left(\sin\left(\frac{x}{2}\right)\right)^{(r+1-v)p}} \right]^{1/p} \times$$

$$\times \left\{ \int_0^\pi \left| \sum_{j=1}^{k} \alpha_j \left(j + \frac{1}{2}\right)^v \sin\left[\left(j + \frac{1}{2}\right)x + \frac{v\pi}{2}\right] \right|^q dx \right\}^{1/q}.$$

Since

$$\int_{\pi/k}^{\pi} \frac{dx}{\left(\sin\left(\frac{x}{2}\right)\right)^{(r+1-v)p}} \leq \frac{\pi k^{(r+1-v)p-1}}{(r+1-v)p-1} \leq \frac{\pi}{p-1} k^{(r+1-v)p-1},$$

we have:

$$V_k \leq \left(\frac{\pi}{p-1}\right)^{1/p} (k^{(r+1-v)p-1})^{1/p} \times$$

$$\times \left\{ \int_0^{\pi} \left| \sum_{j=1}^{k} \alpha_j \left(j+\frac{1}{2}\right)^v \sin\left[\left(j+\frac{1}{2}\right)x + \frac{v\pi}{2}\right] \right|^q dx \right\}^{1/q}.$$

Then using the Hausdorff-Young inequality we get:

$$\left\{ \int_0^{\pi} \left| \sum_{j=1}^{k} \alpha_j \left(j+\frac{1}{2}\right)^v \sin\left[\left(j+\frac{1}{2}\right)x + \frac{v\pi}{2}\right] \right|^q dx \right\}^{1/q} = O_p \left[\left(\sum_{j=1}^{k} |\alpha_j|^p j^{vp} \right)^{1/p} \right].$$

Finally,

$$V_k = O_p \left[(k^{(r+1-v)p-1})^{1/p} \left(\sum_{j=1}^{k} |\alpha_j|^p \, j^{vp} \right)^{1/p} \right] =$$

$$= O_p \left[(k^{(r+1)p-1})^{1/p} \left(\sum_{j=1}^{k} |\alpha_j|^p \right)^{1/p} \right] =$$

$$= O_p \left[k^{r+1} \left(\frac{1}{k} \sum_{j=1}^{k} |\alpha_j|^p \right)^{1/p} \right],$$

where O_p depends only on p.

Lemma 2. [2]. *Let r be a nonnegative integer, and $x \in (0, \pi]$, where $n \geq 1$. Then the r-th derivate of the Dirichlet's kernel is*

$$D_n^{(r)}(x) = \sum_{k=0}^{r-1} \frac{\left(n+\frac{1}{2}\right)^k \sin\left[\left(n+\frac{1}{2}\right)x + \frac{k\pi}{2}\right]}{\left[\sin\left(\frac{x}{2}\right)\right]^{r+1-k}} \varphi_k(x) + \frac{\left(n+\frac{1}{2}\right)^r \sin\left[\left(n+\frac{1}{2}\right)x + \frac{r\pi}{2}\right]}{2\sin\left(\frac{x}{2}\right)},$$

where the same φ_k denotes various analytical function of x, independent of n.

Lemma 3. *Let $\{\alpha_j\}_{j=0}^{k}$ be a sequence of real numbers. Then the following relation holds for $v = 0, 1, 2, \ldots r$, $\alpha \geq 0$ and $r \in \{0, 1, 2, \ldots [\alpha]\}$*

$$V_k = \int_{\pi/(k+1)}^{\pi} \left| \sum_{j=0}^{k} \alpha_j \frac{\left(j+\frac{1}{2}\right)^v \sin\left[\left(j+\frac{1}{2}\right)x + \frac{v\pi}{2}\right]}{\left(\sin\left(\frac{x}{2}\right)\right)^{r+1-v}} \right| dx =$$

$$= O_p \left((k+1)^{1+\alpha} \left[(k+1)^{p(r-\alpha)-1} \sum_{j=0}^{k} |\alpha_j|^p \right]^{1/p} \right),$$

where O_p depends only on p.

Proof. Applying the Lemma 1, we get:

$$V_k = O_p\left[(k+1)^{r+1}\left(\frac{1}{k+1}\sum_{j=0}^{k}|\alpha_j|^p\right)^{1/p}\right] =$$

$$= O_p\left((k+1)^{1+\alpha}\left[(k+1)^{p(r-\alpha)-1}\sum_{j=0}^{k}|\alpha_j|^p\right]^{1/p}\right),$$

where O_p depends only on p.

Lemma 4. *Let the sequence of real numbers $\{a_j\}_{j=0}^{k}$ belongs to the class $S_{p\alpha r}$, $1 < p \le 2$, $\alpha \ge 0$, $r \in \{0,1,2,\dots[\alpha]\}$. Then the following relation holds*

$$\int_0^{\pi}\left|\sum_{j=0}^{k}\frac{\Delta a_j}{A_j}D_j^{(r)}(x)\right|dx = O_p\big((k+1)^{\alpha+1}\big),$$

where O_p depends only on p.

Proof. We have:

$$\int_0^{\pi}\left|\sum_{j=0}^{k}\frac{\Delta a_j}{A_j}D_j^{(r)}(x)\right|dx = \int_0^{\pi/(k+1)} + \int_{\pi/(k+1)}^{\pi} = I_k + J_k\,.$$

Appliying the inequality $D_n^{(r)}(x) = O(n^{r+1})$, we obtain

$$I_k \le \gamma k^r\sum_{j=0}^{k}\frac{|\Delta a_j|}{A_j} \le \gamma(k+1)^{r+1}\left(\frac{1}{k+1}\sum_{j=0}^{k}\frac{|\Delta a_j|^p}{A_j^p}\right)^{1/p} =$$

$$= \gamma(k+1)^{1+\alpha}\left[(k+1)^{p(r-\alpha)-1}\sum_{j=0}^{k}\frac{|\Delta a_j|^p}{A_j^p}\right] = O\big((k+1)^{\alpha+1}\big)\,.$$

For the second integral we applying Lemma 2 and Lemma 3, where

$$\alpha_j = \frac{\Delta a_j}{A_j}, \quad j = 0,1,2,\dots,k\,,$$

and we get:

$$J_k = O_p\big((k+1)^{1+\alpha}\big)\,.$$

Finally the inequality is satisfied.

Lemma 5. *Let the sequence of real numbers* $\{a_n\}$ *belongs to the class* $S_{p\alpha r}$, $1 < p \le 2$, $\alpha \ge 0$, $r \in \{0, 1, 2, \dots [\alpha]\}$. *Then the following limit holds:*

$$A_N \int_0^\pi \left| \sum_{j=0}^{N} \frac{\Delta a_j}{A_j} D_j^{(r)}(x) \right| dx = o(1)\,, \quad N \to \infty\,.$$

Proof. Applying the Lemma 4, yields:

$$A_N \int_0^\pi \left| \sum_{j=0}^{N} \frac{\Delta a_j}{A_j} D_j^{(r)}(x) \right| dx = O_p\big(A_N (N+1)^{1+\alpha}\big) = o(1)\,, \quad N \to \infty\,.$$

Lemma 6. *Let* r *be a non-negative integer. Then for all* $0 < |t| \le \pi$ *and all* $n \ge 1$ *the following estimate holds*

$$|\overline{D}_n^{(r)}(t)| \le \frac{4n^r \pi}{|t|} + O\Big(\frac{1}{|t|^{r+1}}\Big)\,.$$

Proof. Let $E_n(t) = \frac{1}{2} + \sum_{k=1}^{n} e^{ikt}$ and $E_{-n}(t) = \frac{1}{2} + \sum_{k=1}^{n} e^{-ikt}$.

Firstly we shall prove the following two estimates

$$|E_{-n}^{(r)}(t)| \le \frac{4\pi n^r}{|t|} \tag{2.1}$$

$$|\tilde{D}^{(r)}(t)| \le \frac{4n^r \pi}{|t|} \tag{2.2}$$

The case $r = 0$ is trivial. Really, since $E_n(t) = D_n(t) + i\tilde{D}_n(t)$, we have

$$|E_n(t)| \le |D_n(t)| + |\tilde{D}_n(t)| \le \frac{\pi}{2|t|} + \frac{\pi}{|t|} = \frac{3\pi}{|t|} < \frac{4\pi}{|t|}$$

$$|E_{-n}(t)| = |E_n(-t)| < \frac{4\pi}{|t|}\,.$$

Let $r \ge 1$. Applying Abels's transformation, we have:

$$E_n^{(r)}(t) = i^r \sum_{k=1}^{n} k^r e^{ikt} = i^r \left[\sum_{k=1}^{n-1} \Delta(k^r)\Big(E_k(t) - \frac{1}{2}\Big) + n^r \Big(E_n(t) - \frac{1}{2}\Big) \right]$$

$$|E_n^{(r)}(t)| \le \sum_{k=1}^{n-1} [(k+1)^r - k^r]\Big(\frac{1}{2} + |E_k(t)|\Big) + n^r \Big(|E_n(t)| + \frac{1}{2}\Big) \le$$

$$\le \Big(\frac{\pi}{2|t|} + \frac{3\pi}{2|t|}\Big) \left\{ \sum_{k=1}^{n-1} [(k+1)^r - k^r] + n^r \right\} = \frac{4\pi\, n^r}{|t|}\,.$$

Since $E_{-n}^{(r)}(t) = E_n^{(r)}(-t)$, we obtain (2.1).

Applying the inequality (2.1) and the equation

$$2\,i\tilde{D}_n^{(r)}(t) = E_n^{(r)}(t) - E_{-n}^{(r)}(t)\,,$$

we obtain

$$|\tilde{D}_n^{(r)}(t)| = |i\tilde{D}_n^{(r)}(t)| \le \frac{1}{2}|E_n^{(r)}(t)| + \frac{1}{2}|E_{-n}^{(r)}(t)| \le \frac{4n^r\pi}{|t|}\,.$$

We note that $|\left(\operatorname{ctg}\frac{t}{2}\right)^{(r)}| = O\left(\frac{1}{|t|^{r+1}}\right)$.

Finally by inequality (2.2), we obtain

$$|\overline{D}_n^{(r)}(t)| \le |\tilde{D}_n^{(r)}(t)| + \frac{1}{2}\left|\left(\operatorname{ctg}\frac{t}{2}\right)^{(r)}\right| \le \frac{4n^r\pi}{|t|} + O\Big(\frac{1}{|t|^{r+1}}\Big)\,.$$

3. Proofs of the Theorems

3.1. Proof of the Theorem 1

We have:

$$\sum_{k=1}^{n}|\Delta(k^r a_k)| = \sum_{k=1}^{n}|[(k+1)^r a_{k+1} - k^r a_{k+1}] + [k^r a_{k+1} - k^r a_k]| =$$

$$= \sum_{k=1}^{n}|[\Delta(k^r)a_{k+1} + k^r\Delta a_k| \le r\sum_{k=1}^{n}k^{r-1}|a_{k+1}| + \sum_{k=1}^{n}k^r|\Delta a_k|\,.$$

Applying Abel's transformation, we have:

$$\sum_{k=1}^{n}k^{r-1}|a_{k+1}| = \sum_{k=1}^{n-1}\Delta|a_{k+1}|\sum_{j=1}^{k}j^{r-1} + |a_{n+1}|\sum_{j=1}^{n}j^{r-1} \le$$

$$\le \sum_{k=1}^{n-1}|\Delta a_{k+1}|\,k^r + |a_{n+1}|\,n^r =$$

$$= \sum_{k=1}^{n-1}|\Delta a_{k+1}|\,k^r + n^r|\Delta a_{n+1} + \Delta a_{n+2} + \cdots| \le$$

$$\le \sum_{k=1}^{n-1}|\Delta a_{k+1}|\,k^r + (n+1)^r\sum_{k=n+1}^{\infty}|\Delta a_k| \le$$

$$\le \sum_{k=1}^{n-1}|\Delta a_{k+1}|\,k^r + \sum_{k=n+1}^{\infty}k^r|\Delta a_k|\,,$$

and

$$\sum_{k=1}^{n} k^r|\Delta a_k| = \sum_{k=1}^{n-1}(\Delta A_k)\sum_{j=1}^{k}\frac{|\Delta a_j|}{A_j}j^r + A_n\sum_{j=1}^{n}\frac{|\Delta a_j|}{A_j}j^r \le$$

$$\le \sum_{k=1}^{n-1}(\Delta A_k)k^{1+\alpha}\left[k^{p(r-\alpha)-1}\sum_{j=1}^{k}\frac{|\Delta a_j|^p}{A_j^p}\right]^{1/p} +$$

$$+n^{1+\alpha}A_n\left[n^{p(r-\alpha)-1}\sum_{j=1}^{n}\frac{|\Delta a_j|^p}{A_j^p}\right] =$$

$$= O(1)\left[\sum_{k=1}^{n-1}(\Delta A_k)k^{1+\alpha} + n^{1+\alpha}A_n\right] =$$

$$= O(1)\left\{\sum_{k=1}^{n}[k^{\alpha+1}-(k-1)^{\alpha+1}]A_k - n^{1+\alpha}A_n + n^{1+\alpha}A_n\right\} =$$

$$\le O(1)(\alpha+1)\sum_{k=1}^{n}k^{\alpha}A_k\,. \tag{3.1}$$

Thus

$$\sum_{k=1}^{\infty}|\Delta(k^r a_k)| \le O_{r,\alpha}\left(\sum_{k=1}^{\infty}k^{\alpha}A_k\right) < \infty \quad \text{i.e} \quad \lim_{n\to\infty} S_n^{(r)}(x) = f^{(r)}(x)\,.$$

Applying Abel's transformation, we have: $f(x) = \sum_{k=0}^{\infty}\Delta a_k D_k(x)$ $(*)$.

From inequality $|D_n^{(r)}(x)| \le C\,\dfrac{n^r}{x}$, we have that $\sum_{k=0}^{\infty}\Delta a_k D_k^{(r)}(x)$ is uniformly convergent on any compact subset of $(0,\pi]$.

Thus representation $(*)$ implies

$$f^{(r)}(x) = \sum_{k=0}^{\infty}\Delta a_k D_k^{(r)}(x)\,.$$

Now applications of Abel's transformation, Lemma 5 and Lemma 4, yields,

$$\int_0^{\pi}|f^{(r)}(x)|dx = \int_0^{\pi}\left|\sum_{k=0}^{\infty}\Delta a_k D_k^{(r)}(x)\right|dx \le \lim_{N\to\infty}\sum_{k=0}^{N-1}(\Delta A_k)\int_0^{\pi}\left|\sum_{j=0}^{k}\frac{\Delta a_j}{A_j}\,D_j^{(r)}(x)\right|dx =$$

$$= O_p(1)\lim_{N\to\infty}\sum_{k=0}^{N-1}(\Delta A_k)(k+1)^{\alpha+1} =$$

$$= O_p(1)\lim_{N\to\infty}\left\{\sum_{k=0}^{N}\big[(k+1)^{\alpha+1}-k^{\alpha+1}\big]A_k - (N+1)^{\alpha+1}A_n\right\} =$$

$$= O_{p,\alpha}\left(\sum_{k=0}^{\infty}k^{\alpha}A_k\right)\,.$$

Finally,

$$\int_0^{\pi} |f^{(r)}(x)|dx \le M_{p,\alpha} \sum_{n=0}^{\infty} n^{\alpha} A_n\,,$$

where $M_{p,\alpha}$ depends on p and α.

3.2. Proof of the Theorem 2

Proof. We suppose that $a_0 = 0$ and $A_0 = \max(|a_1|, A_1)$.

Applying the Lemma 6 and the inequality (3.1), we have that the series $\sum_{k=1}^{\infty} \Delta a_k \overline{D}_k^{(r)}(x)$ is uniformly convergent on any compact subset of $[\varepsilon, \pi]$, where $\varepsilon > 0$.

Thus representation

$$g(x) = \sum_{k=0}^{\infty} \Delta a_k \overline{D}_k(x)\,.$$

implies that

$$g^{(r)}(x) = \sum_{k=0}^{\infty} \Delta a_k \overline{D}_k^{(r)}(x)\,.$$

Then,

$$\int_{\pi/(m+1)}^{\pi} |g^{(r)}(x)|dx \le \sum_{j=1}^{m} \int_{\pi/(j+1)}^{\pi/j} \left| \sum_{k=0}^{j-1} \Delta a_k \overline{D}_k^{(r)}(x) \right| dx +$$

$$O\left(\sum_{j=1}^{m} \int_{\pi/(j+1)}^{\pi/j} \left| \sum_{k=j}^{\infty} \Delta a_k \overline{D}_k^{(r)}(x) \right| dx \right)\,. \tag{3.2}$$

Let

$$I_1 = \sum_{j=1}^{m} \int_{\pi/(j+1)}^{\pi/j} \left| \sum_{k=0}^{j-1} \Delta a_k \overline{D}_k^{(r)}(x) \right| dx\,, \quad I_2 = \sum_{j=1}^{m} \int_{\pi/(j+1)}^{\pi/j} \left| \sum_{k=j}^{\infty} \Delta a_k \overline{D}_k^{(r)}(x) \right| dx\,.$$

Applyyng the well-known expansion

$$\operatorname{ctg} \frac{x}{2} = \frac{2}{x} + \sum_{n=1}^{\infty} \frac{4x}{x^2 - 4n^2\pi^2}$$

it is not difficult to proof the following estimate:

$$\left(\operatorname{ctg}\frac{x}{2}\right)^{(r)} = \frac{2(-1)^r r!}{x^{r+1}} + O(1)\,, \quad x \in (0, \pi]\,.$$

Thus

$$\overline{D}_n^{(r)}(x) = \frac{(-1)^{r+1} r!}{x^{r+1}} + O\big((n+1)^{r+1}\big)\,, \quad x \in (0, \pi]\,.$$

Hence,

$$I_1 = r! \sum_{j=1}^{m} \left| \sum_{k=0}^{j-1} \Delta a_k \right| \int\limits_{\pi/(j+1)}^{\pi/j} \frac{dx}{x^{r+1}} + O\left(\sum_{j=1}^{m} \left[\sum_{k=0}^{j-1} |\Delta a_k| (k+1)^{r+1} \right] \int\limits_{\pi/(j+1)}^{\pi/j} dx \right) =$$

$$= O_\alpha \left(\sum_{j=1}^{m} |a_j| j^{r-1} \right) + O\left(\sum_{j=1}^{m} \sum_{k=0}^{j-1} \frac{(k+1)^{r+1} |\Delta a_k|}{j(j+1)} \right),$$

where O_α depends on α. But

$$\sum_{j=1}^{m} \sum_{k=0}^{j-1} \frac{(k+1)^{r+1} |\Delta a_k|}{j(j+1)} = \sum_{j=1}^{m} \frac{1}{j(j+1)} \sum_{k=0}^{j-1} (k+1)^{r+1} |\Delta a_k| \leq$$

$$\leq \sum_{k=0}^{\infty} (k+1)^{r+1} |\Delta a_k| \sum_{j=k+1}^{\infty} \frac{1}{j(j+1)} =$$

$$= \sum_{k=0}^{\infty} (k+1)^{r} |\Delta a_k| =$$

$$= |\Delta a_0| + \sum_{k=1}^{\infty} (k+1)^{r} |\Delta a_k| \leq$$

$$\leq \sum_{k=1}^{\infty} |\Delta a_k| + 2^r \sum_{k=1}^{\infty} k^r |\Delta a_k| \leq$$

$$\leq (1 + 2^r) \sum_{k=1}^{\infty} k^r |\Delta a_k|$$

Thus,

$$\sum_{j=1}^{m} \sum_{k=0}^{j-1} \frac{|\Delta a_k| (k+1)^{r+1}}{j(j+1)} \leq (1 + 2^r) \sum_{k=1}^{\infty} k^r |\Delta a_k| \, .$$

Therefore,

$$I_1 = O_\alpha \left(\sum_{j=1}^{m} |a_j| j^{r-1} \right) + O_\alpha \left(\sum_{k=1}^{\infty} k^r |\Delta a_k| \right) \, ,$$

i.e. by inequality (3.1),

$$I_1 = O_\alpha \left(\sum_{j=1}^{m} |a_j| j^{r-1} \right) + O_\alpha \left(\sum_{k=1}^{\infty} k^\alpha A_k \right) \, . \tag{3.3}$$

Applying the Abel's transformation, we have:

$$I_2 \leq \sum_{j=1}^{m} \left[\sum_{k=j}^{\infty} (\Delta A_k) \, J_k + A_j T_j \right] \, ,$$

where

$$J_k = \int\limits_{\pi/(j+1)}^{\pi} \left| \sum_{i=0}^{k} \frac{\Delta a_i}{A_i} \overline{D}_i^{(r)}(x) \right| dx$$

and

$$T_j = \int\limits_{\pi/(j+1)}^{\pi/j} \left| \sum_{i=0}^{j-1} \frac{\Delta a_i}{A_i} \overline{D}_i^{(r)}(x) \right| dx$$

Applying the Holder-Hausdorff-Young technique (see the proof of Lemma 4), we obtain $J_k = O_{p,\alpha}\big((k+1)^{\alpha+1}\big)$, where $O_{p,\alpha}$ depends on α and p. Then by Lemma 6,

$$\begin{aligned} T_j &= O\left(j^r \ln\Big(1+\frac{1}{j}\Big) \left(\sum_{i=0}^{j-1} \frac{|\Delta a_i|}{A_i} \right) \right) + O\left(\sum_{i=0}^{j-1} \frac{|\Delta a_i|}{A_i} \int\limits_{\pi/(j+1)}^{\pi/j} \frac{dx}{x^{r+1}} \right) = \\ &= O\left(j^{\alpha} \left(j^{p(r-\alpha)-1} \sum_{i=0}^{j-1} \frac{|\Delta a_i|^p}{A_i^p} \right)^{1/p} \right) + O_{\alpha}\left(j^{r-1} \sum_{i=0}^{j-1} \frac{|\Delta a_i|}{A_i} \right) = \\ &= O(j^{\alpha}) + O_{\alpha}\left(j^{\alpha} \left(j^{p(r-\alpha)-1} \sum_{i=0}^{j-1} \frac{|\Delta a_i|^p}{A_i^p} \right)^{1/p} \right) = \\ &= O(j^{\alpha}) + O_{\alpha}(j^{\alpha}) = O_{\alpha}(j^{\alpha}) \,. \end{aligned}$$

Thus

$$\begin{aligned} I_2 &\le O_{p,\alpha}(1) \sum_{k=1}^{\infty} (k+1)^{\alpha+1} (\Delta A_k) + O_{\alpha}(1) \sum_{j=1}^{\infty} j^{\alpha} A_j = \\ &= O_{p,\alpha}(1) \sum_{k=1}^{\infty} k^{\alpha} A_k + O_{\alpha}(1) \sum_{j=1}^{\infty} j^{\alpha} A_j = \qquad (3.4) \\ &= O_{p,\alpha} \left(\sum_{k=1}^{\infty} k^{\alpha} A_k \right) , \end{aligned}$$

since $n^{\alpha+1} A_n = o(1)$, $n \to \infty$.

Combining the inequalities (3.2), (3.3) and (3.4), the inequality (1.3) is satisfied.

If $\sum\limits_{n=1}^{\infty} n^{r-1} |a_n| < \infty$, by letting $m \to \infty$ in inequality (1.3), we obtain that the r-th derivate of the series (1.2) is a Fourier series of some $g^{(r)} \in L^1(0,\pi)$ and the inequality (1.4) is satisfied.

Corollary. *Let the coefficients of the series (1.2) belong to the class $S_{p\alpha r}$, $1 < p \le 2$, $\alpha \ge 0$, $r \in \{1, 2, \dots [\alpha]\}$. Then the following inequalty holds:*

$$\int\limits_{0}^{\pi} |g^{(r)}(x)| dx \le O_{p,\alpha} \left(\sum_{j=1}^{\infty} j^{\alpha} A_j \right)$$

where $O_{p,\alpha}$ depends on p and α.

Proof. By inequalities

$$\sum_{n=1}^{m} |a_n|\, n^{r-1} \leq \sum_{n=1}^{\infty} n^{r-1} \sum_{k=n}^{\infty} |\Delta a_k| \leq \sum_{n=1}^{\infty} n^{r-1} \sum_{k=n}^{\infty} A_k =$$

$$= \sum_{k=1}^{\infty} A_k \sum_{n=1}^{k} n^{r-1} \leq \sum_{k=1}^{\infty} k^r A_k \,,$$

and by Theorem 2, we obtain:

$$\int_{\pi/(m+1)}^{\pi} |g^{(r)}(x)| dx \leq O_\alpha \left(\sum_{n=1}^{\infty} n^\alpha A_n \right) + O_{p,\alpha} \left(\sum_{j=1}^{\infty} j^\alpha A_j \right) .$$

Letting $m \to \infty$, the inequality is satisfied.

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