A PERTURBED VERSION OF THE GENERALISED TAYLOR'S FORMULA AND APPLICATIONS

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Abstract. A Perturbed version of the generalised Taylor's formula for Appell-type polynomials and applications are given.

1. Introduction

In [4], Matić et al. introduced the concept of *harmonic sequences of polynomials* by assuming that the polynomial $\{P_n\}_{n\in\mathbb{N}}$ satisfies the condition

$$P_0 = 1,\ \ P_n'(t) = P_{n-1}(t) \quad \text{for all } t\in\mathbb{R} \ \text{ and } \ n\in\mathbb{N}. \tag{1.1}$$

With this assumption, they proved the following generalised Taylor's formula:

Theorem 1. *Let $I\subset\mathbb{R}$ be a closed interval and $a\in I$. If $f: I\to\mathbb{R}$ is any function such that, for some $n\in\mathbb{N}$, $f^{(n)}$ is absolutely continuous, then for any $x\in I$*

$$f(x) = f(a) + \sum_{k=1}^{n} (-1)^{k+1}\left[P_k(x) f^{(k)}(x) - P_k(a) f^{(k)}(a)\right] + R_n(f;a,x), \tag{1.2}$$

where

$$R_n(f;a,x) = (-1)^n \int_a^x P_n(t) f^{(n+1)}(t)\,dt \tag{1.3}$$

and $\{P_n\}_{n\in\mathbb{N}}$ is a harmonic sequence of polynomials.

As examples of such polynomials, they mentioned the following

$$P_n(t) := \frac{1}{n!}(t-x)^n, \ \ t\in\mathbb{R};$$

or

$$P_n(t) := \frac{1}{n!}\left(t - \frac{a+x}{2}\right)^n, \ \ t\in\mathbb{R};$$

or

$$P_n(t) := \frac{(x-a)^n}{n!} B_n\left(\frac{t-a}{x-a}\right),\ x\geq 1,\ P_0(t) = 1,$$

where $B_n(\cdot)$ are the *Bernoulli polynomials,* or

$$P_n(t) := \frac{(x-a)^n}{n!} E_n\left(\frac{t-a}{x-a}\right),\ x\geq 1,\ P_0(t) = 1,$$

where $E_n(\cdot)$ are the *Euler polynomials.*

Matić et al. [4] proved the following general estimation result for the remainder $R_n(f;a,x)$.

Date: May 16, 2002.

1991 *Mathematics Subject Classification.* Primary 26D15; Secondary 26D10.

Key words and phrases. Taylor's formula, Appell polnomials.

Corollary 1. *Under the assumptions of Theorem 1 and if $x \geq a$, then*

$$(1.4)\qquad |R_n(f;a,x)| \leq \begin{cases} \max\limits_{t\in[a,x]} |P_n(t)| \int_a^x \left|f^{(n+1)}(s)\right| ds; \\[2ex] \int_a^x |P_n(s)|\, ds \max\limits_{t\in[a,x]} \left|f^{(n+1)}(t)\right|; \\[2ex] \left(\int_a^x |P_n(s)|^q\, ds\right)^{\frac{1}{q}} \left(\int_a^x \left|f^{(n+1)}(s)\right|^p ds\right)^{\frac{1}{p}}, \\ \qquad \text{where } \frac{1}{p}+\frac{1}{q}=1,\ p>1. \end{cases}$$

Now, if one would choose in (1.2) $f(x) = \int_a^b g(t)\,dt$ and then put $x = b$, one could state the following generalised integration by parts formula

$$(1.5)\qquad \int_a^b g(t)\,dt = \sum_{k=1}^{n} (-1)^{k+1} \left[P_k(b)\, g^{(k-1)}(b) - P_k(a)\, g^{(k-1)}(a)\right] + S_n(f;a,b),$$

where

$$(1.6)\qquad S_n(f;a,b) = (-1)^n \int_a^b P_n(t)\, g(t)\,dt.$$

Using the classical notation for the Lebesgue norms,

$$\|h\|_\infty := ess \sup_{t\in[a,b]} |h(t)|,$$

$$\|h\|_p := \left(\int_a^b |h(t)|^p\,dt\right)^{\frac{1}{p}},\ p \geq 1;$$

the remainder (1.6) may be bounded in the following manner

$$(1.7)\qquad |S_n(g;a,b)| \leq \begin{cases} \|P_n\|_\infty \left\|g^{(n)}\right\|_1 \\[1ex] \|P_n\|_1 \left\|g^{(n)}\right\|_\infty \\[1ex] \|P_n\|_q \left\|g^{(n)}\right\|_p, \ \text{where } \frac{1}{p}+\frac{1}{q}=1,\ p>1. \end{cases}$$

For recent results on Taylor's expansion see [1] and [3].

2. The Results

The following result is a reformulation of Theorem 1 in a more appropriate manner for our further purposes.

Theorem 2. *Assume that the sequence of polynomials $\{P_n(t,x)\}_{x\in\mathbb{N}}$ satisfies the Appell condition* [2]*, i.e.,*

$$(2.1)\qquad \frac{\partial P_n(t,x)}{\partial t} = P_{n-1}(t,x), \quad t,x \in \mathbb{R},\ n\in\mathbb{N} \ \text{ and } \ P_0(t,x) = 1,\ t,x\in\mathbb{R}.$$

Let $I \subset \mathbb{R}$ be a closed interval, $a \in I$. If $f : I \to \mathbb{R}$ is any function such that for some $n \in \mathbb{N}$, $f^{(n)}$ is absolutely continuous, then for any $x \in I$ we have the *representation*

$$f(x) = f(a) + \sum_{k=1}^{n} (-1)^{k+1} \left[P_k(x,x) f^{(k)}(x) - P_k(a,x) f^{(k)}(a) \right] + R_n(f;a,x), \tag{2.2}$$

where

$$R_n(f;a,x) = (-1)^n \int_a^x P_n(t,x) f^{(n+1)}(t)\, dt. \tag{2.3}$$

Using the notations

$$\|g\|_{[a,b],p} := \left| \int_a^b |g(t)|^p \, dt \right|^{\frac{1}{p}}, \quad p \geq 1$$

and

$$\|g\|_{[a,b],\infty} := ess \sup_{\substack{t \in [a,b] \\ (t \in [b,a])}} |g(t)|,$$

where a, b are arbitrary real numbers, then we can state the following corollary (see [4, Corollary 1])

Corollary 2. *With the assumptions in Theorem 2, we have the bounds for the remainder* $R_n(f;a,x)$

$$|R_n(f;a,x)| \leq \begin{cases} \|P_n(\cdot,x)\|_{[a,x],\infty} \left\| f^{(n+1)} \right\|_{[a,x],1}; \\ \|P_n(\cdot,x)\|_{[a,x],q} \left\| f^{(n+1)} \right\|_{[a,x],p}, & p > 1, \ \frac{1}{p} + \frac{1}{q} = 1, \\ & \text{if } f^{(n+1)} \in L_p[a,x]; \\ \|P_n(\cdot,x)\|_{[a,x],1} \left\| f^{(n+1)} \right\|_{[a,x],\infty}, & \text{if } f^{(n+1)} \in L_\infty[a,x]. \end{cases} \tag{2.4}$$

The following perturbed version of the above result also holds.

Theorem 3. *Let* $P_n(t,x)$, f, a, x *be as in Theorem 2. Then for any* $y \in I$, *we have the representation:*

$$f(x) = f(a) + \sum_{k=1}^{n} (-1)^{k+1} \left[P_k(x,x) f^{(k)}(x) - P_k(a,x) f^{(k)}(a) \right] + (-1)^n \left[P_{n+1}(x,x) - P_{n+1}(a,x) \right] f^{(n+1)}(y) + S_n(f;a,x,y), \tag{2.5}$$

where the new remainder $S_n(f;a,x,y)$ *can be represented in the following manner*

$$S_n(f;a,x,y) = (-1)^n \int_a^x P_n(t,x) \left[f^{(n+1)}(t) - f^{(n+1)}(y) \right] dt. \tag{2.6}$$

Proof. We have

$$\begin{aligned} R_n(f;a,x) &= (-1)^n \int_a^x P_n(t,x) f^{(n+1)}(t)\,dt \\ &= (-1)^n \int_a^x P_n(t,x) f^{(n+1)}(y)\,dt \\ &\quad + (-1)^n \int_a^x P_n(t,x) \left[f^{(n+1)}(t) - f^{(n+1)}(y)\right] dt \\ &= (-1)^n P_{n+1}(t,x) f^{(n+1)}(y)\Big|_a^x + S_n(f;a,x,y) \\ &= (-1)^n \left[P_{n+1}(x,x) - P_{n+1}(a,x)\right] f^{(n+1)}(y) + S_n(f;a,x,y). \end{aligned}$$

Using (2.2) we deduce (2.5). ∎

Remark 1. *Some particular interesting cases are for $y = a$ and $y = x$.*

The following corollary for functions f for which $f^{(n+1)}$ is Hölder continuous holds.

Corollary 3. *If the function $f^{(n+1)} : I \to \mathbb{R}$ is of $r - H-$Hölder type on I, i.e.,*

$$\left|f^{(n+1)}(s) - f^{(n+1)}(u)\right| \leq H|s-u|^r \quad \text{for all } s,u \in I, \tag{2.7}$$

where $H > 0$ and $r \in [0,1]$ are given, then for any y between a and x, we have the bound

$$|S_n(f;a,x,y)| \leq \begin{cases} \frac{H}{r+1}\left[|y-a|^{r+1} + |x-y|^{r+1}\right] \|P_n(\cdot,x)\|_{[a,x],\infty}; \\ \\ \frac{H}{(rp+1)^{\frac{1}{p}}}\left[|y-a|^{rp+1} + |x-y|^{rp+1}\right]^{\frac{1}{p}} \|P_n(\cdot,x)\|_{[a,x],q}, \ p>1, \ \frac{1}{p}+\frac{1}{q}=1; \\ \\ H\max\left\{|y-a|^r, |x-y|^r\right\} \|P_n(\cdot,x)\|_{[a,x],1}. \end{cases} \tag{2.8}$$

Proof. As $f^{(n+1)}$ is of Hölder type, we have

$$\begin{aligned} |S_n(f;a,x,y)| &\leq \left|\int_a^x \left||P_n(t,x)|\left[f^{(n+1)}(t) - f^{(n+1)}(y)\right]\right| dt\right| \\ &\leq H\left|\int_a^x |P_n(t,x)|\,|t-y|^r\,dt\right| =: A(x,a). \end{aligned}$$

Obviously,

$$\begin{aligned} A(x,a) &\leq H\|P_n(\cdot,x)\|_{[a,x],\infty}\left|\int_a^x |t-y|^r\,dt\right| \\ &= H\|P_n(\cdot,x)\|_{[a,x],\infty}\frac{|y-a|^{r+1} + |x-y|^{r+1}}{r+1}, \end{aligned}$$

which proves the first part of (2.8).

Using Hölder's integral inequality, we have:

$$\begin{aligned} A(x,a) &\le H\left\|P_n(\cdot,x)\right\|_{[a,x],q}\left|\int_a^x |t-y|^{rp}\,dt\right|^{\frac{1}{p}} \\ &= H\left\|P_n(\cdot,x)\right\|_{[a,x],q}\left[\frac{|y-a|^{rp+1}+|x-y|^{rp+1}}{rp+1}\right]^{\frac{1}{p}} \\ &= \frac{H}{(rp+1)^{\frac{1}{p}}}\left\|P_n(\cdot,x)\right\|_{[a,x],q}\left[|y-a|^{rp+1}+|x-y|^{rp+1}\right]^{\frac{1}{p}} \end{aligned}$$

and the second inequality in (2.8) is also proved.

Finally,

$$\begin{aligned} A(x,a) &\le H\max_{\substack{t\in[a,x]\\(t\in[x,a])}}|t-y|^r\cdot\left\|P_n(\cdot,x)\right\|_{[a,x],1} \\ &\le H\max\left\{|y-a|^r,|x-y|^r\right\}\left\|P_n(\cdot,x)\right\|_{[a,x],1} \end{aligned}$$

and the corollary is proved. ∎

Remark 2. *We have the following inequalities:*

$$\begin{aligned} &\left|f(x)-\left[f(a)+\sum_{k=1}^{n}(-1)^{k+1}\left[P_k(x,x)f^{(k)}(x)-P_k(a,x)f^{(k)}(a)\right]\right.\right. \\ &\left.\left.+(-1)^n\left[P_{n+1}(x,x)-P_{n+1}(a,x)\right]f^{(n+1)}(a)\right]\right| \\ &\le B(x,a):=\begin{cases} \dfrac{H|x-a|^{r+1}}{r+1}\left\|P_n(\cdot,x)\right\|_{[a,x],\infty}; \\ \dfrac{H|x-a|^{r+\frac{1}{p}}}{(rp+1)^{\frac{1}{p}}}\left\|P_n(\cdot,x)\right\|_{[a,x],q},\ p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ H|x-a|^r\left\|P_n(\cdot,x)\right\|_{[a,x],1}; \end{cases} \end{aligned} \tag{2.9}$$

and

$$\begin{aligned} &\left|f(x)-\left[f(a)+\sum_{k=1}^{n}(-1)^{k+1}\left[P_k(x,x)f^{(k)}(x)-P_k(a,x)f^{(k)}(a)\right]\right.\right. \\ &\left.\left.+(-1)^n\left[P_{n+1}(x,x)-P_{n+1}(a,x)\right]f^{(n+1)}(x)\right]\right|\le B(x,a) \end{aligned} \tag{2.10}$$

and

$$\begin{aligned} &\left|f(x)-\left[f(a)+\sum_{k=1}^{n}(-1)^{k+1}\left[P_k(x,x)f^{(k)}(x)-P_k(a,x)f^{(k)}(a)\right]\right.\right. \\ &\left.\left.+(-1)^n\left[P_{n+1}(x,x)-P_{n+1}(a,x)\right]f^{(n+1)}\left(\frac{a+x}{2}\right)\right]\right|\le\frac{1}{2^r}B(x,a), \end{aligned} \tag{2.11}$$

where f and P_n are as in Corollary 3.

3. Some Particular Cases

It is natural now to investigate the case when $f^{(n+1)}$ is assumed to be absolutely continuous.

The following result holds.

Corollary 4. *Let $P_n(t,x)$, f,a,x be as in Theorem 1 and $f^{(n+1)}$ is absolutely continuous on I. Then we have the inequality*

$$|S_n(f;a,x,y)|$$

$$\leq \begin{cases} \left[\frac{1}{4}(x-a)^2+\left(y-\frac{a+x}{2}\right)^2\right] \\ \qquad \times \|P_n(\cdot,x)\|_{[a,x],\infty}\left\|f^{(n+2)}\right\|_{[a,x],\infty}, & \text{if } f^{(n+2)}\in L_\infty[a,x]; \\ \left[\frac{|y-a|^{\beta+1}+|x-y|^{\beta+1}}{\beta+1}\right]^{\frac{1}{\beta}}\|P_n(\cdot,x)\|_{[a,x],\alpha}\left\|f^{(n+2)}\right\|_{[a,x],\infty}, & \text{if } f^{(n+2)}\in L_\infty[a,x]; \\ & \alpha>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1; \\ \left[\frac{1}{2}|x-a|+\left|y-\frac{a+x}{2}\right|\right]\|P_n(\cdot,x)\|_{[a,x],1}\left\|f^{(n+2)}\right\|_{[a,x],\infty} & \text{if } f^{(n+2)}\in L_\infty[a,x]; \\ \left[\frac{|x-y|^{\frac{1}{q}+1}+|y-a|^{\frac{1}{q}+1}}{\frac{1}{q}+1}\right]\|P_n(\cdot,x)\|_{[a,x],\infty}\left\|f^{(n+2)}\right\|_{[a,x],p}, & p>1,\ \frac{1}{p}+\frac{1}{q}=1, \\ & f^{(n+2)}\in L_p[a,x]; \\ \left[\frac{|x-y|^{\frac{\delta}{q}+1}+|y-a|^{\frac{\delta}{q}+1}}{\frac{\delta}{q}+1}\right]^{\frac{1}{\delta}}\|P_n(\cdot,x)\|_{[a,x],\gamma}\left\|f^{(n+2)}\right\|_{[a,x],p}, \\ p>1,\ \frac{1}{p}+\frac{1}{q}=1,\ \delta>1,\ \frac{1}{\delta}+\frac{1}{\gamma}=1,\ f^{(n+2)}\in L_p[a,x]; \\ \left[\frac{1}{2}|x-a|+\left|y-\frac{a+x}{2}\right|\right]^{\frac{1}{\delta}}\|P_n(\cdot,x)\|_{[a,x],1}\left\|f^{(n+2)}\right\|_{[a,x],p}, & p>1,\ \frac{1}{p}+\frac{1}{q}=1, \\ & f^{(n+2)}\in L_p[a,x]; \\ \|P_n(\cdot,x)\|_{[a,x],1}\left\|f^{(n+2)}\right\|_{[a,x],1}. \end{cases}$$

Proof. Since $f^{(n+1)}$ is absolutely continuous, then $f^{(n+1)}$ is a.e. differentiable on

$$f^{(n+1)}(t)-f^{(n+1)}(y)=\int_y^t f^{(n+2)}(u)\,du.$$

We have

$$\left|f^{(n+1)}(t)-f^{(n+1)}(y)\right|\leq\begin{cases}|t-y|\left\|f^{(n+2)}\right\|_{[y,t],\infty} & \text{if } f^{(n+2)}\in L_\infty[a,x] \\ |t-y|^{\frac{1}{q}}\left\|f^{(n+2)}\right\|_{[y,t],p} & \text{if } f^{(n+2)}\in L_p[a,x] \\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1, \\ \left\|f^{(n+2)}\right\|_{[y,t],1}. \end{cases}.$$

Using the representation (2.6), we may get that

$$|S_n(f;a,x,y)| \le \begin{cases} \left|\int_a^x |P_n(t,x)|\,|t-y|\,\left\|f^{(n+2)}\right\|_{[y,t],\infty} dt\right| =: M_1(a,x) \\ \left|\int_a^x |P_n(t,x)|\,|t-y|^{\frac{1}{q}}\left\|f^{(n+2)}\right\|_{[y,t],p} dt\right| =: M_2(a,x) \\ \left|\int_a^x |P_n(t,x)|\left\|f^{(n+2)}\right\|_{[y,t],1} dt\right| =: M_3(a,x) \end{cases}.$$

Since

$$\left\|f^{(n+2)}\right\|_{[y,t],\infty} \le \left\|f^{(n+2)}\right\|_{[a,x],\infty},$$

then we get

$$\begin{aligned} & M_1(a,x) \\ \le\; & \left\|f^{(n+2)}\right\|_{[a,x],\infty}\left|\int_a^x |P_n(t,x)|\,|t-y|\,dt\right| \\ \le\; & \left\|f^{(n+2)}\right\|_{[a,x],\infty} \times \begin{cases} \|P_n(\cdot,x)\|_{[a,x],\infty}\left[\frac{1}{4}(x-a)^2 + \left(y-\frac{a+x}{2}\right)^2\right] \\ \|P_n(\cdot,x)\|_{[a,x],\alpha}\left[\frac{|y-a|^{\beta+1}+|x-y|^{\beta+1}}{\beta+1}\right]^{\frac{1}{\beta}}, \ \alpha>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1; \\ \|P_n(\cdot,x)\|_{[a,x],1}\left[\frac{1}{2}|x-a| + \left|y-\frac{a+x}{2}\right|\right]. \end{cases} \end{aligned}$$

Since

$$\left\|f^{(n+2)}\right\|_{[y,t],p} \le \left\|f^{(n+2)}\right\|_{[a,x],p},$$

then we get

$$\begin{aligned} & M_2(a,x) \\ \le\; & \left\|f^{(n+2)}\right\|_{[a,x],p}\left|\int_a^x |P_n(t,x)|\,|t-y|^{\frac{1}{q}}\,dt\right| \\ \le\; & \left\|f^{(n+2)}\right\|_{[a,x],p} \times \begin{cases} \|P_n(\cdot,x)\|_{[a,x],\infty}\left[\frac{|x-y|^{\frac{1}{q}+1}+|y-a|^{\frac{1}{q}+1}}{\frac{1}{q}+1}\right] \\ \|P_n(\cdot,x)\|_{[a,x],\gamma}\left[\frac{|x-y|^{\frac{\delta}{q}+1}+|y-a|^{\frac{\delta}{q}+1}}{\frac{\delta}{q}+1}\right]^{\frac{1}{\delta}}, \ \gamma>1,\ \frac{1}{\delta}+\frac{1}{\gamma}=1, \\ \|P_n(\cdot,x)\|_{[a,x],1}\left[\frac{1}{2}|x-a| + \left|y-\frac{a+x}{2}\right|\right]^{\frac{1}{q}} \end{cases} \end{aligned}$$

and since

$$\left\|f^{(n+2)}\right\|_{[y,t],1} \le \left\|f^{(n+2)}\right\|_{[a,x],1},$$

then

$$M_3(a,x) \le \left\|f^{(n+2)}\right\|_{[a,x],1}\|P_n(\cdot,x)\|_{[a,x],1}.$$

and the corollary is proved. ∎

Remark 3. *We have the following inequality*

$$
\begin{aligned}
& \left| f(x) - \left[f(a) + \sum_{k=1}^{n} (-1)^{k+1} \left[P_k(x,x) f^{(k)}(x) - P_k(a,x) f^{(k)}(a) \right] \right.\right. \\
& \left.\left. + (-1)^n \left[P_{n+1}(x,x) - P_{n+1}(a,x) \right] f^{(n+1)}(a) \right] \right| \\
\leq\ & L(x,a) \\
:=\ & \begin{cases}
\frac{1}{2}(x-a)^2 \|P_n(\cdot,x)\|_{[a,x],\infty} \left\|f^{(n+2)}\right\|_{[a,x],\infty}, & \text{if } f^{(n+2)} \in L_\infty[a,x]; \\
\frac{1}{(\beta+1)^{\frac{1}{\beta}}} |x-a|^{1+\frac{1}{\beta}} \|P_n(\cdot,x)\|_{[a,x],\alpha} \left\|f^{(n+2)}\right\|_{[a,x],\infty}, & \text{if } f^{(n+2)} \in L_\infty[a,x]; \\
 & \text{and } \alpha > 1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1; \\
|x-a| \|P_n(\cdot,x)\|_{[a,x],1} \left\|f^{(n+2)}\right\|_{[a,x],\infty} & \text{if } f^{(n+2)} \in L_\infty[a,x]; \\
\frac{q|x-a|^{\frac{1}{q}+1}}{q+1} \|P_n(\cdot,x)\|_{[a,x],\infty} \left\|f^{(n+2)}\right\|_{[a,x],p}, & p>1,\ \frac{1}{p}+\frac{1}{q}=1, \\
 & f^{(n+2)} \in L_p[a,x]; \\
\frac{q^{\frac{1}{\delta}} |x-a|^{\frac{1}{q}+\frac{1}{\delta}}}{(\delta+q)^{\frac{1}{\delta}}} \|P_n(\cdot,x)\|_{[a,x],\gamma} \left\|f^{(n+2)}\right\|_{[a,x],p}, & p>1,\ \frac{1}{p}+\frac{1}{q}=1, \\
\delta > 1,\ \frac{1}{\delta}+\frac{1}{\gamma}=1,\ f^{(n+2)} \in L_p[a,x]; & \\
|x-a|^{\frac{1}{q}} \|P_n(\cdot,x)\|_{[a,x],1} \left\|f^{(n+2)}\right\|_{[a,x],p}, & p>1,\ \frac{1}{p}+\frac{1}{q}=1, \\
 & f^{(n+2)} \in L_p[a,x];
\end{cases}
\end{aligned}
$$

and

$$
\begin{aligned}
& \left| f(x) - \left[f(a) + \sum_{k=1}^{n} (-1)^{k+1} \left[P_k(x,x) f^{(k)}(x) - P_k(a,x) f^{(k)}(a) \right] \right.\right. \\
& \left.\left. + (-1)^n \left[P_{n+1}(x,x) - P_{n+1}(a,x) \right] f^{(n+1)}(x) \right] \right| \\
\leq\ & L(x,a)
\end{aligned}
$$

and

$$
\begin{aligned}
& \left| f(x) - \left[f(a) + \sum_{k=1}^{n} (-1)^{k+1} \left[P_k(x,x) f^{(k)}(x) - P_k(a,x) f^{(k)}(a) \right] \right.\right. \\
& \left.\left. + (-1)^n \left[P_{n+1}(x,x) - P_{n+1}(a,x) \right] f^{(n+1)}\left(\frac{a+x}{2}\right) \right] \right| \\
\leq\ & \frac{1}{2} L(x,a).
\end{aligned}
$$

If we consider the polynomial $P_n(t,x) := \frac{1}{n!}(t-x)^n$, then we have, by (2.5), the following perturbed version of Taylor's formula

$$
f(x) = f(a) + \sum_{k=1}^{n} \frac{(x-a)^k}{k!} f^{(k)}(a) + \frac{(x-a)^{n+1}}{(n+1)!} f^{(n+1)}(y) + S_n^T(f;a,x,y) \tag{3.1}
$$

where the remainder $S_n^T(f;a,x,y)$ is given by

$$S_n^T(f;a,x,y)=\frac{(-1)^n}{n!}\int_a^x (t-x)^n\left[f^{(n+1)}(t)-f^{(n+1)}(y)\right]dt.$$

We have

$$\|P_n(\cdot,x)\|_{[a,x],1}=\frac{1}{n!}\left|\int_a^x |t-x|^n\,dt\right|=\frac{|x-a|^{n+1}}{(n+1)!},$$

$$\|P_n(\cdot,x)\|_{[a,x],q}=\frac{1}{n!}\left|\int_a^x |t-x|^{nq}\,dt\right|^{\frac{1}{q}}=\frac{|x-a|^{n+\frac{1}{q}}}{n!\,(nq+1)^{\frac{1}{q}}},\ q\in(1,\infty),$$

$$\|P_n(\cdot,x)\|_{[a,x],\infty}=\frac{1}{n!}\sup_{\substack{t\in[a,x]\\(t\in[x,a])}}|t-x|^n=\frac{|x-a|^n}{n!}$$

for all $a,x\in I$.
Using (2.8), we may state that

$$\left|S_n^T(f;a,x,y)\right| \tag{3.2}$$
$$\leq\begin{cases}\dfrac{H}{(r+1)(n+1)!}|x-a|^{n+1}\left[|y-a|^{r+1}+|x-y|^{r+1}\right]; \\[2ex] \dfrac{H|x-a|^{n+\frac{1}{q}}}{n!\,(rp+1)^{\frac{1}{p}}(nq+1)^{\frac{1}{q}}}\left[|y-a|^{rp+1}+|x-y|^{rp+1}\right]^{\frac{1}{p}},\ p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\[2ex] H\dfrac{|x-a|^n}{n!}\max\left\{|y-a|^r,|x-y|^r\right\}.\end{cases}$$

for all y between x and a.

The following particular inequalities which follow by directly evaluating $A(x,a)$ (defined in Corollary 2) are useful:

$$\left|f(x)-\left[f(a)+\sum_{k=1}^{n+1}\frac{(x-a)^k}{k!}f^{(n)}(a)\right]\right| \tag{3.3}$$
$$\leq\quad C(x,a):=\begin{cases}\dfrac{H|x-a|^{n+r+2}}{(r+1)(n+1)!}; \\[2ex] \dfrac{H|x-a|^{n+r+1}}{n!\,(rp+1)^{\frac{1}{p}}(nq+1)^{\frac{1}{q}}}, \\[2ex] H\dfrac{|x-a|^{n+r}}{n!},\end{cases}$$

provided that $f^{(n+1)}$ is of $r-H-$Hölder type on I.

Example 1. *We also have*

$$\left|f(x)-\left[f(a)+\sum_{k=1}^{n}\frac{(x-a)^k}{k!}f^{(k)}(a)+\frac{(x-a)^{n+1}}{(n+1)!}f^{(n+1)}(x)\right]\right| \tag{3.4}$$
$$\leq\quad C(x,a)$$

and

$$\left|f(x)-\left[f(a)+\sum_{k=1}^{n}\frac{(x-a)^{k}}{k!}f^{(k)}(a)+\frac{(x-a)^{n+1}}{(n+1)!}f^{(n+1)}\left(\frac{a+x}{2}\right)\right]\right| \le \frac{1}{2^{r}}C(x,a), \tag{3.5}$$

where $f^{(n+1)}$ *is of* $r-H-$*Hölder type on* I*.*

If we consider the polynomial $P_n(t,x) := \frac{1}{n!}\left(t-\frac{a+x}{2}\right)^n$, then, by (2.5), we may state:

$$\begin{aligned} f(x) &= f(a)+\sum_{k=1}^{n}\frac{(x-a)^{k}}{2^{k}k!}\left[f^{(k)}(a)+(-1)^{k+1}f^{(k)}(x)\right] \\ &\quad +\frac{[1+(-1)^{n}](x-a)^{n+1}}{2^{n+1}(n+1)}f^{(n+1)}(y)+S_n^M(f;a,x,y), \end{aligned} \tag{3.6}$$

where the remainder $S_n^M(f;a,x,y)$ is given by

$$S_n^M(f;a,x,y)=\frac{(-1)^n}{n!}\int_a^x\left(t-\frac{a+x}{2}\right)^n\left[f^{(n+1)}(t)-f^{(n+1)}(y)\right]dt.$$

We have

$$\begin{aligned} \|P_n(\cdot,x)\|_{[a,x],1} &= \frac{1}{n!}\left|\int_a^x\left|t-\frac{a+x}{2}\right|^n dt\right| \\ &= \frac{1}{n!}\left|\int_a^{\frac{a+x}{2}}\left|t-\frac{a+x}{2}\right|^n dt+\int_{\frac{a+x}{2}}^x\left|t-\frac{a+x}{2}\right|^n dt\right| \\ &= \frac{1}{n!(n+1)}\left|\frac{|x-a|^{n+1}}{2^{n+1}}+\frac{|x-a|^{n+1}}{2^{n+1}}\right| \\ &= \frac{|x-a|^{n+1}}{(n+1)!2^n}; \end{aligned}$$

$$\begin{aligned} \|P_n(\cdot,x)\|_{[a,x],q} &= \frac{1}{n!}\left|\int_a^x\left|t-\frac{a+x}{2}\right|^{nq} dt\right|^{\frac{1}{q}} \\ &= \frac{1}{n!}\left|\int_a^{\frac{a+x}{2}}\left|t-\frac{a+x}{2}\right|^{nq} dt+\int_{\frac{a+x}{2}}^x\left|t-\frac{a+x}{2}\right|^{nq} dt\right|^{\frac{1}{q}} \\ &= \frac{1}{n!}\left[\frac{|x-a|^{nq+1}}{(nq+1)2^{nq+1}}+\frac{|x-a|^{nq+1}}{(nq+1)2^{nq+1}}\right]^{\frac{1}{q}} \\ &= \frac{1}{2^n n!(nq+1)^{\frac{1}{q}}}|x-a|^{n+\frac{1}{q}}; \end{aligned}$$

and

$$\|P_n(\cdot,x)\|_{[a,x],\infty}=\frac{1}{n!}\sup_{t\in[a,x]}\left|t-\frac{a+x}{2}\right|^n=\frac{|x-a|^n}{2^n n!}.$$

Using (2.8), we may state that:

$$
\left|S_n^M(f;a,x,y)\right| \leq \begin{cases} \dfrac{H|x-a|^{n+1}}{(n+1)!2^n(r+1)}\left[|y-a|^{r+1}+|x-y|^{r+1}\right]; \\[2ex] \dfrac{H|x-a|^{n+\frac{1}{q}}}{(rp+1)^{\frac{1}{p}}(nq+1)^{\frac{1}{q}}2^n n!}\left[|y-a|^{rp+1}+|x-y|^{rp+1}\right]^{\frac{1}{p}}, \\ \qquad\qquad p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\[2ex] H\dfrac{|x-a|^n}{2^n n!}\max\left\{|y-a|^r,|x-y|^r\right\}. \end{cases} \tag{3.7}
$$

The following particular inequalities which follow by directly evaluating $A(x,a)$ are useful:

$$
\begin{aligned}
&\left|f(x)-\left[f(a)+\sum_{k=1}^{n}\frac{(x-a)^k}{2^k k!}\left[f^{(k)}(a)+(-1)^{k+1}f^{(k)}(x)\right]\right.\right. \\
&\left.\left.+\ \frac{[1+(-1)^n](x-a)^{n+1}}{2^{n+1}(n+1)}f^{(n+1)}(a)\right]\right| \\
&\leq\ D(x,a):=\begin{cases} \dfrac{H|x-a|^{n+r+2}}{(n+1)!2^n(r+1)}; \\[2ex] \dfrac{H|x-a|^{n+r+1}}{(rp+1)^{\frac{1}{p}}(nq+1)^{\frac{1}{q}}2^n n!},\ p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\[2ex] H\dfrac{|x-a|^{n+r}}{2^n n!} \end{cases},
\end{aligned} \tag{3.8}
$$

where $f^{(n+1)}$ is of $r-H-$Hölder type on I.

We also have:

$$
\begin{aligned}
&\left|f(x)-\left[f(a)+\sum_{k=1}^{n}\frac{(x-a)^k}{2^k k!}\left[f^{(k)}(a)+(-1)^{k+1}f^{(k)}(x)\right]\right.\right. \\
&\left.\left.+\ \frac{[1+(-1)^n](x-a)^{n+1}}{2^{n+1}(n+1)}f^{(n+1)}(x)\right]\right| \leq D(x,a)
\end{aligned} \tag{3.9}
$$

and

$$
\begin{aligned}
&\left|f(x)-\left[f(a)+\sum_{k=1}^{n}\frac{(x-a)^k}{2^k k!}\left[f^{(k)}(a)+(-1)^{k+1}f^{(k)}(x)\right]\right.\right. \\
&\left.\left.+\ \frac{[1+(-1)^n](x-a)^{n+1}}{2^{n+1}(n+1)}f^{(n+1)}\left(\frac{a+x}{2}\right)\right]\right| \leq \frac{1}{2^r}D(x,a)
\end{aligned} \tag{3.10}
$$

for all $a,x\in I$.

Remark 4. *Similar inequalities may be stated if we use Corollary 4.*

Remark 5. *Similar inequalities can be stated for the other choices of "harmonic polynomials" $P_n(t,x)$ considered in* [4]. *As we do not have explicit or at least bounds for $\|P_n(\cdot,x)\|_p$, $p\in[1,\infty]$, we omit the details.*

The following corollaries are given for particular Peano kernels based on the order $a<y<x$.

Under the assumptions of Theorem 2, let a point $y\in I\subset\mathbb{R}$ such that $a<y<x$. If we consider the polynomial $P_n(t,x):=\frac{1}{n!}(t-x)^n$, by (2.5) we have the perturbed version of Taylor's formula:

$$f(x)=f(a)+\sum_{k=1}^{n}\frac{(x-a)^k}{k!}f^{(k)}(a)+\frac{(x-a)^{n+1}}{(n+1)!}f^{(n+1)}(y)+\tilde{S}_n^T(f;a,y,x), \tag{3.11}$$

where the remainder $\tilde{S}_n^T(f;a,y,x)$ is given by

$$\tilde{S}_n^T(f;a,y,x):=\frac{(-1)^n}{n!}\int_a^x(t-x)^n\left[f^{(n+1)}(t)-f^{(n+1)}(y)\right]dt. \tag{3.12}$$

The following corollary holds for functions f for which $f^{(n+1)}$ is Hölder continuous.

Corollary 5. *If the function $f^{(n+1)}:I\to\mathbb{R}$ is of $r-H-$Hölder type on I, we have, for $H>0$ and $r\in[0,1]$, the bound*

$$\left|\tilde{S}_n^T(f;a,y,x)\right|\le\frac{H}{n!}(x-y)^{n+r+1}\left[\Psi_\alpha(n+1,r+1)+B(n+1,r+1)\right], \tag{3.13}$$

where

$$\alpha=\frac{y-a}{x-y}>0,$$
$$B(l,s)=\int_0^1 t^{l-1}(1-t)^{s-1}dt,\quad l,s>0$$

is the classical Beta function and

$$\Psi_r(l,s)=\int_0^r t^{l-1}(1+t)^{s-1}dt,\quad l,s>0$$

is a real positive valued integral.

Proof. As $f^{(n+1)}$ is of Hölder type, we have

$$\left|\tilde{S}_n^T(f;a,y,x)\right|\le\frac{H}{n!}\int_a^x|t-x|^n|t-y|^r dt=\tilde{A}(a,x),$$

where

$$\tilde{A}(a,x)=\frac{H}{n!}\left[\int_a^y(x-t)^n(y-t)^r dt+\int_y^x(x-t)^n(t-y)^r dt\right].$$

Making the substitutions

$$t=y-(x-y)w\quad\text{and}\quad t=y+(x-y)w$$

into the first and second integrals respectively, we obtain

$$A(a,x)=\frac{H}{n!}(x-y)^{n+r+1}\left[\Psi_\alpha(n+1,r+1)+B(n+1,r+1)\right]$$

and Corollary 5 follows. ∎

Remark 6. *When $y=a$ the following inequality holds, provided that $f^{(n+1)}$ is of $r-H-$Hölder type on I*

$$\left|f(x)-\sum_{k=0}^{n+1}\frac{(x-a)^k}{k!}f^{(n)}(a)\right|\leq\frac{H(x-a)^{n+r+1}}{n!}B(n+1,r+1).$$

Corollary 6. *If $f^{(n+1)}:I\to\mathbb{R}$ is absolutely continuous on I, then we have*

$$\left|f(x)-f(a)-\sum_{k=1}^{n}\frac{(x-a)^k}{k!}f^{(k)}(a)-\frac{(x-a)^{n+1}}{(n+1)!}f^{(n+1)}(y)\right|$$
$$<\left\|f^{(n+2)}\right\|_\infty\left(\frac{(y-a)(x-a)^{n+1}}{(n+1)!}-\frac{(x-a)^{n+2}}{(n+2)!}+\frac{2(x-y)^{n+2}}{(n+2)!}\right),$$

where $f^{n+2}\in L_\infty[a,x]$.

Proof. From Corollary 4

$$\begin{aligned}\int_a^x|P_n(t,x)||t-y|\,dt &= \int_a^y\frac{|t-x|^n|t-y|}{n!}dt+\int_y^x\frac{|t-x|^n|t-y|}{n!}dt\\ &= \int_a^y\frac{(x-t)^n(y-t)}{n!}dt+\int_y^x\frac{(x-t)^n(t-y)}{n!}dt\\ &= \frac{(y-a)(x-a)^{n+1}}{(n+1)!}-\frac{(x-a)^{n+2}}{(n+2)!}+\frac{2(x-y)^{n+2}}{(n+2)!},\end{aligned}$$

and Corollary 6 follows. ∎

Now consider the polynomial

$$P_n(t,x):=\frac{1}{n!}\left(t-\frac{a+x}{2}\right)^n$$

and from (2.5) we may state

$$\begin{aligned}M\quad:&=f(x)-f(a)-\sum_{k=1}^{n}\frac{(x-a)^k}{2^k k!}\left[f^{(k)}(a)+(-1)^{k+1}f^{(k)}(x)\right]\\ &\quad-\frac{[1+(-1)^n](x-a)^{n+1}}{2^{n+1}(n+1)!}f^{(n+1)}(y)\\ &= \tilde{S}_n^M(f;a,y,x),\end{aligned}$$

where the remainder $\tilde{S}_n^M(f;a,y,x)$ is given by

$$\tilde{S}_n^M(f;a,y,x)=\frac{(-1)^n}{n!}\int_a^x\left(t-\frac{a+x}{2}\right)^n\left[f^{(n+1)}(t)-f^{(n+1)}(y)\right]dt.$$

Corollary 7. *If $f^{(n+1)}:I\to\mathbb{R}$ is absolutely continuous on I and $a<y<x$, then we have*

$$\left|\tilde{S}_n^M(f;a,y,x)\right|\leq\frac{2\left\|f^{(n+2)}\right\|_\infty}{(n+1)!}\left(\frac{x-a}{2}\right)^{n+1}\left|y-\frac{a+x}{2}\right|.$$

Proof. Consider

$$I=\int_a^x\frac{1}{n!}\left|t-\frac{a+x}{2}\right|^n|t-y|\,dt.$$

If $\frac{a+x}{2} \in (a,y)$, then

$$\begin{aligned} n!I_1 &= \int_a^{\frac{a+x}{2}} \left(\frac{a+x}{2}-t\right)^n (y-t)\,dt + \int_{\frac{a+x}{2}}^{y} \left(t-\frac{a+x}{2}\right)^n (y-t)\,dt \\ &\quad + \int_y^x \left(t-\frac{a+x}{2}\right)^n (y-t)\,dt \\ &= \int_a^{\frac{a+x}{2}} \left(\frac{a+x}{2}-t\right)^n (y-t)\,dt + \int_{\frac{a+x}{2}}^{x} \left(t-\frac{a+x}{2}\right)^n (y-t)\,dt. \end{aligned}$$

Integration by parts leads to the result

$$I_1 = \frac{2}{(n+1)!}\left(\frac{x-a}{2}\right)^{n+1}\left(y-\frac{a+x}{2}\right).$$

If $\frac{a+x}{2} \in [y,x)$, then also

$$\begin{aligned} n!I_2 &= \int_a^{y} \left(\frac{a+x}{2}-t\right)^n (t-y)\,dt + \int_{y}^{\frac{a+x}{2}} \left(\frac{a+x}{2}-t\right)^n (t-y)\,dt \\ &\quad + \int_{\frac{a+x}{2}}^x \left(t-\frac{a+x}{2}\right)^n (t-y)\,dt \\ &= \int_a^{\frac{a+x}{2}} \left(\frac{a+x}{2}-t\right)^n (t-y)\,dt + \int_{\frac{a+x}{2}}^{x} \left(t-\frac{a+x}{2}\right)^n (t-y)\,dt. \end{aligned}$$

Integration by parts leads to the result:

$$I_2 = \frac{2}{(n+1)!}\left(\frac{x-a}{2}\right)^{n+1}\left(-y+\frac{a+x}{2}\right).$$

Utilising I_1 and I_2 we see that Corollary 7 follows. ∎

Remark 7. *When $y=a$ the following inequality holds*

$$\left| f(x) - f(a) - \sum_{k=1}^{n} \frac{(x-a)^k}{2^k k!}\left[f^{(k)}(a) + (-1)^{k+1} f^{(k)}(x)\right] - \frac{\left[1+(-1)^n\right](x-a)^{n+1}}{2^{n+1}(n+1)!} f^{(n+1)}(a) \right| < \frac{2\left\|f^{(n+2)}\right\|_\infty}{(n+1)!}\left(\frac{x-a}{2}\right)^{n+2}.$$

References

[1] G.A. ANASTASSIOU and S.S. DRAGOMIR, On some estimates of the remainder in Taylor's formula, *J. of Math. Anal. & Appl.,* **263** (2001), 246-263.

[2] P. APPELL, Sur une classe de polnômes, *Ann. Sci. Ecole Norm. Sup.,* **9**(2) (1880), 119-144.

[3] S.S. DRAGOMIR, An improvement of the remainder estimate in the generalized Taylor formula, *RGMIA Res. Rep. Coll.,* **3**(1) (2000), Article 1.

[4] M. MATIĆ, J.E. PEČARIĆ, and N. UJEVIĆ, On new estimation of the remainder in generalized Taylor's formula, *Math. Ineq. Appl.,* **2**(3) (1999), 343-361.

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