GENERALISED TAYLOR'S FORMULA WITH ESTIMATES OF THE REMAINDER

P. CERONE

ABSTRACT. Generalised Taylor's formulae are obtained utilising an integral remainder in which the kernel is comprised of a product of two polynomials, each of which satisfy the Appell condition $w'_k = w_{k-1}$. Bounds are obtained in terms of the Lebesgue norms. Prior results are shown to be recaptured as special cases of the current development. Perturbed Taylor's formulae are also investigated with this general setting complete with bounds.

1. INTRODUCTION

A number of authors have recently obtained generalisations of the traditional Taylor series expansion of a function $f(x)$ about a point a assuming sufficient differentiability. Estimates of bounds on the remainder have also been procured.

Before proceeding further let us introduce some notation. We shall term polynomials of degree k, W_k as Appell type and say $W_k \in \mathcal{A}$ if they satisfy the condition

$$W'_k = \xi_k W_{k-1}(t), \; W_0(t) = 1, \quad t \in \mathbb{R}. \tag{1.1}$$

These are so named since Appell studied (1.1) with $\xi_k = k$ in 1880 (see [1]). Polynomials satisfying (1.1) with $\xi_k = 1$ have been termed harmonic polynomials in Matić et al. [7] however a simple scaling will demonstrate that these are Appell.

The following results we obtained by Matić et al. [7] where $P_n(t)$ satisfy (1.1) with $\xi_k = 1$.

Theorem 1. *Let $\{P_n\}_{n\in\mathbb{N}}$ be a sequence of polynomials, that satisfy*

$$P'_n(t) = P_{n-1}(t), \; P_0(t) = 1, \; t \in \mathbb{R}, \; n \in \mathbb{N}, \; n \geq 1. \tag{1.2}$$

Further, let $I \subset \mathbb{R}$ be a closed interval and $a \in I$. If $f : I \to \mathbb{R}$ is any function such that for some $n \in \mathbb{N}$, $f^{(n)}$ is absolutely continuous, then for any $x \in I$

$$f(x) = T_n(f; a, x) + R_n(f; a, x) \tag{1.3}$$

where

$$T_n(f; a, x) = f(a) + \sum_{k=1}^{n} (-1)^{k+1} \left[P_k(x) f^{(k)}(x) - P_k(a) f^{(k)}(a) \right], \tag{1.4}$$

$$R_n(f; a, x) = (-1)^n \int_a^x P_n(t) f^{(n+1)}(t) \, dt. \tag{1.5}$$

They also pointed out the following bounds for the remainder $R_n(f, \cdot, \cdot)$.

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Corollary 1. *With the above assumptions, we have the estimations*

$$|R_n(f;a,x)| \leq \begin{cases} \|P_n\|_\infty \left\|f^{(n+1)}\right\|_1 & \text{provided} \quad f^{(n+1)} \in L_1[a,x], \\ \|P_n\|_q \left\|f^{(n+1)}\right\|_p & \text{provided} \quad f^{(n+1)} \in L_p[a,x],\ p>1,\ \frac{1}{p}+\frac{1}{q}=1, \\ \|P_n\|_1 \left\|f^{(n+1)}\right\|_\infty & \text{provided} \quad f^{(n+1)} \in L_\infty[a,x], \end{cases} \tag{1.6}$$

where $x \geq a$ and $\|\cdot\|_s$ $(1 \leq s \leq \infty)$ are the usual $s-$Lebesgue norms. That is,

$$\|g\|_s := \left(\int_a^x |g(t)|^s\, dt\right)^{\frac{1}{s}}, \quad s \in [1,\infty)$$

and

$$\|g\|_\infty := ess \sup_{t\in[a,x]} |g(t)|.$$

We introduce superscripts for $T(f;a,x)$ and $R_n(f;a,x)$ as given in (1.4) and (1.5) respectively to reflect the particular polynomial $P_n(t)$ involved. Let

$$P_n^{c_\lambda}(t) = \frac{(t-\theta(\lambda))^n}{n!},\ \theta(\lambda) = \lambda a + (1-\lambda)x,\ \ \lambda \in [0,1], \tag{1.7}$$

$$P_n^B(t) = \frac{(x-a)^n}{n!} B_n\left(\frac{t-a}{x-a}\right), \tag{1.8}$$

and

$$P_n^E(t) = \frac{(x-a)^n}{n!} E_n\left(\frac{t-a}{x-a}\right) \tag{1.9}$$

represent polynomials involving; a convex combination of the end points, Bernoulli polynomials and Euler polynomials respectively.

With the polynomials (1.7) – (1.9) then from (1.4) and (1.5)

$$T_n^{c_\lambda}(f;a,x) = f(a) + \sum_{k=1}^{n} (-1)^{k+1} \frac{(x-a)^k}{k!} \left[\lambda^k f^{(k)}(x) + (-1)^{k+1}(1-\lambda)^k f^{(k)}(a)\right], \tag{1.10}$$

$$T_n^B(f;a,x) = f(a) + \frac{x-a}{2}\left[f'(x) + f'(a)\right] - \sum_{k=1}^{\left[\frac{n}{2}\right]} \frac{(x-a)^{2k}}{(2k)!} B_{2k}\left[f^{(2k)}(x) - f^{(2k)}(a)\right], \tag{1.11}$$

$$T_n^E(f;a,x) = f(a) + 2\sum_{k=1}^{\left[\frac{n+1}{2}\right]} \frac{(x-a)^{2k-1}\left(4^k-1\right)}{(2k)!} B_{2k}\left[f^{(2k-1)}(x) + f^{(2k-1)}(a)\right] \tag{1.12}$$

and

$$(1.13)\qquad \begin{aligned} R_n^{c_\lambda}(f;a,x) &= \frac{(-1)^{n+1}}{n!}\int_a^x (t-\theta(\lambda))^n f^{(n+1)}(t)\,dt, \\ \theta(\lambda) &= \lambda a + (1-\lambda)x, \quad \lambda \in [0,1], \end{aligned}$$

$$(1.14)\qquad R_n^B(f;a,x) = (-1)^{n+1}\frac{(x-a)^n}{n!}\int_a^x B_n\left(\frac{t-a}{x-a}\right) f^{(n+1)}(t)\,dt,$$

$$(1.15)\qquad R_n^E(f;a,x) = (-1)^{n+1}\frac{(x-a)^n}{n!}\int_a^x E_n\left(\frac{t-a}{x-a}\right) f^{(n+1)}(t)\,dt,$$

where $B_n(\cdot)$ are the Bernoulli polynomials, $B_n = B_n(0)$ the Bernoulli numbers and $E_n(\cdot)$ the Euler polynomials.

The above expressions (1.10) – (1.15) were obtained by Matić et al. [7] for the Bernoulli and Euler polynomials but only for the equivalent of $\lambda = 0$ and $\frac{1}{2}$ in (1.7).

Cerone and Dragomir [4] obtained the following theorem which follows directly from Corollary 1 with $P_n^{c_\lambda}(t)$ as given by (1.7) and using (1.10) and (1.13).

Theorem 2. *Assume that f is as in Theorem 1, then we have*

$$(1.16)\qquad \begin{aligned} & |f(x) - T_n^{c_\lambda}(f;a,x)| \\ = & |R_n^{c_\lambda}(f;a,x)| \\ \le & \begin{cases} \frac{1}{n!}(x-a)^n\left[\frac{1}{2}+\left|\lambda-\frac{1}{2}\right|\right]^n \left\|f^{(n+1)}\right\|_1 \quad \text{if } f^{(n+1)} \in L_1[a,x]; \\ \frac{1}{n!(nq+1)^{\frac{1}{q}}}(x-a)^{n+\frac{1}{q}}\left[(1-\lambda)^{nq+1}+\lambda^{nq+1}\right]^{\frac{1}{q}}\left\|f^{(n+1)}\right\|_p \\ \qquad \text{if } f^{(n+1)} \in L_p[a,x], \ p>1, \ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{1}{(n+1)!}(x-a)^{n+1}\left[(1-\lambda)^{n+1}+\lambda^{n+1}\right]\left\|f^{(n+1)}\right\|_\infty, \\ \qquad \text{if } f^{(n+1)} \in L_\infty[a,x]. \end{cases} \end{aligned}$$

It was also noted that since $h_1(\lambda) = \left[\frac{1}{2}+\left|\lambda-\frac{1}{2}\right|\right]^n$, $h_2(\lambda) = \left[(1-\lambda)^{nq+1}+\lambda^{nq+1}\right]^{\frac{1}{q}}$ and $h_3(\lambda) = (1-\lambda)^{n+1}+\lambda^{n+1}$ are convex and symmetrical about $\frac{1}{2}$, then

$$\inf_{\lambda\in[0,1]} h_i(\lambda) = h_i\left(\frac{1}{2}\right), \quad i = 1,2,3.$$

Hence the best inequality possible in the sense of providing the tightest bound is

$$\left|f(x)-T_n^{c_{\frac{1}{2}}}(f;a,x)\right| \leq \begin{cases} \dfrac{1}{2^n n!}(x-a)^n \left\|f^{(n+1)}\right\|_1; \\[2ex] \dfrac{1}{n!(nq+1)^{\frac{1}{q}}2^n}(x-a)^{n+\frac{1}{q}}\left\|f^{(n+1)}\right\|_p, \\ \qquad \text{if } f^{(n+1)}\in L_p[a,x], \ p>1, \ \frac{1}{p}+\frac{1}{q}=1; \\[2ex] \dfrac{1}{(n+1)!2^n}(x-a)^{n+1}\left\|f^{(n+1)}\right\|_\infty, \text{ if } f^{(n+1)}\in L_\infty[a,x]. \end{cases} \tag{1.17}$$

Taking $\lambda=0$ in (1.16) produces the classical Taylor series expansion in terms of the $L_p[a,x]$, $p\geq 1$, Lebesgue norms for the bounds (see for example, Dragomir [6]). That is,

$$\left|f(x)-\left[f(a)+\sum_{k=1}^{n}\frac{(x-a)^k}{k!}f^{(k)}(a)\right]\right| \leq \begin{cases} \dfrac{(x-a)^n}{n!}\left\|f^{(n+1)}\right\|_1, & f^{(n+1)}\in L_1[a,x]; \\[2ex] \dfrac{(x-a)^{n+\frac{1}{q}}}{n!(nq+1)^{\frac{1}{q}}}\left\|f^{(n+1)}\right\|_p, & f^{(n+1)}\in L_p[a,x], \\ & p>1, \ \frac{1}{p}+\frac{1}{q}=1; \\ \dfrac{(x-a)^{n+1}}{(n+1)!}\left\|f^{(n+1)}\right\|_\infty, & f^{(n+1)}\in L_\infty[a,x]. \end{cases} \tag{1.18}$$

Recently in [5] Dragomir introduced a perturbed Taylor's formula using the Grüss inequality for the Chebychev functional. Matić et al. [3] obtained generalised Taylor's formulae involving expansions in terms of general polynomials satisfying (1.2) producing in particular Theorem 1 and Corollary 1 above. They also examined perturbed versions of (1.3), namely

$$\begin{aligned} f(x) &= T_n(f;a,x) \\ &\quad +(-1)^n\left[P_{n+1}(x)-P_{n+1}(a)\right]\left[f^{(n)};a,x\right]+\rho_n(f;a,x), \end{aligned} \tag{1.19}$$

where

$$\left[f^{(n)};a,x\right]:=\frac{f^{(n)}(x)-f^{(n)}(a)}{x-a}, \quad \text{the divided difference,} \tag{1.20}$$

$$\rho_n(f;a,x) \quad \text{is the remainder.} \tag{1.21}$$

Dragomir [5] developed an estimate of the remainder using the Grüss inequality for $P_n(t)=\frac{(t-x)^n}{n!}$, Matić et al. [7] used a premature or pre-Grüss argument to procure bounds on $\rho_n^{c_{\frac{1}{2}}}(f;a,x)$, $\rho_n^{c_0}(f;a,x)$, $\rho_n^{c_B}(f;a,x)$ and $\rho_n^{c_E}(f;a,x)$. Dragomir [6] obtained tighter bounds for the same polynomial generators of the perturbed Taylor series for $f^{(n+1)}\in L_2(I)$ with $x,a\in I\subseteq\mathbb{R}$. In the paper [4], Cerone and

Dragomir procured bounds on $\rho_n(f;a,x)$ in terms of $\Delta-$seminorms resulting from the Chebychev functional and Korkine's identity which are used to produce (1.18).

In two recent papers, Cerone [2] and [3] developed quadrature rules utilizing Peano kernels comprised of the product of polynomials satisfying the Appell condition (1.2).

It is the intention that in the current paper the results of Matić et al. [7], as exemplified in Theorem 1 and Corollary 1, be extended to polynomials that in themselves do not satisfy (1.2) but are comprised of products of polynomials that do. That is, develop the results of Cerone [2], [3] to obtain generalised Taylor series involving different polynomials complete with an estimate of bounds for the remainder. Perturbed Taylor series will also be analysed.

2. Results from Product Appell Polynomials

We commence by developing an identity.

Lemma 1. *Let p_k, $q_k \in \mathcal{A}$ for $k \in \mathbb{N}$ and so are sequences of Appell polynomials satisfying (1.2).*
Further, let I be a closed interval and $a \in I$ then if $f : I \to \mathbb{R}$ is such that $f^{(n)}$ is absolutely continuous on I, the following identity holds for any $x \in I$

$$f(x) = \tau_n(f;a,x) + R_n(f;a,x), \tag{2.1}$$

where

$$\tau_n(f;a,x) = f(a) + \frac{1}{\binom{n}{m}} \sum_{k=1}^{n} (-1)^{k+1} \Big[K_n^{(k)}(x) f^{(n-k)}(x) + (-1)^{k+1} - K_n^{(k)}(a) f^{(n-k)}(a) \Big], \tag{2.2}$$

$$R_n(f;a,x) = \frac{(-1)^n}{\binom{n}{m}} \int_a^x K_n(t) f^{(n+1)}(t)\,dt \tag{2.3}$$

with

$$K_n(t) = p_{n-m}(t)\, q_m(t), \quad t \in [a,x] \tag{2.4}$$

and

$$K_n^{(k)}(t) = \sum_{j=L}^{U} \binom{k}{j} p_{n-m-j}(t)\, q_{j-k+m}(t), \tag{2.5}$$

$$U = \min\{k, n-m\}, \quad L = \max\{0, k-m\}. \tag{2.6}$$

Proof. Consider

$$(-1)^n \int_a^x K_n(t) f^{(n+1)}(t)\,dt$$

then repeated integration by parts gives

$$(-1)^n \int_a^x K_n(t) f^{(n+1)}(t)\,dt = \sum_{k=0}^{n-1} (-1)^k K_n^{(k)}(t) f^{(n-k)}(t)\Big|_a^x + \int_a^x K_n^{(n)}(t) f'(t)\,dt. \tag{2.7}$$

Now, using the Leibnitz rule for differentiation of a product gives from (2.2)

$$K_n^{(k)}(t) = \sum_{j=0}^{k} \binom{k}{j} \frac{d^j}{dt^j} p_{n-m}(t) \frac{d^{k-j}}{dt^{k-j}} q_m(t). \tag{2.8}$$

Further, since $p_{n-m}(\cdot)$ and $q_m(\cdot)$ are Appell polynomials satisfying (1.2), then,

$$W_k^{(j)}(t) = \begin{cases} W_{k-j}(t), & j \le k \\ \\ 0, & j > k \end{cases}$$

and so from (2.8)

$$K_n^{(k)}(t) = \sum_{j=\max\{0,k-m\}}^{\min\{k,n-m\}} \binom{k}{j} p_{n-m-j}(t) q_{j-k+m}(t). \tag{2.9}$$

Also, for $k = m$ we deduce from (2.9) that $j = n - m$ since the subscripts of p and q are non negative giving

$$K_n^{(n)}(t) = \binom{n}{m}. \tag{2.10}$$

Using (2.9) and (2.10) in (2.8) readily produces the desired result (2.3) after minor manipulation. ∎

Remark 1. *If we take $m = 0$ then $K_n(t) = p_n(t)$ and the generalised Taylor's formula, (1.3) – (1.5), of Matić et al.* [7] *is recaptured. If we further take $p_n(t) = P_n^{c_0}(t) = \frac{(t-x)^n}{n!}$ then the classical Taylor's formula results. Further, if $p_n(t) = P_n^{c_\lambda}(t)$, $P_n^B(t)$ or $P_n^E(t)$, as defined by (1.7) – (1.9), then the results (1.10) – (1.15) are obtained and so recapturing existing work.*

Theorem 3. *Let f satisfy the assumptions of Lemma 1. Then the following estimations holds:*

$$\begin{aligned} & |f(x) - \tau_n(f;a,x)| \\ = & |R_n(f;a,x)| \\ \le & \begin{cases} Q_n(1,x) \left\| f^{(n+1)} \right\|_\infty, & f^{(n+1)} \in L_\infty[a,x]; \\ \\ Q_n(q,x) \left\| f^{(n+1)} \right\|_p, & f^{(n+1)} \in L_p[a,x], \\ & p > 1,\ \frac{1}{p} + \frac{1}{q} = 1; \\ \frac{1}{\binom{n}{m}} \sup_{t \in [a,x]} |K_n(t)| \left\| f^{(n+1)} \right\|_1, & f^{(n+1)} \in L_1[a,x], \end{cases} \end{aligned} \tag{2.11}$$

where

$$Q_n(r,x) = \frac{1}{\binom{n}{m}} \left(\int_a^x |K_n(t)|^r \, dt \right)^{\frac{1}{r}},$$

with $K_n(t)$ as defined by (2.4).

Proof. The estimations are a simple consequence of Hölder's inequalities and properties of the integral and absolute value. ∎

Remark 2. *The results of Lemma 1 and Theorem 3 are quite general being capable of recapturing prior generalised Taylor formulae as particular cases of the current work. Theorem 3 provides bounds on the remainder $R_n(f;a,x)$ defined in terms of $K_n(t)$ which is comprised of the product of Appell polynomials satisfying the* *conditions (1.2). The bounds provided by (2.11) may be evaluated either analytically using the properties of the specific polynomials in question or else numerically.*

The following corollary gives an example for a particular n^{th} degree polynomial.

Corollary 2. *Let the conditions of Lemma 1 hold. The following result is then valid. Namely, for* $\theta \in [a,x]$

$$\begin{aligned} &\left|f(x)-\tau_n^*(f;a,x)\right| \\ =\;& \left|R_n^*(f;a,x)\right| \\ \le\;& \begin{cases} Q_n^*(1,x)\left\|f^{(n+1)}\right\|_\infty, \; f^{(n+1)}\in L_\infty[a,x]; \\[2mm] Q_n^*(q,x)\left\|f^{(n+1)}\right\|_p, \; f^{(n+1)}\in L_p[a,x], \; p>1, \; \frac{1}{p}+\frac{1}{q}=1; \\[2mm] \dfrac{(x-a)^{n-m}}{n!2^m}\left[x-a+2\left|\theta-\dfrac{a+x}{2}\right|\right]^m \left\|f^{(n+1)}\right\|_1, \\ \qquad\qquad f^{(n+1)}\in L_1[a,x], \end{cases} \end{aligned} \tag{2.12}$$

where

$$\begin{aligned} &\tau_n^*(f;a,x) \\ =\;& f(a)+\frac{1}{\binom{n}{m}}\sum_{k=1}^{n}(-1)^{k+1}\left[\kappa_n^{(k)}(x)\,f^{(n-k)}(x)-\kappa_n^{(k)}(a)\,f^{(n-k)}(a)\right], \end{aligned} \tag{2.13}$$

$$\kappa_n(t)=\frac{(t-x)^{n-m}}{(n-m)!}\cdot\frac{(t-\theta)^m}{m!}, \; \theta\in[a,x], \tag{2.14}$$

$$\kappa_n^{(k)}(t)=\sum_{j=L}^{U}\binom{k}{j}\frac{(t-x)^{n-m-j}}{(n-m-j)!}\cdot\frac{(t-\theta)^{j-k+m}}{(j-k+m)!}, \tag{2.15}$$

with $U=\min\{k,n-m\}, \; L=\max\{0,k-m\}$.
Further,

$$\begin{aligned} & n!Q_n^*(r,x) \\ =\;& \begin{cases} \left[\dfrac{(x-a)^{rn+1}}{rn+1}\right]^{\frac{1}{r}}, \; \theta=x \\[3mm] (x-\theta)^{rn+1}\left[\chi\left(r(n-m),rm,\dfrac{\theta-a}{x-a}\right)\right. \\ \qquad\qquad \left.+B\left(r(n-m)+1,rm+1\right)\Big)\right]^{\frac{1}{r}}, \; \theta\in[a,x) \end{cases}, \end{aligned} \tag{2.16}$$

with

$$\chi(\alpha,\beta,X)=\int_0^X (u+1)^\alpha u^\beta du$$

and

$$B(\alpha,\beta)=\int_0^1 (1-u)^{\alpha-1}u^{\beta-1}du,$$

the Euler beta function.

Proof. For $K_n(t)$ of (2.4) given by $\kappa_n(t)$ of (2.14) we have from (2.1) the identity in (2.12) with $\tau_n^*(f;a,x)$ as given by (2.13).

The bounds are obtained utilising (2.11).

For $f^{(n+1)} \in L_1[a,x]$ then we require to determine from the third inequality in (2.11), with $K_n(t) = \kappa_n(t)$ defined in (2.14). Namely,

$$\begin{aligned}\frac{1}{\binom{n}{m}} \sup_{t\in[a,x]} |\kappa_n(t)| &= \frac{1}{\binom{n}{m}} \sup_{t\in[a,x]} |t-x|^{n-m} |t-\theta|^m \\ &= \frac{1}{\binom{n}{m}} \sup_{t\in[a,x]} (x-t)^{n-m} |t-\theta|^m \\ &= \frac{(x-a)^{n-m}}{n!} [\max\{\theta-a, x-\theta\}]^m,\end{aligned}$$

which produces the third inequality of (2.12) on using the result $2\max\{A,B\} = A+B+|A-B|$.

Now, for $f^{(n+1)} \in L_p[a,x]$, $1 \le p < \infty$ then from (2.11) with $K_n(t)$ given by $\kappa_n(t)$ of (2.14)

$$\begin{aligned}Q_n^*(r,x) &= \frac{1}{\binom{n}{m}} \left(\int_a^x |\kappa_n(t)|^r dt\right)^{\frac{1}{r}} \\ &= \frac{1}{n!}\left(\int_a^x (x-t)^{r(n-m)} |t-\theta|^{rm} dt\right)^{\frac{1}{r}}.\end{aligned} \tag{2.17}$$

Firstly, if $\theta = x$ then

$$Q_n^*(r,x) = \frac{1}{n!}\left(\int_a^x (x-t)^{rn} dt\right)^{\frac{1}{r}} = \frac{1}{n!}\left[\frac{(x-a)^{rn+1}}{rn+1}\right]^{\frac{1}{r}}. \tag{2.18}$$

For $\theta \in [a,x)$ then from (2.16),

$$\begin{aligned}&Q_n^*(r,x) \\ &= \frac{1}{n!}\left[\int_a^\theta (x-t)^{r(n-m)} (\theta-t)^{rm} dt + \int_\theta^x (x-t)^{r(n-m)} (t-\theta)^{rm} dt\right]^{\frac{1}{r}} \\ &: = \frac{1}{n!}[I(a,\theta,x) + J(a,\theta,x)]^{\frac{1}{r}}.\end{aligned} \tag{2.19}$$

Now, taking $u = \theta - t$ then

$$I(a,\theta,x) = \int_0^{\theta-a} (u+x-\theta)^{r(n-m)} u^{rm} du$$

which, with a further substitution of $u = (x-\theta)v$ gives

$$I(a,\theta,x) = (x-\theta)^{rn+1} \int_0^{\frac{\theta-a}{x-a}} (v+1)^{r(n-m)} v^{rm} dv. \tag{2.20}$$

Further, the substitution $(x-\theta)w = t-\theta$ produces from (2.17)

$$J(a,\theta,x) = (x-\theta)^{rn+1} \int_0^1 (1-w)^{r(n-m)} w^{rm} dw. \tag{2.21}$$

Combining (2.17) – (2.21) produces (2.16) and thus completing the proof of the theorem. ∎

Remark 3. *Different choices of m, n and θ recapture earlier results. If $m = n$ then $\kappa_n(t)$ of (2.14) becomes $P_n^{c_\lambda}(t)$ given by (1.7) and the resulting Taylor expansion and bounds are procured. Taking $\theta = x$ in (2.15) gives $P_n^{c_0}(t)$ from (1.7) producing agreement between the results (1.18) and (2.11).*

The tightest bounds from (2.11) are obtained if we take $\theta = \frac{a+x}{2}$. The following corollary then holds.

Corollary 3. *Let the conditions of Lemma 1 and Corollary 2 hold, then*

$$\left| f(x) - \left[f(a) + \frac{1}{\binom{n}{m}} \sum_{k=1}^{n} (-1)^{k+1} \left[\tilde{\kappa}_n^{(k)}(x) f^{(n-k)}(x) - \tilde{\kappa}_n^{(k)}(a) f^{(n-k)}(a) \right] \right] \right|$$

$$\leq \begin{cases} B_1(x) \left\| f^{(n+1)} \right\|_\infty, & f^{(n+1)} \in L_\infty[a,x]; \\ B_q(x) \left\| f^{(n+1)} \right\|_p, & f^{(n+1)} \in L_p[a,x], \\ & p > 1, \ \frac{1}{p} + \frac{1}{q} + 1; \\ \frac{(x-a)^n}{n! 2^m} \left\| f^{(n+1)} \right\|_1, & f^{(n+1)} \in L_1[a,x], \end{cases}$$

where

$$\begin{aligned} \tilde{\kappa}_n^{(k)}(x) &= \binom{k}{n-m} \left(\frac{x-a}{2} \right)^{n-k}, \\ \tilde{\kappa}_n^{(k)}(a) &= \sum_{j=L}^{U} \binom{k}{j} \frac{(-1)^{n-k} (x-\theta)^{n-k}}{2^{j-k+m} (n-m-j)! (j-k+m)!}, \end{aligned}$$

with

$$U = \min\{k, n-m\}, \quad L = \max\{0, k-m\}$$

and

$$\begin{aligned} B_1(x) &= \left(\frac{x-a}{2} \right)^{n+1} \left[\chi\left(n-m, m, \frac{1}{2} \right) + \frac{1}{(n+1)\binom{n}{m}} \right], \\ B_q(x) &= \left(\frac{x-a}{2} \right)^{n+\frac{1}{q}} \left[\chi\left(q(n-m), qm, \frac{1}{2} \right) + B(q(n-m), qm+1) \right]^{\frac{1}{q}}. \end{aligned}$$

Further, $\chi(\alpha, \beta, X)$ and $B(\alpha, \beta)$ are as defined in Corollary 2.

Proof. Taking $\theta = \frac{a+x}{2}$ in Corollary 2 produces the results stated after some minor algebraic manipulation. ∎

3. Perturbed Taylor's Formula

Perturbed Taylor series may be obtained utilising the well known Chebychev functional and its properties including identities and bounds. It has a very extensive and long history, see for example [8, pp. 295-310].

For $g, h : I \to \mathbb{R}$ which are both integrable as is their product, then

$$\mathfrak{T}(g,h) = \mathcal{M}(gh) - \mathcal{M}(g)\mathcal{M}(h) \tag{3.1}$$

is the well known Chebychev functional, where $\mathcal{M}(g) = \frac{1}{x-a}\int_a^x g(t)\,dt$ is the integral mean and $x > a$.

Further, a number of sharp bounds for $|\mathfrak{T}(g,h)|$ exist, under various assumptions about g and h, including (see [3] for example)

$$|\mathfrak{T}(g,h)| \leq \begin{cases} [\mathfrak{T}(g,g)]^{\frac{1}{2}}[\mathfrak{T}(h,h)]^{\frac{1}{2}}, & g,h \in L_2(I) \\ \dfrac{A_u - A_l}{2}[\mathfrak{T}(g,g)]^{\frac{1}{2}}, & A_l \leq h(t) \leq A_u,\ t \in [a,x] \\ \left(\dfrac{A_u - A_l}{2}\right)\left(\dfrac{B_u - B_l}{2}\right), & B_l \leq g(t) \leq B_u,\ t \in [a,x] \quad \text{(Grüss)} \end{cases} \tag{3.2}$$

The bounds in (3.2) are from top to bottom in order of increasing coarseness. They utilise the Korkine identity, namely

$$\mathfrak{T}(g,h) := \frac{1}{2(x-a)^2}\int_a^x\int_a^x (h(t)-h(s))(g(t)-g(s))\,dtds \tag{3.3}$$

for their development.

As mentioned in the introduction, the identity (1.19) was introduced by Dragomir [5] for $P_n(x) = \frac{(t-x)^n}{n!}$ and bounds were obtained on the remainder utilising the Grüss inequality, the third in (3.2). Matić et al. [5] capitalised on the fact that $[\mathfrak{T}(P_n,P_n)]^{\frac{1}{2}}$ can be evaluated explicitly for particular polynomials $P_n(t)$ and so utilised the second inequality in (3.2), termed by them as the *pre-Grüss* inequality, to obtain bounds.

It should be emphasised that (1.19) comes from (3.1) and (3.2) where

$$\rho_n(f;a,x) = (x-a)\,\mathfrak{T}\left(P_n, f^{(n+1)}\right). \tag{3.4}$$

Specifically, Matić et al. [7] obtain the bound

$$|\rho_n(f;a,x)| \leq (x-a)\frac{\left|\Gamma_{n+1}(x) - \gamma_{n+1}(x)\right|}{2}\sqrt{\mathfrak{T}(P_n,P_n)}, \tag{3.5}$$

where

$$\gamma_{n+1}(x) := \inf_{t\in[a,x]} f^{(n+1)}(t),\quad \Gamma_{n+1}(x) = \sup_{t\in[a,x]} f^{(n+1)}(t).$$

For $f^{(n+1)} \in L_2(I)$ and $x > a$, Dragomir [6] obtained, using the first inequality in (3.2) and (3.4),

$$|\rho_n(f;a,x)| \leq (x-a)[\mathfrak{T}(P_n,P_n)]^{\frac{1}{2}}\,\sigma\left(f^{(n+1)};a,x\right), \tag{3.6}$$

where

$$\begin{aligned} \sigma\left(f^{(n+1)};a,x\right) &= \left[\mathfrak{T}\left(f^{(n+1)},f^{(n+1)}\right)\right]^{\frac{1}{2}} \\ : &= \left[\frac{1}{x-a}\left\|f^{(n+1)}\right\|_2^2 - \left(\left[f^{(n)};a,x\right]\right)^2\right]^{\frac{1}{2}} \end{aligned} \tag{3.7}$$

with

$$\left[f^{(n)};a,x\right] = \mathcal{M}\left(f^{(n+1)}\right) = \frac{1}{x-a}\int_a^x f^{(n+1)}(t)\,dt = \frac{f^{(n)}(x) - f^{(n)}(a)}{x-a}.$$

Let $\beta_n = [\mathfrak{T}(P_n, P_n)]^{\frac{1}{2}}$, then Matić et al. [7] showed that

$$\beta_n^B = (x-a)^{n+1} \left(\frac{|B_{2n}|}{(2n)!} \right)^{\frac{1}{2}}$$

and

$$\beta_n^E = 2(x-a)^{n+1} \left\{ \frac{(4^{n+1}-1)|B_{2m+2}|}{(2n+2)!} - \left[\frac{2(2^{n+2}-1)B_{n+2}}{(n+1)!} \right]^2 \right\}^{\frac{1}{2}}.$$

where the polynomials $P_n^B(t)$ and $P_n^E(t)$ involving the Bernoulli and Euler polynomials are present, respectively.

Consider now the polynomial $P_n^{c_\lambda}(t)$ as defined in (1.7), then

$$\beta_n^{c_\lambda} = [\mathfrak{T}(P_n^{c_\lambda}, P_n^{c_\lambda})]^{\frac{1}{2}}$$

and so from (3.1)

$$[\beta_n^{c_\lambda}]^2 = \mathcal{M}\left((P_n^{c_\lambda})^2 \right) - (\mathcal{M}(P_n^{c_\lambda}))^2. \tag{3.8}$$

Now, from (1.7),

$$\begin{aligned} n!(x-a)\mathcal{M}(P_n^{c_\lambda}) &= \int_a^x (t-\theta(\lambda))^n dt \\ &= \int_{-(1-\lambda)(x-a)}^{\lambda(x-a)} u^n du, \end{aligned}$$

$$n!(x-a)\mathcal{M}(P_n^{c_\lambda}) = \left[\lambda^{n+1} + (-1)^n (1-\lambda)^{n+1} \right] \frac{(x-a)^{n+1}}{n+1}. \tag{3.9}$$

Similarly,

$$\begin{aligned} (n!)^2 (x-a)\mathcal{M}\left((P_n^{c_\lambda})^2 \right) &= \int_a^x (t-\theta(\lambda))^{2n} dt \\ &= \left[\lambda^{2n+1} + (1-\lambda)^{2n+1} \right] \frac{(x-a)^{2n+1}}{2n+1}. \end{aligned} \tag{3.10}$$

Thus substitution of (3.9) and (3.10) into (3.8) gives, upon simplification

$$\beta_n^{c_\lambda} = \frac{(x-a)^n}{(n+1)!\sqrt{2n+1}} \gamma(\lambda), \tag{3.11}$$

where

$$\begin{aligned} \gamma^2(\lambda) &= (n+1)^2 \Big[\lambda^{2n+1} + (1-\lambda)^{2n+1} \\ &\quad - (2n+1) \left[\lambda^{n+1} + (-1)^n (1-\lambda)^{n+1} \right]^2 \Big]. \end{aligned} \tag{3.12}$$

We note that $\gamma(\lambda)$ is convex so that the tightest bound is obtained for $\lambda = \frac{1}{2}$. Also, taking either $\lambda = 0$ or 1 gives

$$\beta_n^{c_0} = \frac{(x-a)^n}{(n+1)!\sqrt{2n+1}} = \beta_n^{c_1}. \tag{3.13}$$

Further,

$$\beta_n^{c_{\frac{1}{2}}} = \frac{(x-a)^n}{(n+1)!\sqrt{2n+1}} \cdot \frac{1}{2^n} \left[n^2 + \frac{2n+1}{2} \left(1 - (-1)^n\right) \right]^{\frac{1}{2}},$$

which may be written more succinctly as

$$\beta_n^{c_{\frac{1}{2}}} = \frac{(x-a)^n}{(n+1)!\sqrt{2n+1}} \cdot \frac{1}{2^n} \begin{cases} n, & n \text{ even} \\ n+1, & n \text{ odd.} \end{cases} \tag{3.14}$$

The result (3.13) was obtained by Dragomir [5]. It was also obtained in Matić et al. [7] as was an equivalent result for $\beta_n^{c_{\frac{1}{2}}}$. The expression (3.14) shows that the bound is slightly tighter for n even than it is for n odd since $\frac{n}{n+1} < 1$.

Thus we have the perturbed Taylor series expansion (1.19) with error $\rho_n(f;a,x)$ bounded by (3.5) for $\gamma(x) \le f^{n+1}(t) \le \Gamma(x)$, $t \in [a,x]$ and by (3.5) for $f^{(n+1)} \in L_2(I)$. Here the bound is in terms of

$$\beta_n = \left[\mathfrak{T}(P_n, P_n)\right]^{\frac{1}{2}} \tag{3.15}$$

which is dependent on the particular polynomials used.

It should be noted that the Grüss inequality (the third in (3.2)) has not been emphasised for two main reasons. Namely, it is coarser than the other two and, for $P_n(t)$ involving the Bernoulli and Euler polynomials the bounds are not useful. However, for $P_n^{c_\lambda}$ as given by (1.7), we have

$$\phi_n(x) \le P_n^{c_\lambda} \le \Phi_n(x), \quad t \in [a,x],$$

where

$$\phi_n(x) = \inf_{t\in[a,x]} p_n(t) = \begin{cases} 0, & n \text{ even} \\ -(1-\lambda)^n \dfrac{(x-a)^n}{n!}, & n \text{ odd} \end{cases}$$

and

$$\Phi_n(x) = \sup_{t\in[a,x]} p_n(t) = \begin{cases} \dfrac{(x-a)^n}{n!} \left[\max\{\lambda, 1-\lambda\}\right]^n, & n \text{ even} \\ \lambda^n \dfrac{(x-a)^n}{n!}, & n \text{ odd.} \end{cases}$$

Hence, from (3.15)

$$\beta_n^{c_\lambda} \le \frac{\Phi_n(x) - \phi_n(x)}{2} = \frac{(x-a)^n}{2n!} \begin{cases} \left[\dfrac{1}{2} + \left|\lambda - \dfrac{1}{2}\right|\right]^n, & n \text{ even} \\ \lambda^n + (1-\lambda)^n, & n \text{ odd.} \end{cases}$$

The following theorem gives perturbed Taylor formulae involving product Appell polynomials.

Theorem 4. *Let p_k, $q_k \in \mathcal{A}$ for $k \in \mathbb{N}$ so that they are sequences of Appell polynomials satisfying (1.2). Further, let I be a closed interval and $a \in I$ then, for $f : I \to \mathbb{R}$ and $f^{(n)}$ absolutely continuous on I, the following inequalities hold. Namely, for $x \in I$,*

$$\left| f(x) - \tau_n(f;a,x) - \frac{(-1)^n}{\binom{n}{m}} U_n(x) \left[f^{(n)};a,x\right] \right| \tag{3.16}$$

$$
\begin{aligned}
&\leq (x-a)\, b_n(x)\, \sigma\left(f^{(n+1)}; a, x\right), && f^{(n+1)} \in L_2[I], \\
&\leq (x-a)\, b_n(x) \left(\frac{\Gamma_{n+1}(x) - \gamma_{n+1}(x)}{2}\right), && \gamma_{n+1}(x) \leq f^{(n+1)}(t) \leq \Gamma_{n+1}(x), \\
& && t \in [a, x] \\
&\leq \frac{(x-a)}{\binom{n}{m}} \cdot \frac{\left(\Phi_{n+1}(x) - \phi_{n+1}(x)\right)}{2} && \phi_{n+1}(x) \leq K_n(t) \leq \Phi_{n+1}(x), \\
&\qquad \times \frac{\left(\Gamma_{n+1}(x) - \gamma_{n+1}(x)\right)}{2}, && t \in [a, x]
\end{aligned}
$$

where $\tau_n(f; a, x)$ is as defined in (2.2),

$$U_n(x) = \int_a^x K_n(t)\, dt = \int_a^x p_{n-m}(t)\, q_m(t)\, dt, \tag{3.17}$$

$$\left[f^{(n)}; a, x\right] = \frac{f^{(n)}(x) - f^{(n)}(a)}{x-a}, \tag{3.18}$$

$$b_n(x) = \frac{1}{\binom{n}{m}} \left[\frac{1}{x-a} \int_a^x K_n^2(t)\, dt - \left(\frac{U_n(x)}{x-a}\right)^2\right]^{\frac{1}{2}}, \tag{3.19}$$

and $\sigma\left(f^{(n+1)}; a, x\right)$ being as defined by (3.7).

Proof. Associating $g(t)$ with $\frac{(-1)^n}{\binom{n}{m}} K_n(t)$ and $h(t)$ with $f^{(n+1)}(t)$ then from (3.1) we obtain

$$
\begin{aligned}
&\mathfrak{T}\left(\frac{(-1)^n}{\binom{n}{m}} K_n(t), f^{(n+1)}(t)\right) \\
&= \mathcal{M}\left(\frac{(-1)^n}{\binom{n}{m}} K_n(t)\, f^{(n+1)}(t)\right) - \mathcal{M}\left(\frac{(-1)^n}{\binom{n}{m}} K_n(t)\right) \mathcal{M}\left(f^{(n+1)}(t)\right)
\end{aligned}
$$

and thus, from (2.3)

$$
\begin{aligned}
&(x-a)\, \mathfrak{T}\left(\frac{(-1)^n}{\binom{n}{m}} K_n(t), f^{(n+1)}(t)\right) \\
&= R_n(f; a, x) - \frac{(-1)^n}{\binom{n}{m}} U_n(x) \left[f^{(n)}; a, x\right]
\end{aligned} \tag{3.20}
$$

where $U_n(x)$ and $\left[f^{(n)}; a, x\right]$ are given by (3.17) and (3.18).

Substituting (2.1) in (3.20) produces

$$
\begin{aligned}
&f(x) - \tau_n(f; a, x) - \frac{(-1)^n}{\binom{n}{m}} U_n(x) \left[f^{(n)}; a, x\right] \\
&= (x-a)\, \mathfrak{T}\left(\frac{(-1)^n}{\binom{n}{m}} K_n(t), f^{(n+1)}(t)\right),
\end{aligned}
$$

which upon taking the modulus and utilising (3.2) gives the results (3.16) where $b_n(x) = \frac{1}{\binom{n}{m}} \left[\mathfrak{T}(K_n(t), K_n(t))\right]^{\frac{1}{2}}$, since $\mathfrak{T}(ag, ah) = a^2 \mathfrak{T}(g, h)$, a constant. ∎

The following corollary provides a particular result for a specific n^{th} degree polynomial.

Corollary 4. *Let the conditions of Theorem 4 hold. The following result is then valid. Namely, for $\theta \in [a,x]$*

$$\left| f(x) - \tau_n^*(f;a,x) - \frac{(-1)^n}{\binom{n}{m}} U_n^*(x)\left[f^{(n)};a,x\right] \right| \tag{3.21}$$

$$\begin{aligned} &\le (x-a)\, b_n^*(x)\, \sigma\left(f^{(n+1)};a,x\right), && f^{(n+1)} \in L_2[I], \\ &\le (x-a)\, b_n^*(x)\left(\frac{\Gamma_{n+1}(x) - \gamma_{n+1}(x)}{2}\right), && \gamma_{n+1}(x) \le f^{(n+1)}(t) \le \Gamma_{n+1}(x), \\ & && t \in [a,x] \\ &\le (x-a) \cdot \frac{\left(\Phi_{n+1}^*(x) - \phi_{n+1}^*(x)\right)}{2} && \phi_{n+1}^*(x) \le \frac{\kappa_n(t)}{2} \le \Phi_{n+1}^*(x), \\ &\qquad \times \frac{\left(\Gamma_{n+1}(x) - \gamma_{n+1}(x)\right)}{2}, && t \in [a,x] \end{aligned}$$

where

$$\begin{aligned} &\tau_n^*(f;a,x) \\ &\quad = f(a) + \frac{1}{\binom{n}{m}} \sum_{k=1}^{n} (-1)^{k+1}\left[\kappa_n^{(k)}(x) f^{(n-k)}(x) - \kappa_n^{(k)}(a) f^{(n-k)}(a)\right] \end{aligned} \tag{3.22}$$

$$\kappa_n(t) = \frac{(t-x)^{n-m}}{(n-m)!} \cdot \frac{(t-\theta)^m}{m!}, \quad \theta \in [a,x], \tag{3.23}$$

$$\kappa_n^{(k)}(t) = \sum_{j=L}^{U} \binom{k}{j} \frac{(t-x)^{n-m-j}}{(n-m-j)!} \cdot \frac{(t-\theta)^{j-k+m}}{(j-k+m)!} \tag{3.24}$$

with $U = \min\{k, n-m\}$, $L = \max\{0, k-m\}$.
Further,

$$\begin{aligned} &\frac{U_n^*(x)}{\binom{n}{m}} \\ &= \begin{cases} (-1)^{n+1} \frac{(x-a)^{n+1}}{(n+1)!}, & \theta = x, \\ (-1)^{n-m} \frac{(x-\theta)^{n+1}}{n!}\left[\frac{1}{\binom{n}{m}} - B\left(n-m+1, m+1, \frac{a-\theta}{x-\theta}\right)\right], & \theta \in [a,x) \end{cases} \end{aligned} \tag{3.25}$$

$$b_n^*(x) = \frac{1}{\binom{n}{m}}\left[\mathcal{M}^*\left(\kappa_n^2\right) - \left(\frac{U_n^*(x)}{x-a}\right)^2\right]^{\frac{1}{2}}, \tag{3.26}$$

$$\begin{aligned} &\mathcal{M}^*\left(\kappa_n^2\right) \\ &= \begin{cases} (-1)^{n+1} \frac{(x-a)^{2n}}{(n-m)!m!(2n+1)}, & \theta = x, \\ \frac{(x-\theta)^{2n+1}}{(n-m)!m!(x-a)}\left[\frac{1}{\binom{2n}{2m}} - B\left(2(n-m)+1, 2m+1, \frac{a-\theta}{x-\theta}\right)\right], & \theta \in [a,x) \end{cases} \end{aligned} \tag{3.27}$$

and

$$B(\alpha,\beta;X)=\int_0^X(1-u)^{\alpha-1}u^{\beta-1}du\quad\textit{with}\quad B(\alpha,\beta;1)=\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}. \tag{3.28}$$

Finally,

$$\binom{n}{m}\phi_n^*(x)=\inf_{t\in[a,x]}\kappa_n(t)\quad\textit{and}\quad\binom{n}{m}\Phi_n^*(x)=\sup_{t\in[a,x]}\kappa_n(t).$$

Proof. The above results are a direct consequence of Theorem 4 for $K_n(t)=\kappa_n(t)$ as given by (3.23). A star is used to signify this particular $K_n(t)$. Now, from (3.17),

$$\frac{U_n^*(x)}{\binom{n}{m}}=\int_a^x\kappa_n(t)\,dt=\frac{1}{n!}\int_a^x(t-x)^{n-m}\cdot(t-\theta)^m\,dt,\quad\theta\in[a,x]. \tag{3.29}$$

For $\theta=x$ then $U_n^*(x)=(-1)^{n+1}\frac{(x-a)^{n+1}}{(n+1)(n-m)!m!}$ and for $\theta\in[a,x)$ we have

$$\begin{aligned}\frac{U_n^*(x)}{\binom{n}{m}}&=\frac{1}{n!}\int_{a-\theta}^{x-\theta}(u+\theta-x)^{n-m}u^m du\\&=(-1)^{n-m}\frac{(x-\theta)^{n+1}}{n!}\Big[B(n-m+1,m+1,1)\\&\quad-B\left(n-m+1,m+1,\frac{a-\theta}{x-\theta}\right)\Big].\end{aligned}$$

Now, since $B(\alpha,\beta;1)=\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$, then the expression (3.25) is as stated.

We have to determine $b_n^*(x)$. From (3.19) we have

$$b_n^*(x)=\frac{1}{\binom{n}{m}}\left[\frac{1}{x-a}\int_a^x\kappa_n^2(t)\,dt-\left(\frac{U_n^*(x)}{x-a}\right)^2\right]^{\frac{1}{2}}. \tag{3.30}$$

Thus, since $U_n^*(x)$ has already been determined as (3.25) we need to evaluate $\mathcal{M}^*(\kappa_n^2)$. That is, from (3.23)

$$(x-a)\mathcal{M}^*(\kappa_n^2)=\int_a^x\kappa_n^2(t)\,dt$$

and so,

$$(n-m)!m!(x-a)\mathcal{M}^*(\kappa_n^2)=\int_a^x(t-x)^{2(n-m)}(t-\theta)^{2m}dt:=J(\theta). \tag{3.31}$$

For $\theta=x$ we have

$$J(x)=\int_a^x(t-x)^{2n}dt=\frac{(x-a)^{2n+1}}{(2n+1)!}. \tag{3.32}$$

For $\theta\neq x$, that is for $\theta\in[a,x)$ we have

$$\begin{aligned}J(\theta)&=\int_{a-\theta}^{x-\theta}(u-(x-\theta))^{2(n-m)}u^{2m}du\\&=(x-\theta)^{2n+1}\int_{\frac{a-\theta}{x-\theta}}^1(1-v)^{2(n-m)}v^{2m}dv\\&=(x-\theta)^{2n+1}\left[\frac{1}{\binom{2n}{2m}}-B\left(2(n-m)+1,2m+1,\frac{a-\theta}{x-\theta}\right)\right].\end{aligned} \tag{3.33}$$

Combining (3.32) and (3.21) with (3.31) followed by substitution in (3.30) gives (3.26). ∎

Remark 4. *If $m = n$ then $\kappa_n(t) = P_n^{c_\lambda}(t)$ with $\theta = \lambda a + (1-\lambda)x$, then $b_n^*(x)$ is equivalent to $\beta_n^{c_\lambda}$ given in (3.11) with (3.12). Taking $\theta = x$ will reproduce the results of Dragomir* [5] *involving bounds for the perturbed traditional Taylor representation.*

Remark 5. *It is possible to obtain bounds for perturbed Taylor formulae by using the Chebychev inequality and an inequality due to Lupaş. The Chebychev inequality (see* [9, p. 207]*) states that for $g, h : [a, x] \to \mathbb{R}$ absolutely continuous and $g'(\cdot)$, $h'(\cdot)$ bounded then*

$$|\mathfrak{T}(g,h)| \le \frac{1}{12}(x-a)^2 \sup_{t\in[a,x]} |g'(t)| \sup_{t\in[a,x]} |h'(t)|.$$

However $\mathfrak{T}^2(g,h) \le \mathfrak{T}(g,g)\,\mathfrak{T}(h,h)$ and so

$$|\mathfrak{T}(g,h)| \le \frac{x-a}{\sqrt{12}} \sup_{t\in[a,x]} |g'(t)| \sqrt{\mathfrak{T}(h,h)}. \tag{3.34}$$

The Lupaş result (see [3, p. 210]) states that if $g, h : (a, x) \to \mathbb{R}$ are locally absolutely continuous on $I = (a, x)$ and $g', h' \in L_2(I)$ then

$$|\mathfrak{T}(g,h)| \le \frac{(x-a)^2}{\pi^2} \|g'\|_2^\dagger \|h'\|_2^\dagger,$$

where

$$\|f\|_2^\dagger := \left(\frac{1}{x-a}\int_a^x |f(t)|^2 \, dt\right)^{\frac{1}{2}}, \quad f \in L_2(I).$$

Following a similar procedure as above gives

$$|\mathfrak{T}(g,h)| \le \frac{x-a}{\pi} \|g'\|_2^\dagger \sqrt{\mathfrak{T}(h,h)}. \tag{3.35}$$

Taking $g(t) = f^{(n+1)}(t)$ and $h(t) = P_n(t)$ or $K_n(t)$ in (3.34) and (3.35) would produce further bounds for Taylor expansions. These would however, require further conditions on $f^{(n+2)}(t)$ being bounded when utilizing the result (3.34) and $f^{(n+2)} \in L_2(I)$ for (3.35). The earlier results were in terms of conditions on $f^{(n+1)}(t)$ rather than $f^{(n+2)}(t)$. This will not be pursued further here.

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School of Communications and Informatics, Victoria University of Technology, PO Box 14428, Melbourne City MC, Victoria 8001, Australia

E-mail address: pc@matilda.vu.edu.au

URL: http://sci.vu.edu.au/staff/peterc.html